Reliability estimation and parameter estimation for inverse Weibull distribution under different loss functions

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Abstract

In this paper, the classical and Bayesian estimators of the unknown parameters and the reliability function of the inverse Weibull distribution are considered. The maximum likelihood estimators (MLEs) and modified maximum likelihood estimators (MMLEs) are used in the classical parameter estimation. Bayesian estimators of the parameters are obtained by using symmetric and asymmetric loss functions under informative and non-informative priors. Bayesian computations are derived by using Lindley approximation and Markov chain Monte Carlo (MCMC) methods. The asymptotic confidence intervals are constructed based on the maximum likelihood estimators. The Bayesian credible intervals of the parameters are obtained by using the MCMC method. Furthermore, the performances of these estimation methods are compared concerning their biases and mean square errors through a simulation study. It is seen that the Bayes estimators perform better than the classical estimators. Finally, two real-life examples are given for illustrative purposes.

Keywords: Lindley's approximation; loss function; MCMC method; parameter estimation; reliability estimation.

1. Introduction

Weibull distribution is one of the most important distributions in reliability theory, risk analysis, actuarial sciences, and engineering and energy studies, (Lawless 2003; Tang 2004; Teimouri *et al*., 2013). This distribution has many forms. In this study, we focus on the Inverse Weibull (IW) distribution. IW distribution provides flexibility for modeling the long-tailed right-skewed data, see (Rinne, 2009). Therefore, IW distribution can be used in many real-life problems. It is a special case of the generalized extreme value distribution and is known as type 2 extreme value or the Fréchet distribution in the literature (Johnson *et al*., 1994). Also, Drapella (1993), Mudholkar & Kollia (1994) and Murthy *et al*. (2004) proposed the names complementary Weibull, reciprocal Weibull, and reverse Weibull for IW distribution. Over the decade, the inverse Weibull distribution has been studied by many researchers. Maswadah (2003) presented the conditional confidence intervals for unknown parameters and the reliability of the inverse Weibull distribution based on censored generalized order statistics. Calabria & Pulcini (1994) considered the Bayesian parameter estimation of the IW distribution under informative priors. Helu & Samawi (2015) presented a Bayes estimator for estimating the parameters of the IW distribution based on progressively first failure censoring using Lindley's methods. Kundu & Howlader (2010) discussed Bayesian inference of the inverse Weibull distribution for type-II censored data. Erto showed (1986) that the IW model provides a good fit for various survival data. Bi & Gui (2017) considered the problem of estimating stress-strength reliability for IW distribution.

In the recent past, many researchers have compared various parameter estimation methods for estimating the unknown parameters and reliability estimation of the different distributions. Remarkable among them are Guure *et al*. (2014) for Weibull failure time distribution; Singh *et al*. (2015) for extension of exponential distribution; Rastogi & Merovci (2018) for Weibull Rayleigh distribution; Farahani & Khorram (2014) for weighted exponential distribution and so on. Also, in recent years, Bayesian prediction plays an important role in different areas of applied statistics. Therefore, the problem of Bayesian prediction has been discussed by several authors (Koç & Cengiz, 2020; Kishorilaz & Mukhopadhyay, 2018; Rashid, 2019) and so on. Ramos *et al*. (2018) used the classical and Bayesian methods to estimate the parameters of inverse Weibull distribution. They considered Monte Carlo Markov Chain (MCMC) methods for Bayesian computation. In the present work, in addition to classical and Bayesian parameter estimation methods, the estimation of reliability function is discussed. The reliability function is widely used in life testing and survival analysis. Also, Bayesian computations are obtained by using Lindley approximation and MCMC methods. In Bayesian parameter estimation, the selection of loss function plays an important role. Therefore, Bayesian estimates are considered under symmetric and asymmetric loss functions. To the best of our knowledge, all these estimators for IW distribution have not been studied yet.

In the study, in the classical approach, the maximum likelihood estimators (MLEs) are considered. Moreover, we propose new estimators which are called the modified maximum likelihood estimators (MMLEs). Asymptotic confidence intervals (ACIs) are obtained for maximum likelihood estimators. In the Bayesian approach, we compute Bayes estimators of the unknown parameters under squared error loss function (SELF), linear exponential (LINEX) loss function, general entropy (GE) loss function, weighted squared loss function (WSELF), and precautionary (PRE) loss function. Based on these loss functions, we find the Bayes estimators of the unknown parameters and reliability function using informative and non-informative priors. The Bayes estimators cannot be expressed in closed form. Therefore, we implement the Markov Chain Monte Carlo (MCMC) and Lindley approximation methods to compute the estimates. Also, we obtain the Bayesian credible intervals (BCIs) by using Gibb's sampling. To evaluate the performance of these estimation methods an extensive Monte-Carlo simulation study is performed. Two real-life examples are analyzed at the end of the study for a better understanding of the methods presented in this paper.

The novelty of this paper comes from the fact that there has been no previous study comparing all these aforementioned estimation methods for inverse Weibull distribution.

The rest of the paper is organized as follows: inverse Weibull distribution is described in Section 2. In Section 3, some classical estimation methods are given to estimate the unknown parameters. In Section 4, Bayes estimators of unknown model parameters are obtained by using Lindley's and

MCMC methods. In Section 5, a simulation study is presented to evaluate the performances of the estimators concerning their biases and mean square errors (MSE). Finally, two real data sets are used for application purposes in Section 6.

2. Inverse Weibull distribution

The probability density function (pdf) of the inverse Weibull distribution with the shape parameter α and the scale parameter β is defined as follows:

$$f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-(x/\beta)^{-\alpha}}, \quad x>0,\ \alpha,\beta>0, \tag{1}$$

and the cumulative density function (CDF)

$$F(x)=e^{-\left(x/\beta\right)^{-\alpha}},\ x>0,\ \ \alpha,\beta>0, \tag{2}$$

where α and β represent shape and scale parameters, respectively.
The reliability function $R(t)$ for any specified time t of IW distribution is given by:

$$R(t)=P(x>t)=1-e^{-(t/\beta)^{-\alpha}},\ \ \alpha,\beta,t>0. \tag{3}$$

For details, see (Rinne, 2009).
Density and reliability plots for different values of α and $\beta=1$in Figure1 and Figure 2, respectively.

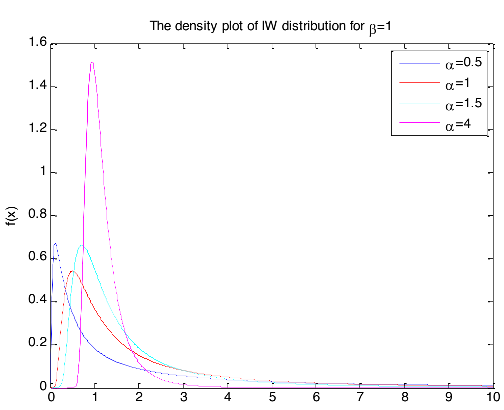

Fig. 1. Density and reliability plots for different values of α =1

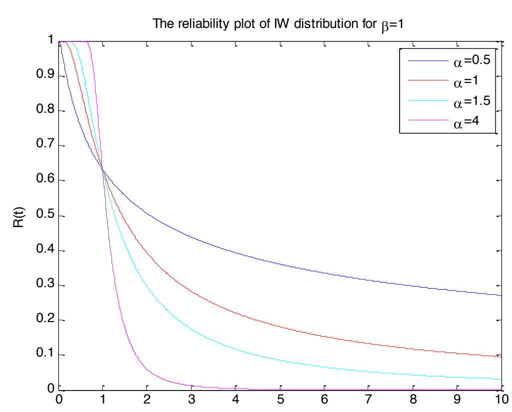

Fig. 2. Density and reliability plots for different values of $\beta=1$

3. The methods of parameter estimation

In this section, we present the parameter estimation methods used in this study.

3.1. Maximum likelihood estimation

The likelihood function of a random sample $X_1, X_2, ..., X_n$ from the IW distribution with pdf (1) is obtained as follows

$$L = \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} (x_i)^{-(\alpha+1)} \exp \sum_{i=1}^{n} -\left(\frac{x_i}{\beta}\right)^{-\alpha}. \tag{4}$$

On taking logarithms of (4), differentiating for α and β, and equating them to zero, we obtain the estimating equations:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + n \ln \beta - \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-\alpha} \ln\left(\frac{x_i}{\beta}\right) = 0, \tag{5}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n\alpha}{\beta} - \alpha \beta^{\alpha-1} \sum_{i=1}^{n} x_i^{-\alpha} = 0. \tag{6}$$

Then, by solving Equation (6), the parameter β is found as:

$$\beta = \left(\frac{\sum_{i=1}^{n} X_i^{-\alpha}}{n} \right)^{\frac{-1}{\alpha}} \tag{7}$$

Let $\hat{\alpha}_{MLE}$ $\hat{\beta}_{MLE}$ and denote the MLEs of α and β, respectively. The $\hat{\alpha}_{MLE}$ is obtained by solving non-linear equations in (5). For more details, see (Calabria & Pulcini 1990). Thereafter, the estimate $\hat{\alpha}_{MLE}$ can be obtained by substituting $\hat{\beta}_{MLE}$ in

$$\hat{\beta}_{MLE} = \left(\frac{\sum_{i=1}^{n} X_i^{-\hat{\alpha}_{MLE}}}{n} \right)^{\frac{-1}{\hat{\alpha}_{MLE}}} \tag{8}$$

Then, by considering Equation (3), the MLE of $R(t)$, say $\hat{R}_{MLE}(t)$, is given by:

$$\hat{R}_{MLE}(t) = 1 - \exp\left[-\left(\frac{t}{\hat{\beta}_{MLE}}\right)\right]^{-\hat{\alpha}_{MLE}}.$$

3.2. Modified maximum likelihood estimators

We propose new estimators for estimating α and β parameters and the reliability function of the inverse Weibull distribution. To obtain the MMLEs of the α, β, and $R(t)$ parameters, we follow the steps below:

Step 1: Let $X_1, X_2, ..., X_n$ is a random sample from the IW distribution with pdf (1). The expected value $\ln(X)$ is given by

$$E(\ln(X)) = \ln(\beta) + \frac{\gamma}{\alpha}.$$

Step 2: By equating the sample expected value of $\ln(X)$ and population expected value of $\ln(X)$ we obtain that

$$\ln(\beta) + \frac{\gamma}{\alpha} = \frac{\sum_{i=1}^{n} \ln(X_i)}{n}.$$

Substitution of β into Equation (7), the resulting equation α becomes

$$\frac{\gamma + \ln n - \ln \sum_{i=1}^{n} X_i^{-\alpha}}{\alpha} = \frac{\sum_{i=1}^{n} \ln(X_i)}{n}. \tag{9}$$

Also, MMLE of the shape parameter α, say $\hat{\alpha}_{MMLE}$,is obtained by solving this equation. Here, $\gamma \cong 0.57722$ is Euler constant.

Hence, by inserting $\hat{\alpha}_{MMLE}$ into equation (8) instead of $\hat{\alpha}$, MMLE of the scale parameter β is obtained as

$$\hat{\beta} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{-\hat{\alpha}_{MMLE}}\right)^{-1/\hat{\alpha}_{MMLE}}. \tag{10}$$

Then, by considering Equation (3), the MMLE estimator $R(t)$ is given as follows:

$$\hat{R}(t) = 1 - \exp\left[-\left(\frac{t}{\hat{\beta}_{MMLE}}\right)\right]^{-\hat{\alpha}_{MMLE}}.$$

4. Bayesian estimation

In this section, we consider the Bayesian estimation of α, β, and the reliability function $R(t)$ of the inverse Weibull distribution. We assume that α and β have two independent gamma prior distributions as in the following.

$v_1(\alpha) \propto \alpha^{a-1} e^{-b\alpha}, \quad \alpha > 0$ and $v_2(\beta) \propto \beta^{c-1} e^{-d\beta}, \quad \beta > 0.$
Here, the hyper-parameters of the prior distribution a,b,c,d are assumed to be known and non negative. Also, a,b,c,d are responsible for providing information about the unknown parameters. Then, the joint prior distribution of parameters α and β is given as:

$$v(\alpha,\beta) \propto \alpha^{a-1} \beta^{c-1} e^{-d\beta - b\alpha}, \quad \alpha,\beta,a,b,c,d \geq 0. \tag{11}$$

The joint posterior density functions of α and β are obtained as follows:

$$\pi(\alpha,\beta|x) = \frac{L(x|\alpha,\beta)v(\alpha,\beta)}{\iint L(x|\alpha,\beta)v(\alpha,\beta)d\alpha d\beta} \propto \alpha^{n+\alpha-1} \beta^{n\alpha+c-1} e^{-d\beta} e^{-b\alpha} e^{-\beta^{\alpha}\sum_{i=1}^{n} x_i^{-\alpha}} \prod_{i=1}^{n} (x_i)^{-(\alpha+1)} \tag{12}$$

In Bayesian parameter estimation, the selection of loss function has an important role. Therefore, we consider the Bayesian estimators for parameters α, β, and the reliability function $R(t)$ of inverse Weibull distribution under symmetric and asymmetric loss functions. One of the most popular loss functions is SELF. This loss function is symmetrical. It gives equal weightage to both underestimation and overestimation. In many practical situations, underestimation may be more serious than overestimation, and vice versa. In such cases, asymmetric loss functions can be taken into account. In this study, we consider the Bayesian estimation of α, β, and the reliability function $R(t)$ of the inverse Weibull distribution under the LINEX, GE, WSELF, and PRE asymmetric loss functions. These loss functions have been proved useful for performing Bayesian analysis in different fields of reliability estimation and life testing problems. For more details about the loss functions (LFs), see (Renjini *et al*., 2016; Ali *et al*., 2013; Helu & Samawi, 2015) among others, and the references cited therein. The Bayes estimators under mentioned loss functions are given in Table 1.

Table 1. Bayes estimator under different loss functions

Loss Function	Bayes Estimator
SELF	$\hat{\theta}_{SELF} = E(\theta\|x).$
LINEX	$\hat{\theta}_{LINEX} = -\frac{1}{k}\ln\left(E\left(e^{-k\theta}\|x\right)\right)$
GE	$\hat{\theta}_{GE} = \left(E\left(\theta^{-k}\|x\right)\right)^{\frac{-1}{k}}$
WSELF	$\hat{\theta}_{WSELF} = \left(E\left(\theta^{-1}\|x\right)\right)^{-1}$
PRE	$\hat{\theta}_{PRE} = \sqrt{E\left(\theta^{2}\|x\right)}$

In Table 1, k given in LINEX and GE loss functions reflects the magnitude and degree of symmetry. Now we obtain Bayes estimators of α, β, and $R(t)$ under the SELF, LINEX, GE, WSELF, and PRE loss functions when prior distribution is taken to be $v(\alpha,\beta)$.

Bayes estimators of α, β and $R(t)$ under the SELF are given as:

$$\hat{\alpha}_{SELF} = E(\alpha|x) = \int_0^\infty\int_0^\infty \alpha\pi(\alpha,\beta|x)d\alpha d\beta, \tag{13}$$

$$\hat{\beta}_{SELF} = E(\beta|x) = \int_0^\infty\int_0^\infty \beta\pi(\alpha,\beta|x)d\alpha d\beta \tag{14}$$

and

$$\hat{R}(t)_{SELF} = E(R(t)|x) = \int_0^\infty\int_0^\infty R(t)\pi(\alpha,\beta|x)d\alpha d\beta \tag{15}$$

respectively.

The Bayesian estimates for the, β, and $R(t)$ under WSELF, LINEX, GE, and PL loss functions can be evaluated similarly.

It is not possible to compute Equations (13)-(15) explicitly, hence, we use two different approximation methods described below.

4.1. Lindley's approximation

Let $u(\alpha,\beta)$ be a function of α and β, then by using Equation (12), the expected value of $u(\alpha,\beta)$ is given by:

$$\hat{u} = E(u(\alpha,\beta)|x) = \frac{\int_0^\infty\int_0^\infty u(\alpha,\beta)v(\alpha,\beta)L(\alpha,\beta;x)d\alpha d\beta}{\int_0^\infty\int_0^\infty v(\alpha,\beta)L(\alpha,\beta;x)d\alpha d\beta}, \tag{16}$$

where $v(\alpha,\beta)$ is the joint prior density function and $L(\alpha,\beta;x)$ is the likelihood function?

The Bayes estimator $u(\alpha,\beta)$ is the solution of Equation (16). However, the Bayes estimator cannot be obtained analytically because $\hat{u}$ given in Equation (16) is in the form of a ratio of two integrals. To overcome this difficulty, we use the Lindley approximation method introduced by Lindley (1980). Then, by using Lindley's approximation method, $\hat{u}$ can be approximated as:

$$\hat{u} \approx \left\{ \begin{array}{l} u(\alpha,\beta) + 0.5\left[u_{11}\sigma_{11} + u_{22}\sigma_{22} + 2u_{12}\sigma_{12} + 2u_1(\sigma_{11}\rho_1 + \sigma_{21}\rho_2) + 2u_2(\sigma_{12}\rho_1 + \sigma_{22}\rho_2)\right] \\ +0.5\left[L_{111}(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}) + L_{112}(3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)) + L_{122}(u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) \right. \\ \left. +3u_2\sigma_{12}\sigma_{22}) + L_{222}(u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2)\right] \end{array} \right\}_{\hat{\alpha},\hat{\beta}} \tag{17}$$

Here, $\hat{\alpha}$ and $\hat{\beta}$ are the ML estimators of α and β respectively,

$$u_1 = \frac{\partial u(\alpha,\beta)}{\partial \alpha},\ u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial \alpha^2},\ u_2 = \frac{\partial u(\alpha,\beta)}{\partial \beta},\ u_{22} = \frac{\partial u(\alpha,\beta)}{\partial \beta^2},\ u_{12} = \frac{\partial^2 u(\alpha,\beta)}{\partial \alpha \partial \beta}$$

$$\rho_1 = \frac{\partial \ln v(\alpha,\beta)}{\partial \alpha} = \frac{a-1}{\alpha} - b,\ \rho_2 = \frac{\partial \ln v(\alpha,\beta)}{\partial \beta} = \frac{c-1}{\beta} - d$$

$$L_{111} = \frac{\partial^3 \ln L}{\partial \beta^3} = \frac{2n}{\alpha^3} + \sum_{i=1}^{n}\left(\ln\left(\frac{x_i}{\beta}\right)\right)^3\left(\frac{x_i}{\beta}\right),$$

$$L_{112} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \beta} = \frac{2}{\beta}\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{-\alpha} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\ln\left(\frac{x_i}{\beta}\right)^2\left(\frac{x_i}{\beta}\right)^{-\alpha},$$

$$L_{122} = \frac{\partial^3 \ln L}{\partial \alpha \partial \beta^2} = \frac{-n}{\beta^2} + \frac{\alpha(\alpha-1)}{\beta^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}\ln\left(\frac{x_i}{\beta}\right) + \frac{1-2\alpha}{\beta^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha},$$

$$L_{222} = \frac{\partial^3 \ln L}{\partial \beta^3} = \frac{2n\alpha}{\beta^3} - \frac{\alpha(\alpha-1)(\alpha-2)}{\beta^3}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}$$

and $\sigma_{ij} = i, j = 1,2$ are the elements of the inverse Fisher information matrix which is given by

$$\sigma_{ij} = \begin{bmatrix} Var\{\hat{\alpha}\} & Cov\{\hat{\alpha},\hat{\beta}\} \\ Cov\{\hat{\alpha},\hat{\beta}\} & Var\{\hat{\beta}\} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 0.608\hat{\alpha}^2 & -0.257\hat{\beta} \\ -0.257\hat{\beta} & \frac{\hat{\beta}^2}{\hat{\alpha}^2}1.109 \end{bmatrix}. \tag{18}$$

4.1.1. Bayes estimators of the α and β using Lindley's approximation

In this subsection, the Bayes estimators of α and β are obtained under SELF, LINEX, GE, WSELF, and PRE loss functions by using the Lindley approximation. From Equation (17), the Bayes estimator of α under SELF-using the Lindley method is obtained as follows:

If $u(\alpha,\beta) = \alpha$, $u_1 = 1$, $u_2 = u_{22} = u_{12} = u_{21} = u_{11} = 0$, then

$$\hat{\alpha}_{SELF}=\hat{\alpha}+\left(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2\right)+0.5\left[\hat{L}_{111}\sigma^2{}_{11}+3\hat{L}_{112}\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{122}\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+L_{222}\hat{\sigma}_{12}\hat{\sigma}_{22}\right].$$

The Bayes estimator of β under SELF is given as:

If $u(\alpha,\beta)=\beta,\ u_2=1,\ u_{22}=u_{12}=u_{21}=u_{11}=0,$ then

$$\hat{\beta}_{SELF}=\hat{\beta}+\left(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2\right)+0.5\left[\hat{L}_{111}\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{112}\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22}+\hat{L}_{222}\hat{\sigma}_{22}^2\right].$$

Similarly, Bayes estimator of α under the LINEX loss function is given by:

If $u(\alpha,\beta)=e^{-k\alpha},\ u_1=-ke^{-k\alpha},\quad u_{11}=k^2\,\mathrm{e}^{-k\alpha},\ u_2=u_{22}=u_{12}=u_{21}=0,$then

$$E\left(e^{-k\alpha}\,|x\right)=e^{-k\hat{\alpha}}+0.5\left(\hat{u}_{11}\hat{\sigma}_{11}\right)+\hat{u}_1\left(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2\right)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2+3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{122}\hat{u}_1\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+\hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}\right].$$

So, the Bayes estimator of α under LINEX loss function is obtained by:

$$\hat{\alpha}_{LINEX}=-\frac{1}{k}\ln E\left(e^{-k\alpha}\,|x\right).$$

The Bayes estimators of β under the LINEX loss function is given as:

If $u(\alpha,\beta)=e^{-k\beta},\ u_2=-ke^{-k\beta},\ u_{22}=k^2\,\mathrm{e}^{-k\beta},\ u_1=u_{11}=u_{12}=u_{21}=0,$ then

$$E\left(e^{-k\beta}\,|x\right)=e^{-k\hat{\beta}}+0.5\left(\hat{u}_{22}\hat{\sigma}_{22}\right)+\hat{u}_2\left(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2\right)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{112}\hat{u}_2\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\sigma_{12}^2\right)+3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22}+\hat{L}_{222}\hat{u}_2\sigma_{22}^2\right].$$

So, the Bayes estimator of β is obtained by:

$$\hat{\beta}_{LINEX}=-\frac{1}{k}\ln E\left(e^{-k\beta}\,|x\right).$$

The Bayes estimator of α under the GE loss function is defined as:

If $u(\alpha,\beta)=\alpha^{-k},\ u_1=-k\alpha^{-(k+1)},\ u_{11}=k\left(k+1\right)\alpha^{-(k+2)},u_2=u_{22}=u_{12}=u_{21}=0,$then

$$E\left(\alpha^{-k}\,|x\right)=\hat{\alpha}^{-k}+0.5\left(\hat{u}_{11}\hat{\sigma}_{11}\right)+\hat{u}_1\left(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2\right)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2+3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{122}\hat{u}_1\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+\hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}\right].$$

Therefore, $\hat{\alpha}_{GE}=\left[E\left(\alpha^{-k}\,|x\right)\right]^{-1/k}$.

The Bayes estimator of β under the general entropy loss function is given by:

If $u(\alpha,\beta)=\beta^{-k}$, $u_2=-k\beta^{-(k+1)}$, $u_{22}=k(k+1)\beta^{-(k+2)}$, $u_1=u_{11}=u_{12}=u_{21}=0$,then

$$E\left(\beta^{-k}|x\right)=\hat{\beta}^{-k}+0.5(\hat{u}_{22}\hat{\sigma}_{22})+\hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{112}\hat{u}_2\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\sigma_{12}^2\right)+3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22}+\hat{L}_{222}\hat{u}_2\sigma_{22}^2\right].$$

So, $\hat{\beta}_{GE}=\left[E\left(\beta^{-k}|x\right)\right]^{-1/k}$.

The Bayes estimator of α under the precautionary loss function is as follows:

If $u(\alpha,\beta)=\alpha^2$, $u_1=2\alpha$, $u_{11}=2$, $u_2=u_{22}=u_{12}=0$,then

$$E\left(\alpha^2|x\right)=\hat{\alpha}^2+0.5(\hat{u}_{11}\hat{\sigma}_{11})+\hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2+3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{122}\hat{u}_1\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+\hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}\right].$$

So, the Bayes estimator of α is $\hat{\alpha}_{PRE}=\sqrt{E\left(\alpha^2|x\right)}$.

The Bayes estimator of β under the precautionary loss function is given by:
If $u(\alpha,\beta)=\beta^2$, $u_1=2\beta$, $u_{11}=2$, $u_2=u_{22}=u_{12}=0$,then

$$E\left(\beta^2|x\right)=\hat{\beta}^2+0.5(\hat{u}_{22}\hat{\sigma}_{22})+\hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{112}\hat{u}_2\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22}+\hat{L}_{222}\hat{u}_2\sigma_{22}^2\right].$$

Hence, $\hat{\beta}_{PRE}=\sqrt{E\left(\beta^2|x\right)}$.

The Bayes estimator of α under the WSELF is given by:

If $u(\alpha,\beta)=\alpha^{-1}$, $u_1=-\alpha^{-2}$, $u_{11}=2\alpha^{-3}$, $u_2=u_{22}=u_{12}=0$,then

$$E\left(\alpha^{-1}|x\right)=\hat{\alpha}^{-1}+0.5(\hat{u}_{11}\hat{\sigma}_{11})+\hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2+3\hat{L}_{112}\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{122}\hat{u}_1\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\hat{\sigma}_{12}^2\right)+\hat{L}_{222}\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22}\right].$$

The Bayes estimator of α is of the following form

$$\hat{\alpha}_{WSELF}=\left[E\left(\alpha^{-1}|x\right)\right]^{-1}.$$

The WSELF with parameter β is given by:

If $u(\alpha,\beta)=\beta^{-1}$, $u_2=-\beta^{-2}$, $u_{22}=2\beta^{-3}$, $u_1=u_{11}=u_{12}=0$,then

$$E\left(\beta^{-1}|x\right)=\hat{\beta}^{-1}+0.5\left(\hat{u}_{22}\hat{\sigma}_{22}\right)+\hat{u}_2\left(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2\right)$$
$$+0.5\left[\hat{L}_{111}\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{L}_{112}\hat{u}_2\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\sigma_{12}^2\right)+3\hat{L}_{122}\hat{\sigma}_{12}\hat{\sigma}_{22}+\hat{L}_{222}\hat{u}_2\sigma_{22}^2\right].$$

Hence, the Bayes estimator of β is

$$\hat{\beta}_{WSELF}=\left[E\left(\beta^{-1}|x\right)\right]^{-1}.$$

4.1.2. Bayes estimator of the reliability function using Lindley's approximation

Estimation of the reliability function is another purpose of this paper. Therefore, we estimate this function under the above loss functions.
We first consider the Bayes estimator of the reliability function under SELF.

$u(\alpha,\beta)=R(t)=1-\mathrm{e}^{-\left(\frac{t}{\beta}\right)^{-\alpha}}=1-\mathrm{e}^{-\left(\frac{\beta}{t}\right)^{\alpha}}$, then the corresponding derivatives are

$$u_1=\frac{\partial u}{\partial\alpha}=\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}\ln\left(\frac{\beta}{t}\right)\mathrm{e}^{-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}},u_{11}=\frac{\partial^2 u}{\partial u\partial\alpha}=e^{-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}}\left(\ln\left(\frac{\beta}{t}\right)\right)^2\left(1-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}\right)\left[\left(\frac{\beta}{t}\right)\right]^{\alpha},$$

$$u_{12}=\frac{\partial^2 u}{\partial\alpha\partial\beta}=\frac{1}{\beta}\mathrm{e}^{-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}}\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}\left[\alpha\ln\left(\frac{\beta}{t}\right)-\alpha\ln\left(\frac{\beta}{t}\right)\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}+1\right]$$

and

$$u_2=\frac{\partial u}{\partial\beta}=\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}\left(\frac{\alpha}{\beta}\right)\mathrm{e}^{-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}},u_{22}=\frac{\partial^2 u}{\partial\beta^2}=\frac{-\alpha}{\beta^2}\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}\mathrm{e}^{-\left[\left(\frac{\beta}{t}\right)\right]^{\alpha}}\left[\alpha\ln\left(\frac{\beta}{t}\right)-\alpha+1\right].$$

The corresponding derivatives ($u_1,u_{11},u_{12},u_2$and u_{12}) can be similarly obtained for WSELF, LINEX, GE, and PL loss functions.
The remaining $l$and $p$terms will be the same as above. Therefore, by considering Equation (17), the Bayes estimators of the reliability function $R(t)$ under SELF-using Lindley method is obtained as:

$$\hat{R}(t)_{SELF}=\left(1-\mathrm{e}^{-\left(t/\hat{\beta}\right)^{\hat{\alpha}}}\right)+0.5\left(\hat{u}_{11}\hat{\sigma}_{11}+\hat{u}_{22}\hat{\sigma}_{22}\right)+\hat{u}_{12}\hat{\sigma}_{12}+\hat{u}_1\left(\hat{\sigma}_{11}\hat{\rho}_1+\hat{\sigma}_{21}\hat{\rho}_2\right)+\hat{u}_2\left(\hat{\sigma}_{12}\hat{\rho}_1+\hat{\sigma}_{22}\hat{\rho}_2\right)$$
$$+0.5\left[\hat{L}_{111}\left(\hat{u}_1\hat{\sigma}_{11}^2+\hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}\right)+\hat{L}_{112}\left(3\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12}+\hat{u}_2\left(\hat{\sigma}_{11}\hat{\sigma}_{22}+2\sigma_{12}^2\right)\right)\right.$$
$$\left.+\hat{L}_{122}\left(\hat{u}_1\left(\hat{\sigma}_{12}\hat{\sigma}_{22}+2\sigma_{12}^2\right)+3\hat{u}_2\hat{\sigma}_{12}\hat{\sigma}_{22}\right)+\hat{L}_{222}\left(\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{22}+\hat{u}_2\sigma_{22}^2\right)\right].$$

The same steps can be applied for WSELF, LINEX, GE, and PL loss functions using their corresponding Bayes estimators of the reliability function $R(t)$. It should be noted that Bayesian estimators of the unknown parameters can be obtained by using Lindley approximation. However, it is not possible to construct a credible interval using this approximation. Therefore, in the next subsection, the Bayesian estimators of α, β, and the reliability function $R(t)$ together with their BCIs are obtained by using posterior distributions.

4.2 MCMC method

In this section, we use the Gibbs sampling method to generate a sample from the posterior distribution. Gibbs sampling is a sub-class of the MCMC method. For more detail about this method, see (Mertopolis *et al*., 1953; Dey *et al*., 2017; Smith & Robert, 1993; Kumar, 2018).

The joint posterior density function of α and β is given in (12). According to this equation, the conditional posterior density functions of parameters α and β are given by

$$\pi_1\left(\alpha\middle|\beta,x\right)\propto\alpha^{n+a-1}\beta^{n\alpha}e^{-b\alpha}e^{-\beta^{\alpha}\sum_{i=1}^{n}x_i^{-\alpha}}\prod_{i=1}^{n}\left(x_i\right)^{-(\alpha+1)} \tag{19}$$

and

$$\pi_2\left(\beta\middle|\alpha,x\right)\propto\beta^{n\alpha+c-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}}e^{-d\beta} \tag{20}$$

respectively. The posterior distributions of α and β from equations (19)-(20) are unknown. Therefore, we can use Metropolis-Hastings (MH) algorithm with normal proposal distribution to generate random samples from (19) and (20), respectively.

The Gibbs sampling method consists of the following steps:

Step 1: Set $j=1$ and let $\alpha_0=\hat{\alpha}\ \beta_0=\hat{\beta}$.

Step 2: Using the M-H algorithm, generate a posterior sample for α_j and β_j from $\pi_1\left(\alpha_{j-1}\middle|\beta_{j-1},x\right)$ and $\pi_2\left(\beta_{j-1}\middle|\alpha_j,x\right)$, respectively.

Step 3: Set $j=j+1$.

Step 4: Repeat Step 2-3, N times to generate samples $(\alpha_1,\beta_1),\ldots,(\alpha_N,\beta_N)$.

Step 5: The Bayes estimator of the parameters α and β and the Reliability function $R(t)$ under all these aforementioned loss functions can be computed from the following formulas.

$$\hat{\alpha}_{SELF}=\frac{1}{N}\sum_{i=1}^{N}\alpha_i,\ \hat{\beta}_{SELF}=\frac{1}{N}\sum_{i=1}^{N}\beta_i \text{ and } \hat{R}(t)_{SELF}=\left(\frac{1}{N}\sum_{i=1}^{N}1-e^{-\left(\frac{\beta_i}{t}\right)^{-\alpha_i}}\right),$$

$$\hat{\alpha}_{LINEX}=\frac{-1}{k}\ln\left(\frac{1}{N}\sum_{i=1}^{N}e^{-k\alpha_i}\right),\ \hat{\beta}_{LINEX}=\frac{-1}{k}\ln\left(\frac{1}{N}\sum_{i=1}^{N}e^{-k\beta_i}\right)$$

and

$$\hat{R}(t)_{LINEX}=\frac{-1}{k}\ln\left(\frac{1}{N}\sum_{i=1}^{N}e^{-k\left(1-e^{-(\beta_i/t)^{-\alpha_i}}\right)}\right),$$

$$\hat{\alpha}_{GE}=\left(\frac{1}{N}\sum_{i=1}^{N}\alpha_i^{-k}\right)^{\frac{-1}{k}},\ \hat{\beta}_{GE}=\left(\frac{1}{N}\sum_{i=1}^{N}\beta_i^{-k}\right)^{\frac{-1}{k}}$$

and

$$\hat{R}(t)_{GE}=\left(\frac{1}{N}\sum_{i=1}^{N}\left(1-e^{-\left(\frac{\beta_i}{t}\right)^{-\alpha_i}}\right)^{-k}\right)^{\frac{-1}{k}},$$

$$\hat{\alpha}_{WSELF}=\left(\frac{1}{N}\sum_{i=1}^{N}\alpha_i^{-1}\right)^{-1},\ \hat{\beta}_{WSELF}=\left(\frac{1}{N}\sum_{i=1}^{N}\beta_i^{-1}\right)^{-1}$$

and

$$\hat{R}(t)_{WSELF}=\left(\frac{1}{N}\sum_{i=1}^{N}\left(1-e^{-\left(\frac{\beta_i}{t}\right)^{-\alpha_i}}\right)^{-1}\right)^{-1},$$

$$\hat{\alpha}_{PRE}=\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\alpha_i^{2}\right)},\ \hat{\beta}_{PRE}=\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\beta_i^{2}\right)}$$

and

$$\hat{R}(t)_{PRE}=\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(1-e^{-\left(\frac{\beta_i}{t}\right)^{-\alpha_i}}\right)^{2}\right)},$$

respectively. The $100(1-\gamma)\%$credible interval of parameters α and β is found by the method of Chen and Shao (1999).

5. Simulation study

In this section, an extensive simulation study is conducted to compare the performances of classical and Bayesian estimators described in Section 3 and Section 4. The performances of all estimators are compared concerning their biases and the mean square errors (MSEs). In both the Bayes and classical approximation, the sample sizes $n=10(5)150$(i.e. $n=10,15,\ 20,...,150$) and the shape parameter α is considered as α=0.5, 1 and 2. Since the simulation results are similar $\alpha=1$and $\alpha=2$, the results are summarized only for, $\alpha=2$.Throughout the study, the scale parameter β is taken to be 1, since all estimators are scale-invariant. The results of simulation for classical estimators (MLEs and MMLEs) under these assumptions are reported in Table 2.

The Gamma prior provides a flexible approach to handling estimation procedures. In other words, it may be used as non-informative prior and informative prior. Therefore, we used the gamma distribution as a prior density function for Bayesian estimation. This paper considers two special cases of the gamma prior corresponding to by $a=b=c=d=0$ and $a,b,c,d\geq 0$. It should be mentioned that $a=b=c=d=0$ the Gamma prior distribution is non-informative prior distribution. For $a,b,c,d\geq 0$, the Gamma prior distribution is referred to as informative prior distribution. First, for non-informative prior, we chose hyper-parameter values $a=b=c=d=0$

and call them NP. Then, for informative prior, we take hyper-parameter values as $a = c = 0.4$, $b = d = 0.2$ and call them GP. The Bayes estimators are computed under SELF, LINEX loss function, GE loss function, WSELF, and PRE loss function. For GE and LINEX loss functions, k is considered as $k = 1.5$. Results are summarized in Table 3 and Table 4. The %95 ACIs and BCIs are presented in Table 5 for parameters α and β. ACIs are obtained by using the sample large theory. It is known as the asymptotic distribution of the MLEs for the unknown parameters. Therefore, $100(1-\gamma)$the ACIs of parameters α and β are constructed as follow:

$$\left(\hat{\alpha} \pm Z_{(\gamma/2)}\sqrt{Var(\hat{\alpha})},\ \hat{\beta} \pm Z_{(\gamma/2)}\sqrt{Var(\hat{\beta})}\right),$$

where $Z_{\gamma/2}$is the upper $(\gamma/2)$ percentile of the standard normal distribution? Table 6-11 gives the MSEs and biases of the reliability function.

From Table 2-11, we can conclude the following:

As the sample size increases, the biases and MSEs of the estimators decrease in all cases. It implies that all the estimators are asymptotically efficient.
All of the estimators usually overestimate α and β. When the classical methods are compared with each other, concerning the MSEs, MMLEs works very well for small $(n \le 15)$ sample size and MLEs outperform the MMLEs for large sample size.
Similarly, if we compare the performance of the Bayes estimators, it is evident that as far as bias is concerned, Bayesian estimation under the GE loss function works the best in most cases. It is followed by Bayesian estimation under the WSELF and LINEX loss function.
The performances of the Bayesian estimation under SELF and PRE loss function are the worst concerning biases and MSEs. Now concerning the MSEs, it is clear that Bayesian estimation under GE loss function outperforms for estimating α parameter. For estimating the β parameter, Bayesian estimation under the LINEX loss function works well.
Among Bayes estimators, under non-informative prior set-up, we observed that the Gibbs sampling method performs well than Lindley's approximation method in most of the cases. Also, under informative prior set-up, Lindley's approximation method demonstrates better performance than Gibbs sampling in almost all cases.
As seen in Table 5, the width of the intervals decreases as the sample size n increases for all cases. Also, the width of the BCIs is smaller than those of the ACIs.
It is evident from Tables 6-7 that the performances of MLE and MMLE are more or less the same. From Table 8 and Table 10 in cases of NP and GP $\alpha = 0.5$, Gibbs sampling under LINEX loss function works well than Lindley's approximation. Similarly, when the Bayesian methods are compared for $\alpha = 2$, in cases of NP and GP, the Lindley method under LINEX loss function performs better than Gibbs sampling. It is clear from Table 9 and Table 11 that as far as bias values are concerned; the Lindley method under SELF and LINEX loss function are somewhat more

efficient than Gibbs sampling, in all cases (NP and GP). Because of MSE values, Gibbs sampling performs slightly better than the Lindley approximation under all loss functions.

When we compare the Bayesian and classical methods for estimating all three parameters α, β and $R(t)$. It is clear that as far as bias and MSEs are concerned. Bayesian methods with convenient prior information (Non-information prior or information prior) and loss function outperform the classical methods in most of the cases. In other cases, it may be preferable to use MMLEs for small sample sizes, i.e., $n \le 15$

6. Application

In this section, we use two real data sets to illustrate how to calculate the estimators of the unknown model parameters. The first data set is called "Example -I" and the second data set is called "Example- II".

6.1. Example- I: The first data set has 708 observations showing hourly wind speed data (m/s) in Turkey (Adilcevaz) during April 2017. The data was taken from the Turkish State Meteorological Service (https://mgm.gov.tr/eng/forecast-cities.aspx).

6.2. Example- II: The second data set the strength measured in GPA for carbon fibers tested under tension at gauge lengths of 20 mm [10]. This data set was also studied by Kundu and Gupta (2006) and Surles and Padgett (2001).

These data sets are supported by the well-known Kolmogorov- Smirnov (K-S) test statistics (K-S=0.4613 and *p*-value=0.0883 for the first data set and K-S =0.0533 and *p*-value=0.0994 for the second data). The *p*-value is large so we cannot reject the null hypothesis. Therefore, it can be concluded that the IW distribution is an appropriate distribution for both data sets. The estimators of the parameters and the reliability function of the inverse Weibull distribution are given in Table 12-15.

In Table 13, Ex-I and Ex-II are presented as ‘Example-I’ and ‘Example –II’, respectively. We use the simulation results given in Section 5 to choose the most appropriate estimators. We observed in the simulation that the estimators obtained by Bayesian methods outperform the classical methods. Also, According to Table 13, the GP for Example-I and Example-II are shorter than the others. Therefore, we recommend using Bayesian estimators in these examples.

Table 2. Simulation results for the classical estimators

		$\hat{\alpha}$				$\hat{\beta}$			
		MLE		MMLE		MLE		MMLE	
α	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.5	15	0.0501	0.0183	0.0389	0.0176	0.2689	0.7052	0.0389	0.0176
	30	0.0247	0.0070	0.0226	0.0070	0.1146	0.2488	0.0226	0.0070
	50	0.0136	0.0038	0.0120	0.0040	0.0582	0.1220	0.0120	0.0040
	100	0.0079	0.0016	0.0074	0.0017	0.0321	0.0528	0.0074	0.0017
2	15	0.2166	0.3036	0.1735	0.2820	0.0218	0.0208	0.0207	0.0210
	30	0.0902	0.1107	0.0898	0.1139	0.0092	0.0097	0.0097	0.0098
	50	0.0610	0.0576	0.0563	0.0590	0.0042	0.0056	0.0060	0.0063
	100	0.0296	0.0278	0.0282	0.0291	0.0028	0.0030	0.0049	0.0031

Table 3. Simulation results for Bayes estimators using Lindley approximation (In Table 3, for LINEX and GE loss function *k* is taken as *k=1.5*)

		SELF		LINEX		GE		WSELF		PRE	
	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
NP $\alpha = 0.5$											
$\hat{\alpha}$	15	0.0223	0.0147	0.0137	0.0135	0.0036	0.0131	0.0041	0.0131	0.0331	0.0159
	30	0.0089	0.0063	0.0049	0.0060	-0.0028	0.0059	-0.0007	0.0059	0.0141	0.0065
	50	0.0039	0.0033	0.0016	0.0032	-0.0033	0.0032	-0.0009	0.0032	0.0070	0.0034
	100	0.0040	0.0016	0.0028	0.0016	0.0003	0.0015	0.0010	0.0015	0.0055	0.0016
$\hat{\beta}$	15	0.4414	1.0669	0.0693	0.2882	0.0403	0.3996	0.0931	0.4520	0.5865	1.4416
	30	0.1871	0.2843	0.0334	0.1330	-0.0037	0.1614	0.0273	0.1741	0.2600	0.3556
	50	0.0102	0.1415	0.0188	0.0912	-0.0113	0.1009	0.0086	0.1057	0.1451	0.1656
	100	0.0527	0.0572	0.0155	0.0462	0.0028	0.0476	0.0076	0.0488	0.0746	0.0629
NP $\alpha = 2$											
$\hat{\alpha}$	15	0.0942	0.2416	-0.0154	0.1936	0.0073	0.2139	0.0216	0.2143	0.1379	0.2616
	30	0.0348	0.0927	-0.0227	0.0826	-0.0117	0.0875	-0.0033	0.0874	0.0558	0.0965
	50	0.0294	0.0531	-0.0062	0.0488	0.0005	0.0508	0.0059	0.0508	0.0419	0.0547
	100	0.0140	0.0267	-0.0040	0.0256	-0.0008	0.0261	0.0021	0.0261	0.0202	0.0271
$\hat{\beta}$	15	0.0275	0.0217	0.0130	0.0193	0.0051	0.0192	0.0094	0.0196	0.0365	0.0230
	30	0.0121	0.0099	0.0050	0.0094	0.0007	0.0093	0.0029	0.0094	0.0166	0.0102
	50	0.0058	0.0057	0.0017	0.0055	-0.0010	0.0061	-0.0004	0.0055	0.0086	0.0058
	100	0.0043	0.0028	0.0022	0.0027	0.0009	0.0027	0.0016	0.0027	0.0057	0.0028
GP $\alpha = 0.5$											
$\hat{\alpha}$	15	0.0275	0.0159	0.0187	0.0145	0.0051	0.0140	0.0088	0.0128	0.0385	0.0173
	30	0.0116	0.0058	0.0075	0.0057	-0.0003	0.0055	0.0019	0.0054	0.0169	0.0060
	50	0.0094	0.0035	0.0070	0.0034	0.0021	0.0033	0.0035	0.0034	0.0126	0.0037
	100	0.0039	0.0016	0.0027	0.0016	0.0002	0.0015	0.0009	0.0015	0.0055	0.0016
$\hat{\beta}$	15	0.4449	0.8953	0.1058	0.3561	0.0539	0.3863	0.1061	0.4150	0.5919	1.3053
	30	0.1990	0.2695	0.0485	0.1247	0.0073	0.1486	0.0393	0.1602	0.2692	0.3442
	50	0.1091	0.1299	0.0287	0.0829	-0.0028	0.0907	0.0175	0.0953	0.1512	0.1530
	100	0.0603	0.0556	0.0230	0.0444	0.0041	0.0454	0.0148	0.0467	0.0821	0.0615
GP $\alpha = 2$											
$\hat{\alpha}$	15	0.1065	0.2535	-0.0013	0.2054	0.0208	0.2252	0.0347	0.2260	0.1502	0.2746
	30	0.0409	0.0997	-0.0166	0.0835	-0.0056	0.0938	0.0028	0.0938	0.0618	0.1038
	50	0.0267	0.0523	-0.0087	0.0483	-0.0022	0.0502	0.0032	0.0542	0.0391	0.0538
	100	0.0112	0.0265	-0.0067	0.0255	-0.0036	0.0261	-0.0008	0.0260	0.0174	0.0268
$\hat{\beta}$	15	0.0327	0.0214	0.0185	0.0190	0.0106	0.0188	0.0149	0.0192	0.0415	0.0227
	30	0.0126	0.0103	0.0056	0.0097	0.0012	0.0097	0.0035	0.0098	0.0171	0.0106
	50	0.0098	0.0058	0.0056	0.0056	0.0029	0.0057	0.0043	0.0056	0.0215	0.0060
	100	0.0041	0.0030	0.0020	0.0028	0.0007	0.0028	0.0014	0.0029	0.0055	0.0030

Table 4. Simulation results for the Bayes estimators using Gibbs sampling

		SELF		LINEX		GE		WSELF		PRE	
	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
NP $\alpha = 0.5$											
$\hat{\alpha}$	15	0.0444	0.0173	0.0354	0.0154	0.0171	0.0140	0.0227	0.0146	0.0549	0.0189
	30	0.0198	0.0065	0.0161	0.0062	0.0079	0.0060	0.0103	0.0061	0.0245	0.0069
	50	0.0103	0.0034	0.0084	0.0033	0.0039	0.0032	0.0052	0.0032	0.0128	0.0035
	100	0.0075	0.0015	0.0067	0.0011	0.0049	0.0015	0.0054	0.0015	0.0086	0.0021
$\hat{\beta}$	15	0.0695	0.1137	0.0075	0.0871	-0.0290	0.0959	0.0091	0.0979	0.1083	0.1262
	30	0.0240	0.0427	-0.0044	0.0383	-0.0227	0.0401	-0.0133	0.0403	0.0426	0.0449
	50	0.0095	0.0256	-0.0070	0.0242	-0.0180	0.0250	-0.0125	0.0250	0.0205	0.0263
	100	0.0660	0.0120	-0.0021	0.0116	-0.0075	0.0118	-0.0048	0.0118	0.0114	0.0121
NP $\alpha = 2$											
$\hat{\alpha}$	15	0.1829	0.2845	0.0467	0.1939	0.0739	0.2329	0.0959	0.2413	0.2234	0.3113
	30	0.0786	0.1010	0.0203	0.0837	0.0310	0.0910	0.0406	0.0926	0.0973	0.1063
	50	0.0566	0.0546	0.0251	0.0489	0.0308	0.0512	0.0360	0.0517	0.0668	0.0564
	100	0.0278	0.0250	0.0154	0.0238	0.0176	0.0243	0.0196	0.0244	0.0319	0.0254
$\hat{\beta}$	15	0.0511	0.1094	-0.0094	0.0830	-0.0460	0.0934	-0.0264	0.0950	0.0894	0.1215
	30	0.0187	0.0421	-0.0094	0.0380	-0.0278	0.0400	-0.0185	0.0401	0.0372	0.0441
	50	0.0072	0.0236	-0.0092	0.0224	-0.0202	0.0231	-0.0147	0.0231	0.0182	0.0242
	100	0.0063	0.0113	-0.0018	0.0109	-0.0071	0.0111	-0.0044	0.0113	0.0117	0.0115
GP $\alpha = 0.5$											
$\hat{\alpha}$	15	0.0507	0.0188	0.0416	0.0168	0.0235	0.0152	0.0291	0.0158	0.0611	0.0206
	30	0.0234	0.0064	0.0197	0.0060	0.0115	0.0057	0.0139	0.0058	0.0280	0.0067
	50	0.0160	0.0036	0.0140	0.0035	0.0095	0.0034	0.0108	0.0034	0.0185	0.0037
	100	0.0078	0.0016	0.0070	0.0015	0.0052	0.0017	0.0057	0.0015	0.0088	0.0016
$\hat{\beta}$	15	0.0846	0.1321	0.0217	0.0974	0.0129	0.1084	0.0068	0.1116	0.1230	0.1466
	30	0.0284	0.0414	0.0002	0.0369	-0.0179	0.0386	-0.0086	0.0388	0.0468	0.0437
	50	0.0123	0.0239	-0.0042	0.0235	-0.0151	0.0231	-0.0096	0.0232	0.0232	0.0246
	100	0.0082	0.0115	0.0001	0.0111	-0.0052	0.0112	-0.0025	0.0112	0.0136	0.0117
GP $\alpha = 2$											
$\hat{\alpha}$	15	0.1925	0.2921	0.0569	0.1964	0.0848	0.2380	0.1067	0.2469	0.2340	0.3199
	30	0.0806	0.1054	0.0229	0,0887	0.0324	0.0959	0.0429	0.0974	0.0991	0.1105
	50	0.0493	0.0527	0.0183	0.0477	0.0238	0.0497	0.0289	0.0502	0.0594	0.0542
	100	0.0209	0.0227	0.0088	0.0219	0.0108	0.0223	0.0129	0.0223	0.0250	0.0230
$\hat{\beta}$	15	0.0692	0.1016	0.0091	0.0780	-0.0264	0.0851	-0.0071	0.0869	0.1069	0.1134
	30	0.0234	0.0439	-0.0046	0.0394	0.0227	0.0413	-0.0135	0.0415	0.0418	0.0462
	50	0.0186	0.0242	0.0020	0.0225	-0.0088	0.0231	-0.0033	0.0232	0.0295	0.0250
	100	0.0076	0.0118	-0.0005	0.0114	0.0059	0.0115	-0.0032	0.0115	0.0129	0.0120

Table 5. 95% confidence/ Bayesian credible intervals for α and β

		ACI		BCI-NP		BCI-GP	
n	α	α	β	α	β	α	β
15	0.5	(0.2981;0.3651)	(0.2380;0.2037)	(0.3645;0.3901)	(0.5516;0.6766)	(0.3694;0.3972)	(0.5702;0.6891)
30		(0.3979;0.4294)	(0.3895;0.5906)	(0.4115;0.4410)	(0.7038;0.8070)	(0.4166;0.4371)	(0.7512;0.8101)
50		(0.8016;0.6836)	(2.6786;1.9979)	(0.8534;0.6912)	(1.8355;1.4649)	(0.8610;0.6938)	(1.8899;1.4655)
100		(0.6272;0.5842)	(1.7274;1.4729)	(0.6339;0.5866)	(1.3374;1.2310)	(0.6451;0.5935)	(1.3454;1.2320)
15	2	(1.2612;1.4459)	(0.7417;0.8139)	(1.4408;1.5899)	(0.5585;0.6637)	(1.4531;1.5761)	(0.5626;0.6847)
30		(1.5851;1.7142)	(0.8599;0.9019)	(1.6642;1.7568)	(0.7375;0.8194)	(1.6515;1.7610)	(0.7578;0.8042)
50		(3.1254;2.7360)	(1.3108;1.2135)	(3.3489;2.7782)	(1.7502;1.4566)	(3.4604;2.7849)	(1.8065; 1.4819)
100		(2.5303;2.3581)	(1.1571;1.1090)	(2.5741;2.3635)	(1.3375;1.3370)	(2.5296;2.3423)	(1.3629;1.2334)

Table 6. Simulation results of the classical estimators for $t = 0.5$

	$\alpha = 0.5$				$\alpha = 2$			
	MLE		MMLE		MLE		MMLE	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
15	0.0129	0.0088	0.0109	0.0087	-0.0020	0.0005	-0.0051	0.0014
30	0.0045	0.0041	0.0042	0.0041	-0.0015	0.0003	-0.0020	0.0003
50	0.0030	0.0026	0.0003	0.0028	-0.0006	0.0002	-0.0008	0.0002
100	0.0016	0.0012	0.0015	0.0012	-0.0002	0.0001	-0.0003	0.0001

Table 7. Simulation results of the classical estimators for $t = 4$

	$\alpha = 0.5$				$\alpha = 2$			
	MLE		MMLE		MLE		MMLE	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
15	-0.0119	0.0117	-0.0083	0.0117	-0.0008	0.0016	0.0068	0.0050
30	-0.0101	0.0053	-0.0093	0.0054	-0.0015	0.0008	-0.0009	0.0008
50	-0.0024	0.0033	-0.0015	0.0035	-0.0001	0.0005	0.0003	0.0006
100	-0.0014	0.0014	-0.0014	0.0015	-0.0009	0.0003	-0.0005	0.0003

Table 8. Simulation results of Lindley approximation for $t = 0.5$

			SELF		LINEX		GE		WSELF		PRE	
LF	n	α	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
NP	15	0.5	-0.0060	0.0076	-0.0086	0.0085	-0.0169	0.0088	-0.0149	0.0085	-0.0012	0.0072
	30		-0.0050	0.0039	-0.0070	0.0040	-0.0110	0.0042	-0.0099	0.0041	-0.0025	0.0037
	50		-0.0051	0.0025	-0.0065	0.0025	-0.0089	0.0026	-0.0081	0.0026	-0.0035	0.0024
	100		-0.0013	0.0012	-0.0021	0.0010	-0.0032	0.0011	-0.0029	0.0012	-0.0006	0.0012
	15	2	-0.0166	0.0012	-0.0046	0.0008	-0.0171	0.0012	-0.0170	0.0012	-0.0164	0.0011
	30		-0.0097	0.0005	-0.0027	0.0004	-0.0101	0.0005	-0.0100	0.0005	-0.0096	0.0005
	50		-0.0057	0.0002	-0.0011	0.0002	-0.0059	0.0002	-0.0058	0.0002	-0.0056	0.0002
	100		-0.0029	0.0001	-0.0005	0.0001	-0.0030	0.0001	-0.0029	0.0001	-0.0028	0.0001
GP	15	0.5	0.0013	0.0070	-0.0014	0.0078	-0.0101	0.0080	-0.0080	0.0078	0.0061	0.0067
	30		-0.0005	0.0035	-0.0026	0.0036	-0.0066	0.0037	-0.0054	0.0037	0.0021	0.0034
	50		-0.0011	0.0022	-0.0025	0.0023	-0.0050	0.0023	-0.0042	0.0023	0.0004	0.0022
	100		0.0002	0.0011	-0.0005	0.0012	-0.0017	0.0012	-0.0013	0.0012	0.0010	0.0011
	15	2	-0.0150	0.0011	-0.0031	0.0008	-0.0155	0.0011	-0.0154	0.0011	-0.0148	0.0010
	30		-0.0095	0.0005	-0.0024	0.0004	-0.0098	0.0005	-0.0097	0.0005	-0.0093	0.0004
	50		-0.0052	0.0002	-0.0007	0.0002	-0.0054	0.0002	-0.0054	0.0002	-0.0052	0.0002
	100		-0.0031	0.0001	-0.0007	0.0001	-0.0032	0.0001	-0.0031	0.0001	-0.0030	0.0001

Table 9. Simulation results of Lindley approximation for $t = 4$

			SELF		LINEX		GE		WSELF		PRE	
LF	n	α	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	15		0.0052	0.0099	-0.0114	0.0106	-0.0324	0.0121	-0.0265	0.0116	0.0130	0.0098
	30		-0.0037	0.0049	-0.0095	0.0051	-0.0207	0.0056	-0.0174	0.0054	0.0029	0.0048
NP	50	0.5	-0.0050	0.0039	-0.0070	0.0040	-0.0110	0.0042	-0.0099	0.0041	-0.0025	0.0037
	100		0.0005	0.0014	-0.0012	0.0014	-0.0047	0.0014	-0.0036	0.0014	0.0025	0.0014
	15		0.0170	0.0023	0.0006	0.0018	-0.0107	0.0015	-0.0067	0.0015	0.0237	0.0029
	30	2	0.0079	0.0009	-0.0007	0.0008	-0.0073	0.0008	-0.0048	0.0008	0.0126	0.0011
	50		0.0058	0.0006	0.0004	0.0005	-0.0038	0.0005	-0.0021	0.0005	0.0090	0.0007
	100		0.0021	0.0003	-0.0006	0.0003	-0.0029	0.0003	-0.0019	0.0003	0.0039	0.0003
	15		0.0055	0.0094	-0.0063	0.0100	-0.0276	0.0113	-0.0215	0.0108	0.0176	0.0095
	30		0.0008	0.0053	-0.0050	0.0054	-0.0162	0.0059	-0.0130	0.0057	0.0073	0.0052
GP	50	0.5	0.0017	0.0032	-0.0017	0.0032	-0.0085	0.0033	-0.0065	0.0033	0.0057	0.0031
	100		0.0022	0.0015	0.0005	0.0015	-0.0029	0.0016	-0.0019	0.0016	0.0042	0.0015
	15		0.0176	0.0024	0.0013	0.0018	-0.0103	0.0015	-0.0062	0.0016	0.0241	0.0029
	30	2	0.0074	0.0010	-0.0012	0.0008	-0.0077	0.0008	-0.0052	0.0008	0.0120	0.0011
	50		0.0055	0.0006	0.0002	0.0005	-0.0041	0.0005	-0.0023	0.0005	0.0087	0.0006
	100		0.0034	0.0003	0.0007	0.0003	-0.0016	0.0003	-0.0006	0.0003	0.0052	0.0003

Table 10. Simulation results of Gibbs sampling method for $t = 0.5$

			SELF		LINEX		GE		WSELF		PRE	
LF	n	α	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	15		0.0085	0.0034	-0.0032	0.0031	-0.0153	0.0038	-0.0265	0.0036	-0.0120	0.0077
	30		0.0029	0.0013	-0.0027	0.0014	-0.0081	0.0014	-0.0059	0.0013	-0.0029	0.0036
NP	50	0.5	0.0005	0.0007	-0.0028	0.0007	-0.0058	0.0008	-0.0046	0.0008	-0.0012	0.0023
	100		0.0011	0.0003	-0.0005	0.0003	-0.0018	0.0003	-0.0012	0.0003	0.0002	0.0012
	15		-0.0279	0.0069	-0.0450	0.0097	-0.0620	0.0157	-0.0540	0.0134	-0.0172	0.0129
	30	2	-0.0144	0.0021	-0.0223	0.0029	-0.0276	0.0038	-0.0247	0.0034	-0.0088	0.0031
	50		-0.0082	0.0008	-0.0125	0.0011	-0.0150	0.0013	-0.0135	0.0012	-0.0050	0.0007
	100		-0.0034	0.0003	-0.0053	0.0003	-0.0062	0.0004	-0.0056	0.0003	-0.0030	0.0003
	15		0.0124	0.0036	0.0008	0.0032	-0.0110	0.0037	-0.0062	0.0036	0.0180	0.0096
	30	0.5	0.0049	0.0013	-0.0007	0.0012	-0.0061	0.0013	-0.0039	0.0013	0.0072	0.0019
	50		0.0026	0.0009	-0.0007	0.0007	-0.0037	0.0008	-0.0024	0.0007	0.0022	0.0012
GP	100		0.0017	0.0003	0.0001	0.0002	-0.0013	0.0003	-0.0007	0.0004	0.0005	0.0006
	15		-0.0223	0.0055	-0.0381	0.0079	-0.0531	0.0130	-0.0459	0.0111	-0.0223	0.0086
	30	2	-0.0128	0.0017	-0.0206	0.0023	-0.0257	0.0031	-0.0228	0.0027	-0.0111	0.0045
	50		-0.0063	0.0007	-0.0105	0.0009	-0.0127	0.0010	-0.0113	0.0009	-0.0075	0.0009
	100		-0.0039	0.0003	-0.0057	0.0004	-0.0066	0.0004	-0.0061	0.0004	-0.0049	0.0001

Table 11. Simulation results of Gibbs sampling method for $t=4$

			SELF		LINEX		GE		WSELF		PRE	
LF	n	α	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	15		-0.0187	0.0060	-0.0239	0.0059	-0.0228	0.0061	-0.0220	0.0061	-0.0224	0.0099
	30		-0.0110	0.0026	-0.0136	0.0024	-0.0133	0.0024	-0.0128	0.0026	-0.0120	0.0058
NP	50	0.5	-0.0042	0.0015	-0.0058	0.0018	-0.0057	0.0014	-0.0054	0.0014	-0.0045	0.0020
	100		-0.0024	0.0006	-0.0033	0.0007	-0.0034	0.0005	-0.0032	0.0006	-0.0004	0.0010
	15		0.0075	0.0030	0.0086	0.0026	0.0040	0.0027	0.0046	0.0024	0.0096	0.0129
	30	2	0.0020	0.0012	0.0028	0.0016	0.0012	0.0018	0.0005	0.0012	0.0028	0.0031
	50		0.0016	0.0008	0.0020	0.0008	0.0004	0.0013	0.0007	0.0010	0.0029	0.0007
	100		-0.0004	0.0003	-0.0004	0.0003	-0.0012	0.0004	-0.0010	0.0003	-0.0009	0.0003
	15		-0.0158	0.0059	-0.0210	0.0057	-0.0198	0.0060	-0.0191	0.0059	-0.0150	0.0060
	30	0.5	-0.0111	0.0027	-0.0138	0.0026	-0.0134	0.0028	-0.0129	0.0026	-0.0120	0.0030
GP	50		-0.0067	0.0015	-0.0084	0.0015	-0.0083	0.0016	-0.0080	0.0016	-0.0065	0.0019
	100		-0.0025	0.0006	-0.0034	0.0006	-0.0035	0.0006	-0.0033	0.0006	-0.0036	0.0006
	15		0.0088	0.0030	0.0100	0.0029	0.0055	0.0027	0.0061	0.0030	0.0093	0.0032
	30	2	0.0036	0.0017	0.0043	0.0014	0.0018	0.0015	0.0021	0.0017	0.0039	0.0020
	50		0.0034	0.0008	0.0037	0.0009	0.0022	0.0008	0.0024	0.0008	0.0035	0.0009
	100		0.0020	0.0004	0.0020	0.0004	0.0012	0.0004	0.0014	0.0004	0.0029	0.0004

Table 12. Classical estimators of the parameters based on real data

	Example-I			Example-II		
Data	α	β	$R(t)$	α	β	$R(t)$
MLE	5.5013	0.3773	0.1914	1.1068	1.1441	0.9179
MMLE	5.6564	0.3764	0.1820	1.1917	1.1019	0.9237

Table 13. Interval estimators of the parameters based on real data

	ACI		BCI-NP		BCI-GP	
	α	β	α	β	α	β
Ex-I	(4.4891;1.0464)	(0.3603;1.0639)	(4.6280;1.0793)	(0.3622;1.0900)	(4.7322;1.0726)	(0.3676;1.0969)
Ex-II	(6.5135;1.1673)	(0.3943;1.2243)	(6.1725;1.1102)	(0.3798;1.2354)	(6.3633;1.1076)	(0.3801;1.2415)

Table 14. Bayesian parameter estimation using Lindley approximation based on real data

	Lindley-NP						Lindley-GP					
	Example-I			Example-II			Example-I			Example-II		
Data	α	β	$R(t)$	α	β	$R(t)$	α	β	$R(t)$	α	β	$R(t)$
SELF	5.4386	0.3772	0.1834	1.1063	1.1439	0.9173	5.4035	0.3775	0.1990	1.1064	1.1440	0.9173
LINEX	5.2786	0.3772	0.1837	1.1055	1.1426	0.9175	5.2551	0.3774	0.1951	1.1056	1.1428	0.9175
GE	5.3814	0.3770	0.1830	1.1051	1.1420	0.9172	5.3484	0.3772	0.1896	1.1053	1.1422	0.9173
WSEF	5.3923	0.3770	0.1831	1.1054	1.1424	0.9172	5.3588	0.3773	0.1915	1.1055	1.1426	0.9173
PRE	5.4627	0.3773	0.1835	1.1068	1.1446	0.9173	5.4272	0.3776	0.2025	1.1069	1.1428	0.9174

Table 15. Bayesian parameter estimation using Gibbs sampling based on real data

	MCMC-NP						MCMC-GP					
	Example-I			Example-II			Example-I			Example-II		
Data	α	β	$R(t)$	α	β	$R(t)$	α	β	$R(t)$	α	β	$R(t)$
SELF	5.5669	0.3751	0.1834	1.1001	1.1273	0.9133	5.7097	0.3754	0.1770	1.1149	1.1279	0.9160
LINEX	5.5595	0.3751	0.1837	1.0997	1.1261	0.9130	5.6935	0.3754	0.1776	1.1143	1.1271	0.9157
GE	5.5646	0.3749	0.1830	1.0995	1.1255	0.9129	5.7049	0.3752	0.1766	1.1140	1.1267	0.9156
WSEF	5.5651	0.3750	0.1831	1.0996	1.1259	0.9130	5.7059	0.3752	0.1767	1.1142	1.1270	0.9157
PRE	5.5678	0.3752	0.1835	1.1003	1.1281	0.9135	5.7116	0.3755	0.1771	1.1153	1.1283	0.9161

7. Conclusion

In this paper, we obtained different methods of estimation of the unknown parameters both with Bayesian and classical approximation. We also discussed the estimation of the reliability function of the IW distribution both from a Bayesian and classical point of view. In classical methods, the parameters α and β were estimated by using MLE and MLE methods. Moreover, we considered the Bayesian estimators of the unknown parameters by using informative and non-informative priors under different loss functions. Furthermore, we compared the performances of the estimators via a simulation study. The simulation results show that Bayesian estimators with NP and GP have higher efficiencies than the classical estimators. Especially, in terms of the MSEs values and LINEX, WSELF, and GE loss function, the method of Bayesian works the best in almost all cases. Given the reliability function (see Table 6-11), Bayesian estimators under the LINEX loss function perform better in most of the cases. When we compare the Bayesian and classical methods, simulation results show that the performances of Bayesian estimators are somewhat more efficient than the MMLEs for small sample sizes. For the moderate and the large sample sizes, Bayesian methods demonstrate better performance than the classical methods. Considering all points, we recommend using Bayesian estimators with a convenient prior distribution and loss function for estimating the parameters α, β, and $R(t)$. In other cases, MMLEs may be preferred for small samples. In the future, this work can be extended to censored data and record values. Also, we evaluated Bayes estimators and their respective bias and MSE values using Lindley and Gibbs sampling methods. However, a comparison of Lindley, Tierney-Kadane, Empirical Bayes, and Gibbs sampling methods will be interesting.

References

Ali, S., Aslam, M. & Kazmi, S. M. A. (2013). A study of the effect of the loss function on Bayes Estimate, posterior risk and hazard function for Lindley distribution. Applied Mathematical Modelling, 37(**8**): 6068-6078.

Bi, Q. & Gui, W. (2017). Bayesian and classical estimation of stress-strength reliability for inverse Weibull lifetime models. Algorithms, 10(**2**): 71.

Calabria, R. &Pulcini, G. (1990). Bayes 2-sample prediction for the inverse Weibull distribution. Communications Statistics Theory and Methods 23(**6**): 1811–1824.

Chen, M. H., & Shao, Q. M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals, Journal of Computational and Graphical Statistics, 8(**1**): pp.69-92.

Dey, S., Zhang, C., Asgharzadeh, A. & Ghorbannezhad, M. (2017). Comparisons of methods of estimation for the NH distribution. Annals of Data Science, 4(**4**): 441-455.

Drapella, A. (1993). The complementary Weibull distribution: unknown or just forgotten? Quality and Reliability Engineering International 9(**4**): 383-385.

Erto, P. (1986). Properties and identification of the inverse Weibull: unknown or just forgotten. Quality and Reliability Engineering International, 9: 383-385.

Farahani, Z. S. M. & Khorram, E. (2014). Bayesian statistical inference for the weighted4 exponential distribution. Communications in Statistics-Simulation and Computation 43.6: 1362-1384.

Guure, C. B., *et al.* (2014). Bayesian parameter and reliability estimate of Weibull failure time distribution. The Bulletin of the Malaysian Mathematical Sciences Society Series. 2(**14**): 611-632.

Helu, A., & Samawi, H. (2015). The Inverse Weibull Distribution as a Failure Model Under Various Loss Functions and Based on Progressive First-Failure Censored Data. Quality Technology & Quantitative Management 12(**4**): 517-535.

Johnson, N. L., Kotz, S. &Balakrishnan, N. (1994). Continuous Univariate Distributions-1, 2nd ed. John Wiley & Sons.

Kishorilal, D. B., & Mukhopadhyay, A. K. (2018). Reliability investigation of diesel engines used in dumpers by the Bayesian approach. Kuwait Journal of Science, 45(**4**).

Koc, T. & Cengiz, M. A. (2020). Investigating Different Priors in Bayesian Continual Reassessment Method. Kuwait Journal of Science, 47(**1**).

Kumar, K. (2018). Classical and Bayesian estimation in log-logistic distribution under random censoring. International Journal of System Assurance Engineering and Management, 9.2: 440-451.

Kundu, D. & Gupta, R. D. (2006). Estimation of P [Y< X] for Weibull distributions. IEEE Trans. Reliability, 55(**2**): 70-280.

Kundu, D. & Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. Computational Statistics & Data Analysis, 54(**6**): 1547-1558.

Lawless, J. F. (2003). Statistical Models and Methods for Lifetime Data. 3rd Edition, John Wiley and Sons, New York.

Lindley, D. V. (1980). Approximate bayesian methods. Trabajos de estadística y de investigaciónoperativa, 31(**1**): 223-245.

Maswadah, M. (2003). Conditional confidence interval estimation for the inverse Weibull distribution based on censored generalized order statistics. Journal of Statistical Computation and Simulation 73(**12**): 887-898.a

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H. & Teller, E. (1953). Equation of state calculations by fast computing machines. The journal of chemical physics, 21(**6**): 1087-1092.

Mudholkar, G. S. & Kollia, G. D. (1994). Generalized Weibull family: a structural analysis. Communications in statistics-theory and methods 23(**4**): 1149-1171.

Murthy, D. N., Bulmer, M. & Eccleston, J. A. (2004). Weibull model selection for reliability modeling. Reliability Engineering & System Safety 86(**3**): 257-267.

Ramos, P. L., Louzada, F., Ramos, E. & Dey, S. (2018). The Fréchet distribution: estimation and application an overview. arXiv preprint arXiv:1801.05327.

Rashid, M. (2019). Socio-economic factors of misconception about HIV/AIDS among ever married women in Punjab: A comparison of non-spatial and spatial hierarchical Bayesian Poisson model. Kuwait Journal of Science, 46(**4**).

Rastogi, M. K. & Merovci, F. (2018). Bayesian estimation for parameters and reliability characteristic of the Weibull Rayleigh distribution. Journal of King Saud University Science 30(**4**): 472-478.

Renjini, K. R., Abdul-Sathar, E. I. & Rajesh, G. (2016). A study of the effect of loss functions on the Bayes estimates of dynamic cumulative residual entropy for Pareto distribution under upper record values. Journal of Statistical Computation and Simulation 86(**2**): 324-339.

Rinne, H. (2009). The Weibull distribution: A Handbook. Genesis 3(1.1): 1-2.

Singh, S. K., Singh, U. & Yadav, A. S. (2015). Reliability estimation and prediction for extension of exponential distribution using informative and non-informative priors. International Journal of System Assurance Engineering and Management, 6(**4**): 466-478.

Smith, A. F. & Roberts, G. O. (1993). Bayesian computation via the Gibbs sampler and related Markov chain Monte Carlo methods. Journal of the Royal Statistical Society: Series B (Methodological), 55.1: 3-23.

Surles, J. G. & Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr type X distribution. Lifetime Data Analysis, 7(**2**): 187-200.

Tang, Y. (2004). Extended Weibull Distributions in Reliability Engineering. A Thesis Submitted to the Department of Industrial & System Engineering, National University of Singapore.

Teimouri, M., Hoseini, S. M. & Nadarajah, S. (2013). Comparison of estimation methods for the Weibull distribution. Statistics, 47(**1**), 93-109.

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