Partition-theoretic interpretations of some q-series identities of Ramanujan

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Abstract

Ramanujan's lost notebook contains several q-series identities, and some of them have theta-function representations. We give partition-theoretic interpretations of some of these identities and prove Ramanujan-type congruences for certain partition functions.

Keywords: Congruence; Jacobi's triple product identity; partition of integer; q-series; Ramanujan's theta-functions.

1. Introduction

Ramanujan's lost notebook contains several q-series identities. Details of these can be found in (Andrews & Berndt, 2009). The purpose of this paper is to give partition interpretations of some of the q-series identities and prove Ramanujan-type congruences for certain par-tition functions.

In the sequel, throughout the paper, we assume that $|q| < 1$ always and as usual, for any complex number a and q, we set

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \text{ for } n \geq 1,$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \qquad (1)$$

For brevity, we write

$$(a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty\cdots(a_m;q)_\infty = (a_1,a_2,a_3,\cdots,a_m;q)_\infty \qquad (2)$$

Several q-series identities in Ramanujan's lost notebook have representations in terms of Ramanujan's theta-function $f(a,b)$ (Berndt ,1991). p. 34, (18.1)) defined by

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1. \qquad (3)$$

In terms of $f(a,b)$, Jacobi's triple product identity (Berndt, 1991, p. 35, Entry 19) can be stated as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = (-a,-b,ab;ab)_\infty. \qquad (4)$$

Three important special cases of $f(a,b)$ are the theta-functions $\phi(q)$, $\psi(q)$ and $f(-q)$ (Berndt, 1991, p. 36, Entry 22), which are given by

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \qquad (5)$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \qquad (6)$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \qquad (7)$$

respectively. Ramanujan also defined the function $\chi(q)$ (Berndt, 1991, p. 39, Entry 22(iv)) as

$$\chi(q) = (-q; q^2)_\infty = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty (q^4; q^4)_\infty}. \quad (8)$$

By using elementary q-analysis, one can easily verify that

$$\phi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}, \ \psi(-q) = \frac{(q; q)_\infty (q^4; q^4)_\infty}{(q^2; q^2)_\infty}$$

and

$$\chi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}. \quad (9)$$

With this background, we now state some q-series identities from (Andrews & Berndt, 2009) for which we will give partition-theoretic interpretations in this paper.

(i) (Andrews & Berndt, 2009, p. 32, Entry 1.7.5):

$$\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n(n+1)/2}}{(q; q)_n (q; q^2)_n} = \frac{\psi(-q^2)}{\phi(-q)}. \quad (10)$$

(ii) (Andrews & Berndt, 2009, p. 32, Entry 1.7.6):

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{(n^2+3n)/2}}{(q; q)_n (q; q^2)_{n+1}} = \frac{f(-q, -q^7)}{\phi(-q)}. \quad (11)$$

(iii) (Andrews & Berndt, 2009, p. 34, Entry 1.7.8):

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+1)/2}}{(q; q)_n (q; q^2)_{n+1}} = \frac{f(-q^3, -q^5)}{\phi(-q)}. \quad (12)$$

(iv) (Andrews & Berndt, 2009, p. 48, Entry 2.3.3):

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n^2 q^{n(n+1)/2}}{(q; q)_n (q; q^2)_{n+1}} = \frac{\psi(q^2)}{\phi(-q)}. \quad (13)$$

(v) (Andrews & Berndt, 2009, p. 65, Entry 3.4.5):

$$\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^n}{(q; q)_{2n+1}} = \frac{\psi(-q^3)}{\phi(-q)}. \quad (14)$$

(vi) (Andrews & Berndt, 2009, p. 85, Entry 4.2.8):

$$\sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n^2}}{(q; q)_n (q; q^2)_n} = \frac{\phi(-q^3)}{\phi(-q)}. \quad (15)$$

(vii) (Andrews & Berndt, 2009, p. 87, Entry 4.2.12):

$$\sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n^2+n}}{(q; q)_n (q; q^2)_{n+1}} = \frac{f(-q, -q^5)}{\phi(-q)}. \quad (16)$$

(viii) (Andrews & Berndt, 2009, p. 88, En-try 4.2.13):

$$\sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n(n+1)}}{(q; q^2)_{2n+1}} = \frac{f(q, q^5)}{\phi(-q^2)}. \quad (17)$$

We end this section by defining partition of a positive integer. A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 3$ has three partitions, namely,

$$3, \quad 2+1, \quad 1+1+1.$$

If $p(n)$ denote the number of partitions of n, then $p(3) = 3$. The generating function for $p(n)$ due to Euler is given by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_\infty}. \quad (18)$$

Ramanujan (Ramanujan, 1919) established fol-lowing beautiful congruences for $p(n)$:

$$p(5n+4) \equiv 0 \ (\mathrm{mod}\ 5),$$

$$p(7n+5) \equiv 0 \pmod 7$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

A part in a partition of n has r colours if there are r copies of each part available and all of them are viewed as distinct objects. For any positive integers n and r, let $p_r(n)$ denote the number of partitions of n where each part may have r distinct colours. For example, if each part in the partition of 3 has two colours, say red (indicated by the suffix r) and green (indicated by the suffix g), then the number of two colour partitions of 3 is 10, namely

$$3_r, \quad 3_g, \quad 2_r+1_r, \quad 2_r+1_g, \quad 2_g+1_g,$$
$$2_g+1_r, \quad 1_r+1_r+1_r, \quad 1_g+1_g+1_g,$$
$$1_r+1_g+1_g, \quad 1_r+1_r+1_g.$$

The generating function of $p_r(n)$ (Berndt & Rankin, 1995) is given by

$$\sum_{n=0}^{\infty} p_r(n) q^n = \frac{1}{(q; q)_\infty^r}. \quad (19)$$

For $r = 1$, $p_1(n)$ is the usual unrestricted partition function $p(n)$ defined in (18).

We also note that, for positive integers k, m and r,

$$\frac{1}{(q^k; q^m)_\infty^r} \quad (20)$$

is the generating function of the number of partitions of a positive integer with parts *congruent to k modulo m* (that is, $\equiv k \pmod{m}$) and each part has r colours. Similarly,

$$\frac{1}{(q^{k_1};q^m)_\infty^2\,(q^{k_2};q^m)_\infty^2}=\frac{1}{(q^{k_1},q^{k_2};q^m)_\infty^2} \tag{21}$$

is the generating function of the number of partitions of positive integer with parts $\equiv k_1$ or $k_2 \pmod{m}$ and each part has two colours.

2. Partition-Theoretic Interpretations of q-Series Identities

In this section, we call q^m as the base of the q-product $(q^k;q^m)_\infty$ for positive integer m.

Theorem 2.1. *Let $A_1(n)$ denote the number of partitions of a positive integer n where the parts are $\equiv 1,3,4,5$ or $7 \pmod{8}$ and each part has two colours except the parts $\equiv 4 \pmod{8}$. Then*

$$\sum_{n=0}^{\infty}A_1(n)q^n=\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_nq^{n(n+1)/2}}{(q;q)_n(q;q^2)_n}=\frac{\psi(-q^2)}{\phi(-q)}.$$

Proof. From (9), we see that

$$\frac{\psi(-q^2)}{\phi(-q)}=\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty}{(q^4;q^4)_\infty(q;q)_\infty^2}. \tag{22}$$

Changing the base q^2 in $(q^2;q^2)_\infty$ to q^8, we obtain

$$(q^2;q^2)_\infty=(q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty=(q^2,q^4,q^6,q^8;q^8)_\infty. \tag{23}$$

Similarly, by changing the bases to q^8 in $(q^4;q^4)_\infty$ and $(q;q)_\infty$, we obtain

$$(q^4;q^4)_\infty=(q^4,q^8;q^8)_\infty \tag{24}$$

and

$$(q;q)_\infty=(q,q^2,q^3,q^4,q^5,q^6,q^7,q^8;q^8)_\infty. \tag{25}$$

Employing (23), (24) and (25) in (22) and simplifying, we arrive at

$$\frac{\psi(-q^2)}{\phi(-q)}=\frac{1}{(q^4;q^8)_\infty(q,q^3,q^5,q^7;q^8)_\infty^2}. \tag{26}$$

Now the desired result follows easily from (10) and (26). □

Theorem 2.2. *Let $A_2(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0 \pmod{8}$ and parts $\equiv 3$ or $5 \pmod{8}$ have two colours. Then*

$$\sum_{n=0}^{\infty}A_2(n)q^n=\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{(n^2+3n)/2}}{(q;q)_n(q;q^2)_{n+1}}=\frac{f(-q,-q^7)}{\phi(-q)}.$$

Proof. Using (4) and (9), we see that

$$\frac{f(-q,-q^7)}{\phi(-q)}=\frac{(q,q^7,q^8;q^8)_\infty(q^2;q^2)_\infty}{(q;q)_\infty^2}. \tag{27}$$

Changing the bases of q-products to q^8, we find that

$$\frac{f(-q,-q^7)}{\phi(-q)}=\frac{(q,q^7,q^8;q^8)_\infty(q^2,q^4,q^6,q^8;q^8)_\infty}{(q,q^2,q^3,q^4,q^5,q^6,q^7,q^8;q^8)_\infty^2}. \tag{28}$$

Simplifying (28), we obtain

$$\frac{f(-q,-q^7)}{\phi(-q)}=\frac{1}{(q,q^2,q^4,q^6,q^7;q^8)_\infty(q^3,q^5;q^8)_\infty^2}. \tag{29}$$

Now the desired result follows easily from (11) and (29). □

Theorem 2.3. *Let $A_3(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0 \pmod{8}$ and parts $\equiv 1$ or $7 \pmod{8}$ have two colours. Then*

$$\sum_{n=0}^{\infty}A_3(n)q^n=\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}}=\frac{f(-q^3,-q^5)}{\phi(-q)}.$$

Proof. Using (4) and (9), we see that

$$\frac{f(-q^3,-q^5)}{\phi(-q)}=\frac{(q^3,q^5,q^8;q^8)_\infty(q^2;q^2)_\infty}{(q;q)_\infty^2}. \tag{30}$$

Changing the bases of q-products to q^8, we find that

$$\frac{f(-q^3,-q^5)}{\phi(-q)}=\frac{(q^3,q^5,q^8;q^8)_\infty(q^2,q^4,q^6,q^8;q^8)_\infty}{(q,q^2,q^3,q^4,q^5,q^6,q^7,q^8;q^8)_\infty^2}. \tag{31}$$

Simplifying (31), we obtain

$$\frac{f(-q^3,-q^5)}{\phi(-q)} = \frac{1}{(q^2,q^3,q^4,q^5,q^6;q^8)_\infty (q,q^7;q^8)^2_\infty}. \quad (32)$$

Now the desired result follows easily from (12) and (32). □

Theorem 2.4. *Let $A_4(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0 \pmod 4$ and each part has two colours. Then*

$$\sum_{n=0}^{\infty} A_4(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)^2_n q^{n(n+1)/2}}{(q;q)_n (q;q^2)_{n+1}} = \frac{\psi(q^2)}{\phi(-q)}.$$

Proof. Using (6) and (9), we see that

$$\frac{\psi(q^2)}{\phi(-q)} = \frac{(q^4;q^4)^2_\infty}{(q;q)^2_\infty}. \quad (33)$$

Changing the base q in $(q;q)_\infty$ to q^4 of (33) and simplifying, we obtain

$$\frac{\psi(q^2)}{\phi(-q)} = \frac{1}{(q,q^2,q^3;q^4)^2_\infty}. \quad (34)$$

Now the desired result follows from (13) and (34). □

Theorem 2.5. *Let $A_5(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0 \pmod{12}$ and parts $\equiv 1,5,7$ or $11 \pmod{12}$ have two colours. Then*

$$\sum_{n=0}^{\infty} A_5(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(q;q)_{2n+1}} = \frac{\psi(-q^3)}{\phi(-q)}.$$

Proof. Using (9), we see that

$$\frac{\psi(-q^3)}{\phi(-q)} = \frac{(q^3;q^3)_\infty (q^{12};q^{12})_\infty (q^2;q^2)_\infty}{(q^4;q^4)_\infty (q;q)^2_\infty}. \quad (35)$$

Changing the bases to q^{12} in (35) and simplifying, we obtain

$$\frac{\psi(-q^3)}{\phi(-q)} = \frac{1}{(q^2,q^3,q^4,q^6,q^8,q^9,q^{10};q^{12})_\infty} \times \frac{1}{(q,q^5,q^7,q^{11};q^{12})^2_\infty}. \quad (36)$$

Now the desired result follows from (14) and (36). □

Theorem 2.6. *Let $A_6(n)$ denote the number of partitions of a positive integer n with parts $\equiv 1,2,4$ or $5 \pmod 6$ and parts $\equiv 1$ or $5 \pmod 6$ have two colours. Then*

$$\sum_{n=0}^{\infty} A_6(n)q^n = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q)_n (q;q^2)_n} = \frac{\phi(-q^3)}{\phi(-q)}.$$

Proof. Using (9), we find that

$$\frac{\phi(-q^3)}{\phi(-q)} = \frac{(q^3;q^3)^2_\infty (q^2;q^2_\infty)}{(q^6;q^6)_\infty (q;q)^2_\infty}. \quad (37)$$

Changing the bases q^2, q and q^3 in (37) to q^6 and simplifying, we obtain

$$\frac{\phi(-q^3)}{\phi(-q)} = \frac{1}{(q^2,q^4;q^6)_\infty (q,q^5;q^6)^2_\infty} \quad (38)$$

Now the desired result follows from (15) and (38). □

Theorem 2.7. *Let $A_7(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0 \pmod 6$ and parts $\equiv 3 \pmod 6$ have two colours. Then*

$$\sum_{n=0}^{\infty} A_7(n)q^n = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2+n}}{(q;q)_n (q;q^2)_{n+1}} = \frac{f(-q,-q^5)}{\phi(-q)}.$$

Proof. Using (4) and (9), we see that

$$\frac{f(-q,-q^5)}{\phi(-q)} = \frac{(q,q^5,q^6;q^6)_\infty (q^2;q^2)_\infty}{(q;q)^2{}_\infty} \quad (39)$$

Changing the bases of q-products to q^6 in (39), we obtain

$$\frac{f(-q,-q^5)}{\phi(-q)} = \frac{(q,q^5,q^6;q^6)_\infty (q^2,q^4,q^6;q^6)_\infty}{(q,q^2,q^3,q^4,q^5,q^6;q^6)^2_\infty}. \quad (40)$$

Simplifying (40), we obtain

$$\frac{f(-q,-q^5)}{\phi(-q)} = \frac{1}{(q,q^2,q^4,q^5;q^6)_\infty (q^3;q^6)^2_\infty}. \quad (41)$$

Now the desired result follows easily from (16) and (41). □

Theorem 2.8. *Let $A_8(n)$ denote the number of partitions of a positive integer n with parts $\not\equiv 0,3,9 \pmod{12}$. Then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q^2)_{2n+1}} = \frac{f(q,q^5)}{\phi(-q^2)}.$$

Proof. From (Berndt, 1991, p. 51, Example(v)), we note that

$$f(q,q^5)=\psi(-q^3)\chi(q). \tag{42}$$

Using (8), (9) and (42), we obtain

$$\frac{f(q,q^5)}{\phi(-q^2)}=\frac{\psi(-q^3)\chi(q)}{\phi(-q^2)}$$
$$=\frac{(q^3,q^3)_\infty(q^{12};q^{12}_\infty}{(q;q)_\infty(q^6;q^6)_\infty}. \tag{43}$$

Changing the bases of q-products to q^{12} in (43) and simplifying, we obtain

$$\frac{f(q,q^5)}{\phi(-q^2)}$$
$$=\frac{1}{(q,q^2,q^4,q^5,q^6,q^7,q^8,q^{10},q^{11};q^{12})_\infty}. \tag{44}$$

The desired result follows easily from (17) and (44). □

3. Ramanujan-Type Congruences for $A_1(n)$, $A_4(n)$, $A_5(n)$ and $A_6(n)$

In this section, the suffixes 'r'and 'g'in parts of partitions will indicate two colours *red* and *green* of the parts of the partitions, respectively.

Theorem 31. *If* $A_1(n)$ *is as defined in Theorem 2.1, then*

$$A_1(2n+1)\equiv 0 \pmod 2.$$

Proof. From (Hirschhorn & Sellers, 2005), we note that

$$\frac{1}{\phi(-q)}=\frac{1}{\phi(q^4)^4}\Big(\phi(q^4)^3+2q\phi(q^4)^2\psi(q^8)$$
$$+4q^2\phi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\Big). \tag{45}$$

Employing (45) in Theorem 2.1, we obtain

$$\sum_{n=0}^{\infty}A_1(n)q^n=\frac{\psi(-q^2)}{\phi(q^4)^4}\Big(\phi(q^4)^3+2q\phi(q^4)^2\psi(q^8)$$
$$+4q^2\phi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\Big)$$
$$\equiv\frac{\psi(-q^2)}{\phi(q^4)} \pmod 2. \tag{46}$$

Since right hand side of (46) contains no terms involving q^{2n+1}, extracting the terms involving q^{2n+1} from (46), we arrive at the desired result. □

Remark 3.2. *We note that* $A_1(3)=6$ *with the relevant partitions given by* 3_r, 3_g, $1_r+1_r+1_r$, $1_r+1_r+1_g$, $1_r+1_g+1_g$ *and* $1_g+1_g+1_g$. *This verifies Theorem 3.1.*

Theorem 33. *If* $A_4(n)$ *is as defined in Theorem 2.4, then*

$$A_4(2n+1)\equiv 0 \pmod 2.$$

Proof. Employing (45) in Theorem 2.4, we obtain

$$\sum_{n=0}^{\infty}A_4(n)q^n=\frac{\psi(q^2)}{\phi(q^4)^4}\Big(\phi(q^4)^3+2q\phi(q^4)^2\psi(q^8)$$
$$+4q^2\phi(q^4)\psi(q^8)^2+8q^3\psi(q^8)^3\Big)$$
$$\equiv\frac{\psi(q^2)}{\phi(q^4)} \pmod 2. \tag{47}$$

Extracting the terms involving q^{2n+1} from (47), we arrive at the desired result. □

Remark 3.4. *We note that* $A_4(3)=10$ *and the relevant partitions are* 3_r, 3_g, 2_r+1_r, 2_g+1_g, 2_r+1_g, 2_g+1_r, $1_r+1_r+1_r$, $1_r+1_r+1_g$, $1_r+1_g+1_g$ *and* $1_g+1_g+1_g$. *This verifies Theorem 3.3.*

Theorem 35. *If* $A_5(n)$ *is as defined in Theorem 2.5, then*

$$(i)\ A_5(3n+1)\equiv 0 \pmod 2,$$
$$(ii)\ A_5(3n+2)\equiv 0 \pmod 4.$$

Proof. From (Hirschhorn & Sellers, 2005), we note that

$$\frac{1}{\phi(-q)}=\frac{\phi(-q^9)}{\phi(-q^3)^4}\Big(\phi(-q^9)^2$$
$$+2q\phi(-q^9)\omega(-q^3)+4q^2\omega(-q^3)^2\Big), \tag{48}$$

where $\omega(-q)=\dfrac{(q;q)_\infty(q^6;q^6)^2_\infty}{(q^2;q^2)_\infty(q^3;q^3)_\infty}$.

Employing (48) in Theorem 2.5, we obtain

$$\sum_{n=0}^{\infty}A_5(n)q^n=\frac{\psi(-q^3)\phi(-q^9)}{\phi(-q^3)^4}\Big(\phi(-q^9)^2$$
$$+2q\phi(-q^9)\omega(-q^3)+4q^2\omega(-q^3)^2\Big). \tag{49}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (49), we arrive at (i) and (ii), respectively. □

Remark 3.6. *We note that,* $A_5(2) = 4$ *with relevant partitions given by* 2, $1_r + 1_r$, $1_r + 1_g$ *and* $1_g + 1_g$. *This verifies Theorem 3.5(ii) for* $n = 0$.

Theorem 3.7. *If* $A_6(n)$ *is as defined in Theorem 2.6, then*

(i) $A_6(3n+1) \equiv 0 \pmod 2$,

(ii) $A_6(3n+2) \equiv 0 \pmod 4$,

(iii) $A_6(9n+3j) \equiv 0 \pmod 3$,

$where \quad j = 1,\ 2,$

(iv) $A_6(9n) \equiv A_6(n) \pmod 3$.

Proof. Employing (48) in Theorem 2.6, we obtain

$$\sum_{n=0}^{\infty} A_6(n)q^n = \frac{\phi(-q^9)}{\phi(-q^3)^3}\Big(\phi(-q^9)^2 +2q\phi(-q^9)\omega(-q^3) + 4q^2\omega(-q^3)^2\Big). \quad (50)$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (50), we arrive at (i) and (ii), respectively.

Next, extracting the terms involving q^{3n} from (50) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} A_6(3n)q^n = \frac{\phi(-q^3)^3}{\phi(-q)^3} \equiv \frac{\phi(-q^9)}{\phi(-q^3)} \pmod 3, \quad (51)$$

where we used the result

$$\phi(-q^3) \equiv \phi(-q)^3 \pmod 3 \quad (52)$$

which follows from (1), (9), and the binomial theorem.

Since right hand side of (51) contains no terms involving q^{3n+1} and q^{3n+2}, so extracting the terms involving q^{3n+j} for j =1, 2 from (51), we arrive at (iii).

To prove (iv), extracting the terms involving q^{3n} from (51), replacing q^3 by q and employing Theorem 2.6, we obtain

$$\sum_{n=0}^{\infty} A_6(9n)q^n \equiv \frac{\phi(-q^3)}{\phi(-q)} = \sum_{n=0}^{\infty} A_6(n)q^n \pmod 3. \quad (53)$$

Extracting the coefficient of q^n on both sides of (53), we arrive at (iv). □

Remark 3.8. *We note that,* $A_6(3) = 6$ *and the relevant partitions are given by* $2 + 1_r$, $2 + 1_g$, $1_r + 1_r + 1_r$, $1_r + 1_r + 1_g$, $1_r + 1_g + 1_g$ *and* $1_g + 1_g + 1_g + 1_g$. *This verifies Theorem 3.7(iii) for* $j = 1$ *and* $n = 0$.

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