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# A novel computational method for solving nonlinear Volterra integro-differential equation

Musa Cakir[1], Baransel Gunes[2,*], Hakki Duru[3]

[1]*Dept. of Mathematics, Faculty of Science, Yuzuncu Yil University, Van, Turkey*

[2]*Dept. of Mathematics, Faculty of Science, Yuzuncu Yil University, Van, Turkey*

[3]*Dept. of Mathematics, Faculty of Science, Yuzuncu Yil University, Van, Turkey*

[*]*Corresponding Author: baransel_gunes_91@hotmail.com*

**Abstract**

In this paper, we study quasilinear Volterra integro-differential equations (VIDEs). Asymptotic estimates are made for the solution of VIDE. Finite difference scheme, which is accomplished by the method of integral identities using interpolating quadrature rules with weight functions and remainder term in integral form, is presented for the VIDE. Error estimates are carried out according to the discrete maximum norm. It is given an effective quasilinearization technique for solving nonlinear VIDE. The theoretical results are performed on numerical examples.

**Keywords:** Error bounds; finite difference method; Volterra integro-differential equation.

## 1. Introduction

In this research, we consider the following quasilinear Volterra integro-differential initial value problem:

$$Lu \coloneqq u'(t) + f\big(t, u(t)\big) + \int_0^t K\big(t, s, u(s)\big)ds = 0, \quad t \in I = (0, T] \tag{1}$$

$$u(0) = A, \tag{2}$$

where $\bar{I} = [0, T]$ and $A$ is a given constant and $f(t, u(t))$ $((t, u) \in \bar{I} \times \mathbb{R})$ and $K\big(t, s, u(s)\big)$ $((t, s, u) \in \bar{I} \times \bar{I} \times \mathbb{R})$ are given sufficiently smooth functions and moreover

$$0 < \alpha \le \frac{\partial f}{\partial u} \le p^* < \infty.$$

Volterra integro-differential equations (VIDEs) are typical mathematical models in many areas of science. The solution of VIDEs has been identified as important in physics, engineering, chemistry and biology.

For instance, Kirchoff's second law is considered as follows:

$$L\frac{d}{dt}I(t) + RI(t) + \frac{1}{C}\int_0^t I(\tau)d\tau = E(t)$$

where $I(t)$ is the current function, $R$ is the resistance, $L$ is the inductance and $C$ is the capacitance (Zill & Warren, 2013). On the other hand, Wilson-Cowan model describes the dynamics of interactions between populations:

$$\frac{\partial E}{\partial t} = -E + \int W_{ee}(x - x')f(E - \theta_1)dx$$
$$- \int W_{ie}(x - x')f(E - \theta_2)dx + \psi_1(x, t)$$
$$\tau\frac{\partial E}{\partial t} = -I + \int W_{ei}(x - x')f(E - \theta_1)dx$$
$$- \int W_{ii}(x - x')f(E - \theta_2)dx + \psi_2(x, t)$$

where initial conditions $E(x, 0) = E_0(x), I(x, 0) = I_0(x)$ and $x \epsilon \mathbb{R}, t > 0$. $E(x, t)$ and $I(x, t)$ represent the activity of a population of excitatory and inhibitory neurons, respectively. $W_{ij}$ describes the strength of connection $\theta_1$, $\theta_2$ and $\tau$ are positive parameters. $\psi_1$ and $\psi_2$ are external inputs (Singh *et al.*, 2018).

For numerical solutions of these equations, numerous methods have been presented by some authors. Adomian decomposition method (Biazar *et al.*, 2003; Goghary *et al.*, 2005; Wazwaz, 2010), Galerkin method (Maleknejad & Kojani, 2004), collocation method (Aguilar & Brunner, 1988; Laib *et al.,* 2018), iterative and non-iterative methods (Ramos, 2009), homotopy perturbation method (Yusufoglu, 2007), Tau method (Mahmoud *et al.,* 2005) and variational iteration method (Wang & He, 2007) are among these. Recently, the studies of these problems have become significant and have been investigated with different properties. Singularly perturbed VIDEs are considered by many authors. An exponentially fitted difference scheme is constructed for these equations (Amiraliyev & Sevgin, 2006). For singularly perturbed VIDEs with smooth kernel, the coupled method (LDG-CFEM) is used (Tao & Zhang, 2019). The difference scheme and its convergence properties are examined for singularly perturbed VIDEs on a graded mesh (Sevgin, 2014). Moreover, more different studies are seen. Finite difference method is examined for first-order VIDEs (Cimen, 2018). In addition, for high-order linear VIDEs, a new numerical method with Taylor polynomials is developed (Laib *et al.,* 2018). Considering more varied methods, implicit Runge-Kutta method is applied for approximate solution of VIDEs (Brunner, 1984). A computational method is developed for numerical solution of a nonlinear VIDE of fractional order (Saeedi & Mohseni, 2011). By using Euler polynomials, VIDEs of pantograph-delay type are examined (Mirzaee *et al*., 2016) and nonlinear Volterra delay integro differential equations have been studied (Gan, 2007). Besides, for high-order integro differential equations with weakly singular kernel, a Galerkin-like method is used (Yüzbaşı & Karaçayır, 2016).

In this study, our purpose is to present finite difference method, which is founded by the method of integral identities using basis functions and interpolating quadrature rules with weight and remainder term in integral form. This method is first applied to nonlinear VIDEs. Unlike other methods, our method yields more accurate results for these equations.

Notations. Throughout the paper, $C$ and $C_0$ are generic positive constants that are independent of the mesh parameter. For any continuous function $v(x)$ defined on the corresponding interval, we use the maximum norm

$$\|v\|_\infty = \max_{[0,T]}|v(x)|.$$

## 2. Continuous problem

In this section, we give some asymptotic behavior of the exact solution, which is needed in the analysis of numerical method.

Lemma 1. Let $p(t), q(t)\epsilon C(\bar{I})$ and $G(t,s)\epsilon C(\bar{I}\times\bar{I})$. Then for the solution $u$ of the problem (1)-(2), the following estimates hold:

$$\|u\|_\infty \le C_0 \tag{3}$$

and

$$\|u'\|_\infty \le C \tag{4}$$

where

$$C_0 = (|A| + \alpha^{-1}\|q\|_\infty)e^{\alpha^{-1}\bar{G}T}$$

$$C = \|q\|_\infty + C_0(p^* + \bar{G}T)$$

and

$$\bar{G} = \max_{\bar{I}\times\bar{I}}|G(t,s)|.$$

Proof. Applying the mean value theorem to functions in Equation (1), we get

$$f(t,u) = f(t,0) + \frac{\partial f(t,\bar{u})}{\partial u}, \qquad \bar{u} = \gamma u, \qquad 0<\gamma<1,$$

$$K\big(t,s,u(s)\big) = K(t,s,0) + \frac{\partial}{\partial u}K(t,s,\dot{u}), \qquad \dot{u} = \theta u, \qquad 0<\theta<1,$$

$$u'(t) + f(t,0) + \frac{\partial f(t,\bar{u})}{\partial u}u + \int_0^t \frac{\partial}{\partial u}K(t,s,\dot{u})u(t)ds$$

$$+\int_0^t K(t,s,0)ds = 0,$$

$$u'(t) + \frac{\partial f(t,\bar{u})}{\partial u} + \int_0^t \frac{\partial}{\partial u}K(t,s,\dot{u})u(t)ds = -f(t,0)$$

$$-\int_0^t K(t,s,0)ds\,.$$

Then, we can write Equation (1) in the form

$$u'(t) + p(t)u(t) + \int_0^t G(t,s)u(s)ds = q(t),\ \ t\epsilon I, \tag{5}$$

where

$$p(t) = \frac{\partial}{\partial u}f(t,\bar{u}),$$

$$G(t,s) = \frac{\partial}{\partial u}K(t,s,\dot{u})$$

and

$$q(t) = -f(t,0) - \int_0^t K\big(t,s,0\big)ds.$$

From (5), we have

$$u(t) = u(0)e^{-\int_0^t p(\eta)d\eta}$$

$$+\int_0^t\left[q(s) + \int_0^s G(s,\xi)u(\xi)d\xi\right]e^{-\int_s^t p(\eta)d\eta}ds.$$

From here, we take

$$|u(t)| \le |A|e^{-\alpha t}$$

$$+\int_0^t\left[q(s) + \int_0^s |G(s,\xi)||u(\xi)|d\xi\right]e^{-\alpha(t-s)}ds$$

$$\le |A|e^{-\alpha t} + \alpha^{-1}\|q\|_\infty(1-e^{-\alpha t})$$

$$+\alpha^{-1}\bar{G}(1-e^{-\alpha t})\int_0^t|u(\xi)|\,d\xi. \tag{6}$$

Applying Gronwall's inequality to inequality (6), we obtain

$$|u(t)| \leq (|A| + \alpha^{-1}\|q\|_{\infty})e^{\alpha^{-1}\bar{G}T},$$

which leads to (3).

Now, we prove inequality (4). From Equation (5), we take

$$|u'(t)| \leq |p(t)||u(t)| + |q(t)|$$

$$+ \int_0^t |G(t,s)||u(s)|ds$$

$$\leq p^* C_0 + \|q\|_{\infty} + \bar{G}C_0 \int_0^t ds$$

$$\leq \|q\|_{\infty} + C_0(p^* + \bar{G}T),$$

which immediately leads to (4).

## 3. Discretization and mesh

$$\varpi_\tau = \{t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N\}$$

is a non-uniform mesh to a set of discrete points. $t_i$ points are called node points and $\tau_i = t_i - t_{i-1}$ is a meshsize. Before describing our numerical method, we define some notation for the mesh functions. For any mesh function $v(x)$ defined on $\varpi_\tau$, we use

$$v_i = v(t_i), \quad v_{\bar{t},i} = \frac{v_i - v_{i-1}}{\tau},$$

$$\|v\|_{\infty} = \|v\|_{\infty,\varpi_\tau} = \max_{0\leq i\leq N}|v_i|.$$

To generate the difference method, we begin the following integral identity

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} Lu dt = 0,$$

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt + \tau_i^{-1} \int_{t_{i-1}}^{t_i} f(t,u)\varphi_i(t)dt$$

$$+\tau_i^{-1} \int_{t_{i-1}}^{t_i} \int_0^t K\big(t,s,u(s)\big)\varphi_i(t)dsdt = 0. \tag{7}$$

Now, we separately evaluate each term of Equation (7). For $\varphi_i(t) = 1$, we use interpolating quadrature rules in Amiraliyev & Mamedov (1995) for each term. For the first term on the right side of Equation (7), we get

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} u'(t)\varphi_i(t)dt = u_{\bar{t},i} + R_i^{(*)},$$

where the truncation error is

$$R_i^{(*)} = -\int_{t_{i-1}}^{t_i} dt\varphi'_i(t) \int_{t_{i-1}}^{t_i} \frac{du(\xi)}{d\xi} T_0(t-\xi)d\xi = 0.$$

For the second term on the right side of Equation (7), we have

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} f(t,u)dt = f(t_i,u_i) + R_i^{(1)},$$

where

$$R_i^{(1)} = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \frac{d}{d\xi} f\big(\xi, u(\xi)\big)d\xi. \tag{8}$$

For the third term on the right side of Equation (7), we obtain

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} \int_0^t K\big(t,s,u(s)\big)dsdt$$

$$= \tau_i^{-1} \int_{t_{i-1}}^{t_i} dt \int_0^{t_i} K\big(t_i,s,u(s)\big)ds$$

$$= \int_0^{t_i} K\big(t_i,s,u(s)\big)ds + R_i^{(2)},$$

where

$$R_i^{(2)} = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \frac{d}{d\xi}\left(\int_0^t K(\xi,s,u(s)ds)\right) d\xi. \tag{9}$$

Using the right side rectangle rule for integral term involving kernel function, we have

$$\int_0^{t_i} K\big(t_i,s,u(s)\big)ds = \sum_{j=1}^{i} \tau_j K_{i,j}\left(t_i,t_j,u_j\right) + R_i^{(3)},$$

where

$$R_i^{(3)} = \sum_{j=1}^{i} \int_{t_{i-1}}^{t_i} \left(\xi - t_{j-1}\right) \frac{d}{d\xi} K\big(t_i,\xi,u(\xi)\big)d\xi. \tag{10}$$

Then, we can write the following difference problem for approximation of VIDE:

$$u_{\bar{t},i} + f(t_i,u_i) + \sum_{j=1}^{i} \tau_j K_{i,j}\left(t_i,t_j,u_j\right) + R_i = 0, \tag{11}$$

$$i = 1,2,\ldots,N,$$

$$u(0) = A, \tag{12}$$

where

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}. \tag{13}$$

Thus, by neglecting $R_i$, we obtain the difference scheme in the following form:

$$y_{\bar{t},i} + f(t_i, y_i) + \sum_{j=1}^{i} \tau_j K_{i,j}\left(t_i, t_j, y_j\right) = 0, \tag{14}$$

$i = 1,2, \dots, , N,$

$$y_0 = A. \tag{15}$$

## 4. Error estimates

For the investigation of the uniform convergence of this method, let $u_i$ be the solution of the problem (1)-(2) and let $y_i$ be the solution of the problem (14)-(15). Error function $z_i = y_i - u_i \;\; i = 0,1, \dots, N$ is the solution of the following discrete problem:

$$lz_i = z_{\bar{t},i} + \left[f\left(t_i, y_i\right) - f(t_i, u_i)\right]$$

$$+\sum_{j=1}^{i} \tau_j \left[K_{i,j}\left(t_i, t_j, y_j\right) - K_{i,j}\left(t_i, t_j, u_j\right)\right] = R_i, \tag{16}$$

$$z_0 = 0, \tag{17}$$

where $R_i$ is given by Equation (13).

Lemma 2. Let $z_i$ be the solution of (16)-(17). Then, the estimate

$$\|z\|_{\infty,\varpi_N} \leq C\|R\|_{\infty,\omega_N} \tag{18}$$

holds.

Proof. Using the mean value theorem for the functions in Equation (16), we take

$$z_{\bar{t},i} + p_i z_i + \sum_{j=1}^{i} \tau_j \frac{\partial \widetilde{K}}{\partial u} z_j = R_i,$$

where

$$p_i = \frac{\partial f}{\partial u}(t_i, \overline{\gamma_i}), \qquad \overline{\gamma_i} = \gamma \overline{y_i}, \qquad 0 < \gamma < 1.$$

$y_i$ are intermediate points called for by the mean value theorem. From here, we have

$$|z_i| \leq \alpha^{-1}\|R\|_{\infty,\omega_N} + \alpha^{-1}\widetilde{K}\sum_{j=1}^{i}|z_j|, \tag{19}$$

$i = 1,2, \dots, N.$

Applying the difference analogue of Gronwall's inequality to (19), we obtain

$$|z_i| \leq \alpha^{-1} e^{\alpha^{-1}\widetilde{K} t_i}\|R\|_{\infty,\omega_N}, \qquad i = 1,2, \dots, N,$$

where

$$\widetilde{K} = \left|\frac{\partial \widetilde{K}}{\partial u}\right|,$$

which implies the validity of (18).

Lemma 3. Assume that the mesh function $g_i$ is the solution of initial value problem

$$l_h g_i \coloneqq g_{\bar{t},i} + a_i g_i = F_i, \;\; i = 1,2 \dots, N, \tag{20}$$

$$g_0 = A, \tag{21}$$

with $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ is a nondecreasing function. Then for the solution of (20)-(21), the following inequality holds:

$$|g_i| \leq |A| + \alpha^{-1}\mathcal{F}_i, \;\; i = 1,2 \dots, N. \tag{22}$$

Proof. First, we note that for the difference operator $l_h g_i$ the maximum principle holds in the form: If any mesh function $l_h g_i \geq 0$, $i = 1,2 \dots, N$ and $g_0 \geq 0$, then $g_i \geq 0$, $i = 1,2 \dots, N$. Since $\mathcal{F}_i$ is a nondecreasing

$$\mathcal{F}_{\bar{t},i} = \frac{\mathcal{F}_i - \mathcal{F}_{i-1}}{\tau} \geq 0,$$

the barrier functions can be written in the following form:

$$\psi_i^{\pm} = \pm g_i + |A| + \alpha^{-1}\mathcal{F}_i.$$

It follows that

$$\psi_0^{\pm} = \pm A + |A| + \alpha^{-1}\mathcal{F}_i \geq 0,$$

$$l_h \psi_i^{\pm} = \pm \mathcal{F}_i + a_i|A| + \alpha^{-1}\mathcal{F}_i \geq F_i + \mathcal{F}_i \geq 0.$$

Thereby, according to the maximum principle, $\psi_i^{\pm} \geq 0$, which proves (22).

Lemma 4. Under the condition of Lemma 1, for the remainder term $R_i$ of scheme (14)-(15), the following estimate satisfies

$$\|R\|_{\infty,\omega_N} \leq C\tau. \tag{23}$$

Proof. Now, we separately evaluate the remainder terms in Equation (13). First, we prove Equation (8). From the relation Equation (8) and taking into account Lemma 1, we obtain

$$|R_i^{(1)}| \leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left|\frac{d}{d\xi} f\left(\xi, u(\xi)\right)\right| d\xi$$

$$\leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left[\left|\frac{\partial}{\partial \xi} f\left(\xi, u(\xi)\right)\right|\right.$$

$$\left. + \left|\frac{\partial}{\partial u} f\left(\xi, u(\xi)\right)\right| |u'(\xi)|\right] d\xi$$

$$\leq C\tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1})|1 + u'(\xi)| d\xi$$

$$\leq C\left\{\tau_i + \int_{t_{i-1}}^{t_i} |u'(\xi)| d\xi\right\},$$

$$\leq C\tau_{i,} \quad i = 1,2,\dots,N. \tag{24}$$

For Equation (9), we have

$$\left|R_i^{(2)}\right| \leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} \left|(\xi - t_{i-1}) \frac{d}{d\xi}\left(\int_0^t K(\xi, s, u(s)) ds\right)\right| d\xi$$

$$\leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left[\left|K(\xi, t, u(t))\right|\right.$$

$$\left. + \int_0^t \left|\frac{\partial}{\partial \xi} K(\xi, s, u(s)) ds\right|\right] d\xi$$

$$\leq C\tau_i, \; i = 1,2,\dots,N. \tag{25}$$

Next, from Equation (10), we get

$$\left|R_i^{(3)}\right| = \sum_{j=1}^{i} \int_{t_{i-1}}^{t_i} (\xi - t_{j-1}) \left|\frac{d}{d\xi} K(t_i, \xi, u(\xi))\right| d\xi$$

$$\leq \sum_{j=1}^{i} \int_{t_{i-1}}^{t_i} (\xi - t_{j-1}) \left[\left|\frac{\partial}{\partial \xi} K(t_i, \xi, u(\xi))\right|\right.$$

$$\left. + \left|\frac{\partial}{\partial u} K(t_i, \xi, u(\xi))\right| |u'(\xi)|\right] d\xi$$

$$\leq C\tau_i, \; i = 1,2,\dots,N. \tag{26}$$

Finally, substituting the inequalities (24), (25) and (26) in Equation (13), we obtain inequality (18).

From the two previous lemmas, we immediately obtain the main result of the following paper.

Theorem 1. Let $u(x)$ be the solution of the problem (1)-(2) and let $y$ be solution of the approximate problem (14)-(15). Then the following estimate satisfies

$$\|y - u\|_{\infty,\varpi_\tau} \leq C\tau_i. \tag{27}$$

Proof. Let $z_i = y_i - u_i \;\; i = 0,1,\dots,N$. Then, from the discrete problem (16)-(17) we can rewrite it in the form

$$z_{\bar{t},i} + p_i z_i + \textstyle\sum_{j=1}^{i} \tau_j \frac{\partial \tilde{K}}{\partial u} z_j = R_i, \tag{28}$$

$$i = 0,1,\dots,N,$$

$$z_0 = 0. \tag{29}$$

After applying Lemma 3 to the discrete problem (28)-(29), we get

$$|z_i| \leq \alpha^{-1}\|R\|_{\infty,\omega_N} + \alpha^{-1}\tilde{K} \sum_{j=1}^{i} |z_j|, \; i = 0,1,\dots,N.$$

From here, by using the difference analogue of Gronwall's inequality, we obtain the following inequality:

$$|z_i| \leq \alpha^{-1} e^{\alpha^{-1}\tilde{K}T} \|R\|_{\infty,\omega_N}$$

$$\leq C\|R\|_{\infty,\omega_N}, \quad i = 1,2,\dots,N. \tag{30}$$

Further, from Lemma 4, we see that

$$\|R\|_{\infty,\omega_N} \leq C\tau. \tag{31}$$

From the inequalities (30) and (31), we arrive at (27).

## 5. Algorithm and numerical results

In this section, we present the numerical results for difference scheme (14)-(15). Because of the nonlinear term, we use the quasilinearization technique. Applying this technique to difference scheme (14)-(15), we obtain

$$y_{\bar{t},i}^{(n)} + f\left(t_i, y_i^{(n-1)}\right) + \frac{\partial}{\partial y} f\left(t_i, y_i^{(n-1)}\right)\left(y_i^{(n)} - y_i^{(n-1)}\right)$$

$$+ \sum_{j=1}^{i} \tau_j \left[K_{ij}\left(t_i, t_j, y_j^{(n-1)}\right)\right.$$

$$\left. + \frac{\partial}{\partial y} K_{ij}\left(t_i, t_j, y_j^{(n-1)}\right)\left(\left(y_j^{(n)} - y_j^{(n-1)}\right)\right)\right] = 0,$$

$$y_0^{(n)} = A, \qquad i = 1,2,\dots,N.$$

If the elimination method is taken into consideration, we have

$$y_i^{(n)} = \frac{A_i y_{i-1}^{(n)} + B_i y_i^{(n-1)} - C_i - \tilde{K}_\iota}{\frac{1}{\tau_i} + \frac{\partial}{\partial y} f(t_i, y_i^{(n-1)}) + \tau_i \frac{\partial}{\partial y} K_{ij}(t_i, t_i, y_i^{(n-1)})},$$

$$y_0^{(n)} = A,$$

where

$$A_i = \frac{1}{\tau_i},$$

$$B_i = \frac{\partial}{\partial y} f\left(t_i, y_i^{(n-1)}\right) + \tau_i \frac{\partial}{\partial y} K_{ij}\left(t_i, t_i, y_i^{(n-1)}\right),$$

$$C_i = f\left(t_i, y_i^{(n-1)}\right) + \tau_i K_{ij}\left(t_i, t_i, y_i^{(n-1)}\right),$$

$$\tilde{K}_\iota = \sum_{j=1}^{i-1} \tau_j K_{ij}\left(t_i, t_j, y_j^{(n-1)}\right).$$

The stopping criterion is

$$\max_i \left| y_i^{(n)} - y_i^{(n-1)} \right| \le 10^{-3}$$

and $y_i^{(0)}$ is the initial process. Then, the difference scheme is tested on the following examples. Exact errors are determined as

$$e^N = |y_i - u_i|,$$

where $u_i$ is the exact solution and $y_i$ is the approximate solution.

Example 1. We consider the following nonlinear Volterra integro-differential equation:

$$u'(t) + \frac{u^2(t)}{2} + u(t) - \frac{1}{2} + \int_0^t u^2(s)ds = 0,$$

with $u(0) = 1$. The exact solution of the equation is $u(t) = e^{-t}$. The computational results are presented in Table 1. In addition, the obtained results from Table 1 are shown in Figure 1.

**Table 1.** Error approximations for Example 1.

| $t_i$ | $y_i$ | $u_i$ | $y_i - u_i$ |
|---|---|---|---|
| 0.000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 0.125 | 0.882496902 | 0.879922493 | 0.002574410 |
| 0.250 | 0.778800783 | 0.778748879 | 0.000051904 |
| 0.375 | 0.687289278 | 0.694229264 | 0.006939985 |
| 0.500 | 0.606530659 | 0.623990032 | 0.017459372 |
| 0.625 | 0.535261428 | 0.565788861 | 0.030527432 |
| 0.750 | 0.472366552 | 0.517629831 | 0.045263278 |
| 0.875 | 0.416862019 | 0.477799240 | 0.060937220 |
| 1.000 | 0.367879441 | 0.444850916 | 0.076971475 |

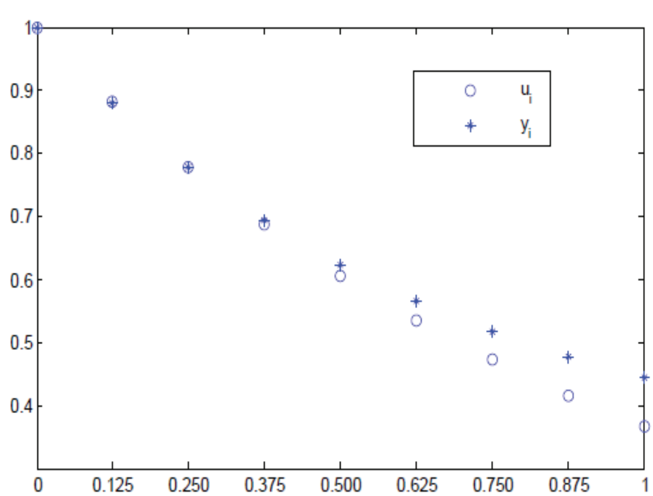


**Fig. 1.** Comparison of $u_i$ and $y_i$ for Example 1.

Example 2. We take into account another nonlinear Volterra integro-differential equation:

$$u'(t) + e^u + u(t) + \int_0^t \tanh(u(s))\, ds = 0, \; u(0) = 1.$$

The exact solution of this equation is unknown. Thus, we use the double-mesh principle to estimate the errors and compute the experimental rates of convergence in the computed solution. That is, we compare the computed solution with the solution on a mesh that is twice as fine (Amiraliyev & Duru, 2005; Cakir & Amiraliyev, 2007). The error estimates are denoted by

$$e^N = \max_i \left| y_i^N - y_i^{2N} \right|.$$

The convergence rates are calculated as follows:

$$p^N = \frac{\ln\left( e^N / e^{2N} \right)}{\ln 2}.$$

Error approximations and convergence rates for Example (2) are given in Tables 2-3.

Error approximations which are obtained from Tables 2-3 are shown in Figures 2-3 for some values of $N$.

**Table 2.** Error approximations and convergence rates for different values of $N$.

| | $N = 8$ | $N = 16$ | $N = 32$ | $N = 64$ |
|---|---|---|---|---|
| $r_0$ | 0.02970507 | 0.01615199 | 0.00845788 | 0.00433473 |
| $r_1$ | 0.01608326 | 0.00845792 | 0.00433440 | 0.00219515 |
| $p$ | 0.88514899 | 0.93333706 | 0.96446294 | 0.98162238 |

**Table 3.** Error approximations and convergence rates for different values of $N$.

| | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|---|---|---|---|---|
| $r_0$ | 0.00219515 | 0.00110470 | 0.00055415 | 0.00027753 |
| $r_1$ | 0.00110470 | 0.00055415 | 0.00027753 | 0.00013888 |
| $p$ | 0.99065946 | 0.99530439 | 0.99763936 | 0.99881619 |

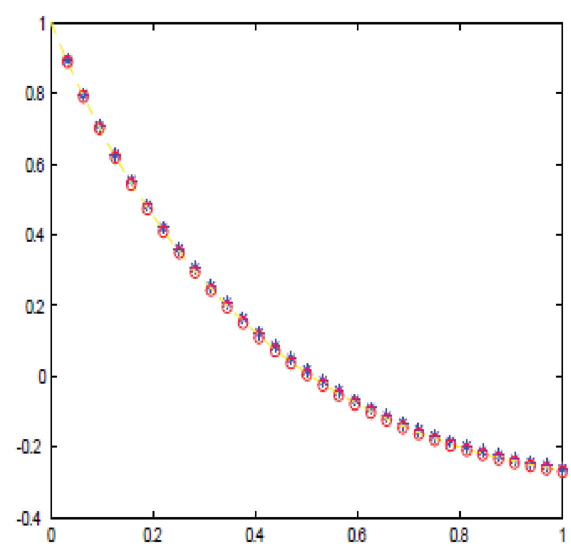


**Fig. 2.** Error approximations for $N = 32$.

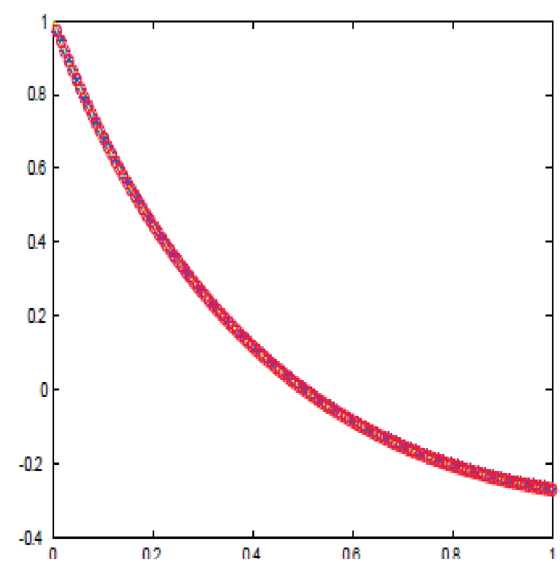


**Fig. 3.** Error approximations for $N = 128$.

Example 3. We tackle the following initial value problem:

$$u'(t) + \cosh(u(t)) + \int_0^t \sin\big(u(s)\big)\, ds = 0,\ u(0) = 1.$$

The exact solution of this equation is unknown. Using the double mesh principle, we find experimental rates of convergence. The computational results are shown in Table 4.

**Table 4.** Error approximations and convergence rates for different values of $N$.

| | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|---|---|---|---|---|
| $r_0$ | 0.00038189 | 0.00019341 | 0.00009733 | 0.00004882 |
| $r_1$ | 0.00019342 | 0.00009733 | 0.00004882 | 0.00002445 |
| $p$ | 0.98140912 | 0.99074969 | 0.99534816 | 0.99765610 |

## 6. Discussion and Conclusion

In this study, we have presented a new and effective numerical method for the quasilinear Volterra integro-differential equations. The stability of the solution for continuous problem was examined by using the maximum principle, and the difference schemes were taken according to the discrete maximum norm. In the numerical algorithm, the quasilinear technique was used because of nonlinear terms. The theoretical results were tested on samples. It is shown from the tables that the numerical results confirm the theoretical results. Numerical investigations can be carried out for the various types such as delay and singularly perturbed equations with different physical properties.

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