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A completely monotonic function involving the gamma and trigamma functions

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Abstract

In this paper the author provides necessary and sufficient conditions on a for the function

$$\frac{1}{2}\ln(2\pi)-x+(x-\frac{1}{2})\ln x-\ln\Gamma(x)+\frac{1}{12}\psi'(x+a)$$

and its negative to be completely monotonic on $(0,\infty)$, where $a\geq 0$ is a real number, $\Gamma(x)$ is the classical gamma function, and $\psi(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function. As applications, some known results and new inequalities are derived.

Keywords: Completely monotonic function; gamma function; inequality; logarithmically completely monotonic function; trigamma function.

1. Introduction

It is well known that the classical Euler gamma function is defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

for $x>0$, that the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function and denoted by

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

for $x>0$, that the derivatives $\psi'(x)$ and $\psi''(x)$ for $x>0$ are respectively called the trigamma and tetragamma functions, and that the derivatives $\psi^{(i)}(x)$ for $i\in\mathrm{N}$ and $x>0$ are called polygamma functions.

We recall from Mitrinović *et al.* (1993) and Widder (1946) that a function $f(x)$ is said to be completely monotonic on an interval *I*, if it has derivatives of all orders on *I* and satisfies

$$0\leq(-1)^n f^{(n)}(x)<\infty \tag{1}$$

for $x\in I$ and all integers $n\geq 0$. If $f(x)$ is non-constant, then the inequality (1) is strict (Dubourdieu, 1939). The class of completely monotonic functions may be characterized by the celebrated Bernstein-Widder Theorem (Widder, 1946), which reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0\leq x<\infty$ is that

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t),$$

where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0\leq x<\infty$.

For $x\in(0,\infty)$ and $a\geq 0$, let

$$F_a(x)=\ln\Gamma(x)-(x-\frac{1}{2})\ln x-\frac{1}{12}\psi'(x+a).$$

Merkle (1998) proved that the function $F_0(x)$ is strictly concave and the function $F_a(x)$ for $a\geq\frac{1}{2}$ is strictly convex on $(0,\infty)$. This was surveyed and reviewed in Qi (2010).

In recent years, some new results on the complete monotonicity of functions involving the gamma and polygamma functions have been obtained (Guo & Qi, 2012a; Guo & Qi, 2012b; Guo & Qi, 2013a; Guo & Qi, 2013b; Guo *et al.,* 2012; Li *et al.,* 2013; Lü *et al.,* 2011; Qi & Berg, 2013; Qi *et al.,* 2013a; Qi *et al.,* 2013b; Qi *et al.,* 2012; Srivastava *et al.,* 2012; Zhao *et al.,* 2011; Zhao *et al.,* 2012b), for example.

The aims of this paper are to generalize the convexity of the function $F_a(x)$ and to derive known results and some new inequalities.

2. Complete monotonicity

The first aim of this paper is to generalize the convexity of $F_a(x)$ to complete monotonicity, which may be stated as Theorem 1 below.

Theorem 1 *For* $x \in (0,\infty)$ *and* $a \geq 0$, *let*

$$f_a(x) = \frac{1}{2}\ln(2\pi) - x + (x - \frac{1}{2})\ln x - \ln\Gamma(x) + \frac{1}{12}\psi'(x+a).$$

Then the functions $f_0(x)$ and $-f_a(x)$ for $a \geq \frac{1}{2}$ are completely monotonic on $(0,\infty)$.

Proof. Using recursion formulas $\Gamma(x+1) = x\Gamma(x)$ and

$$\psi'(x+1) - \psi'(x) = -\frac{1}{x^2}$$

for $x > 0$, an easy calculation yields

$$\begin{aligned} f_a(x) - f_a(x+1) &= 1 \\ &+ (x + \frac{1}{2})\ln(\frac{x}{x+1}) \\ &+ \frac{1}{12}[\psi'(x+a) - \psi'(x+a+1)] \\ &= 1 + (x+\frac{1}{2})\ln(\frac{x}{x+1}) + \frac{1}{12(x+a)^2} \end{aligned}$$

and

$$[f_a(x) - f_a(x+1)]' = \frac{1}{2(x+1)} + \frac{1}{2x} - \frac{1}{6(a+x)^3} + \ln(\frac{x}{x+1}).$$

Utilizing formulas

$$\Gamma(z) = k^z \int_0^\infty t^{z-1} e^{-kt} dt$$

and

$$\ln\frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} du$$

for $Re(z) > 0$, $Re(k) > 0$, $a > 0$ and $b > 0$, (Abramowitz & Stegun, 1972), gives

$$\begin{aligned} &[f_a(x) - f_a(x+1)]' \\ &= \int_0^\infty [\frac{1}{2}e^{-t} + \frac{1}{2} - \frac{1}{12}t^2 e^{-at} \\ &+ \frac{e^{-t}-1}{t}]e^{-xt} dt \\ &\triangleq \int_0^\infty \phi_a(t) e^{-xt} dt. \end{aligned} \tag{2}$$

It is easy to see that

$$\begin{aligned} \phi_0(t) &= -\frac{(t^3 - 6t + 12)e^t - 6(t+2)}{12te^t} \\ &= -\frac{1}{12e^t}\sum_{i=4}^{\infty}\frac{(i-3)(i^2-4)}{i!}t^{i-1} < 0 \end{aligned}$$

and

$$\begin{aligned} \phi_{1/2}(t) &= \frac{6(t-2)e^t - t^3 e^{t/2} + 6(t+2)}{12te^t} \\ &= \frac{1}{12e^t}\sum_{i=5}^{\infty}\frac{(i-2)(3\cdot 2^{i-2} - i^2 + i)}{i!\cdot 2^{i-3}}t^{i-1} > 0 \end{aligned}$$

on $(0,\infty)$, where the inequality $3\cdot 2^{i-2} - i^2 + i > 0$ for $i \geq 5$ may be verified by induction. As a result, the function

$$[f_0(x+1) - f_0(x)]' = f_0'(x+1) - f_0'(x)$$

and

$$\begin{aligned} &[f_{1/2}(x) - f_{1/2}(x+1)]' \\ &= f_{1/2}'(x) - f_{1/2}'(x+1) \end{aligned}$$

are completely monotonic on $(0,\infty)$, that is,

$$\begin{aligned} &(-1)^k[f_0'(x+1) - f_0'(x)]^{(k)} \\ &= (-1)^k f_0^{(k+1)}(x+1) - (-1)^k f_0^{(k+1)}(x) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} &(-1)^k[f_{1/2}'(x) - f_{1/2}'(x+1)]^{(k)} \\ &= (-1)^k f_{1/2}^{(k+1)}(x) - (-1)^k f_{1/2}^{(k+1)}(x+1) \\ &\geq 0 \end{aligned}$$

for $k \geq 0$. By induction, we have

$$\begin{aligned} &(-1)^k f_0^{(k+1)}(x) \leq (-1)^k f_0^{(k+1)}(x+1) \\ &\leq (-1)^k f_0^{(k+1)}(x+2) \\ &\leq (-1)^k f_0^{(k+1)}(x+3) \leq \cdots \\ &\leq (-1)^k \lim_{m\to\infty} f_0^{(k+1)}(x+m) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(-1)^k f_{1/2}^{(k+1)}(x) \geq (-1)^k f_{1/2}^{(k+1)}(x+1) \\ &\geq (-1)^k f_{1/2}^{(k+1)}(x+2) \\ &\geq (-1)^k f_{1/2}^{(k+1)}(x+3) \geq \cdots \\ &\geq (-1)^k \lim_{m\to\infty} f_{1/2}^{(k+1)}(x+m) \end{aligned} \tag{4}$$

for $k \geq 0$.

It is not difficult to obtain

$$f_{a'}(x) = \frac{\psi''(a+x)}{12} - \psi(x) + \ln x - \frac{1}{2x}$$

and

$$f_a^{(i)}(x) = \frac{\psi^{(i+1)}(a+x)}{12} - \psi^{(i-1)}(x) + \frac{(-1)^i(i-2)!}{x^{i-1}} + \frac{(-1)^i(i-1)!}{2x^i}, \quad i \geq 2.$$

In the light of the double inequalities

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x}$$

and

$$\frac{(i-1)!}{x^i} + \frac{i!}{2x^{i+1}} < |\psi^{(i)}(x)| < \frac{(i-1)!}{x^i} + \frac{i!}{x^{i+1}} \tag{5}$$

for $x > 0$ and $i \in \mathrm{N}$, (Guo & Qi, 2011b; Guo & Qi, 2010c; Qi & Guo, 2010a, and Qi *et al.* (2010), we immediately derive

$$\lim_{x\to\infty} f_{a'}(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_a^{(i)}(x) = 0$$

for $i \geq 2$ and $a \geq 0$. Combining this with (3) and (4), we deduce

$$(-1)^k f_0^{(k+1)}(x) \leq 0 \text{ and } (-1)^k f_{1/2}^{(k+1)}(x) \geq 0 \tag{6}$$

for $k \geq 0$ on $(0, \infty)$.

From the formula

$$\ln \Gamma(z) = (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + 2\int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t} - 1} dt \tag{7}$$

for $Re(z) > 0$, (Abramowitz & Stegun, 1972), and the double inequality (5) for $i = 1$, we easily obtain

$$\lim_{x\to\infty} f_a(x) = 0 \tag{8}$$

for $a \geq 0$. Inequalities in (6) imply that the functions $-f_0(x)$ and $f_{1/2}(x)$ are increasing on $(0, \infty)$. Hence, we have

$$f_0(x) > 0 \quad \text{and} \quad f_{1/2}(x) < 0 \tag{9}$$

on $(0, \infty)$.

From (6) and (9), we conclude that the functions $f_0(x)$ and $-f_{1/2}(x)$ are completely monotonic on $(0, \infty)$.

It is clear that

$$-f_a(x) = -f_{1/2}(x) + \frac{1}{12}[\psi'(x + \frac{1}{2}) - \psi'(x + a)].$$

From the facts that the trigamma function

$$\psi'(x) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-xt} dt$$

for $x > 0$, (Abramowitz & Stegun, 1972), is completely monotonic on $(0, \infty)$, that the difference $f(x) - f(x + \alpha)$ for any given real number $\alpha > 0$ of any completely monotonic function $f(x)$ on $(0, \infty)$ is also completely monotonic on $(0, \infty)$, and that the sum of finitely many completely monotonic functions on an interval *I* is still completely monotonic on *I*, it readily follows that the function $-f_a(x)$ for $a > \frac{1}{2}$ is also completely monotonic on $(0, \infty)$. The proof of Theorem 1 is complete.

3. Necessary and sufficient conditions

The second aim of this paper is to answer a natural problem: Find the best constants $\alpha \geq 0$ and $\beta \leq \frac{1}{2}$ such that $f_\alpha(x)$ and $-f_\beta(x)$ are both completely monotonic on $(0, \infty)$.

Theorem 2. *The function* $f_\alpha(x)$ *is completely monotonic on* $(0, \infty)$ *if and only if* $\alpha = 0$, *and so is the function* $-f_\beta(x)$ *if and only if* $\beta \geq \frac{1}{2}$.

Proof. The first proof. The conclusion that the function $\phi_a(t)$ defined in (2) is positive or negative on $(0, \infty)$ is equivalent to

$$\begin{aligned} a &\gtreqless -\frac{1}{t} \ln[\frac{12}{t^2}(\frac{e^{-t}+1}{2} + \frac{e^{-t}-1}{t})] \\ &\triangleq -\varphi(t) = -\frac{1}{t} \ln \varphi_1(t), \quad t > 0. \end{aligned} \tag{10}$$

By the L'Hôspital rule, we have

$$\begin{aligned} \lim_{t\to 0^+} \varphi_1(t) &= 6 \lim_{t\to 0^+} \frac{t(e^{-t}+1) + 2(e^{-t}-1)}{t^3} \\ &= 2 \lim_{t\to 0^+} \frac{e^{-t}(e^t - t - 1)}{t^2} = 1 \end{aligned}$$

and $\lim_{t\to\infty} \varphi_1(t) = 0$. Hence, the function $\varphi(t)$ can be represented as

$$\begin{aligned} \varphi(t) &= \frac{\ln \varphi_1(t) - \ln \varphi_1(0)}{t} = \frac{1}{t} \int_0^t \frac{\varphi_1'(u)}{\varphi_1(u)} du \\ &= -\frac{1}{t} \int_0^t \frac{2(u-3)e^u + u^2 + 4u + 6}{u[(u-2)e^u + u + 2]} du \\ &\triangleq -\frac{1}{t} \int_0^t \varphi_2(u) du \end{aligned}$$

for $t > 0$. Since

$$-\{u^2[(u-2)e^u+u+2]^2\}\varphi_2'(u)$$
$$=2(u^2-6u+6)e^{2u}$$
$$+(u^4+8u^2-24)e^u+2(u^2+6u+6)$$
$$=\sum_{i=8}^{\infty}\frac{2^{i-1}(i^2-13i+24)+i^4}{i!}u^i$$
$$+\sum_{i=8}^{\infty}\frac{-6i^3+19i^2-14i-24}{i!}u^i$$
$$>0$$

for $u>0$, where

$$2^{i-1}(i^2-13i+24)+i^4-6i^3$$
$$+19i^2-14i-24$$
$$=(1+1)^{i-1}(i^2-13i+24)+i^4$$
$$-6i^3+19i^2-14i-24$$
$$>i(i^2-13i+24)+i^4-6i^3$$
$$+19i^2-14i-24$$
$$=(i-5)i^3+6i^2+2(5i-12)>0$$

for $i\ge 8$, the function $\varphi_2(u)$ is strictly decreasing on $(0,\infty)$, and, by Qi *et al.* (1999) and Qi & Zhang (1999), the arithmetic mean

$$-\varphi(t)=\frac{1}{t}\int_0^t\varphi_2(u)du$$

is strictly decreasing, and $\varphi(x)$ is strictly increasing, on $(0,\infty)$. From the L'Hôspital rule and limits

$$\lim_{u\to 0^+}\varphi_2(u)=\frac{1}{2}\quad\text{and}\quad\lim_{u\to\infty}\varphi_2(u)=0,$$

we obtain

$$\lim_{t\to 0^+}\varphi(t)=-\frac{1}{2}\quad\text{and}\quad\lim_{t\to\infty}\varphi(t)=0.$$

As a result, from (10), it follows that

1. when $a=0$, the function $[f_a(x+1)-f_a(x)]'$ is completely monotonic;
2. when $a\ge\frac{1}{2}$, the function $[f_a(x)-f_a(x+1)]'$ is completely monotonic.

Along with the corresponding argument as in the proof of Theorem 1, we obtain that the sufficient condition for $f_a(x)$ or $-f_a(x)$ to be completely monotonic on $(0,\infty)$ is $a=0$ or $a\ge\frac{1}{2}$ respectively.

Conversely, if $-f_a(x)$ is completely monotonic on $(0,\infty)$, then $f_a(x)$ is increasing and negative on $(0,\infty)$, so $x^2f_a(x)<0$ on $(0,\infty)$. From the double inequality

$$\frac{1}{2x^2}-\frac{1}{6x^3}<\frac{1}{x}-\psi'(x+1)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}$$

on $(0,\infty)$, (Qi *et al.,* 2005), it is easy to see that

$$\lim_{x\to\infty}\{x^2[\psi'(x)-\frac{1}{x}]\}=\frac{1}{2}. \tag{11}$$

Using the asymptotic formula

$$\ln\Gamma(z)\sim\ (z-\frac{1}{2})\ln z-z+\frac{1}{2}\ln(2\pi)$$
$$+\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}-\cdots$$

as $z\to\infty$ in $|\arg z|<\pi$, see [Abramowitz & Stegun, 1972], gives

$$\lim_{x\to\infty}\{x^2[\frac{1}{2}\ln(2\pi)-x+(x-\frac{1}{2})\ln x$$
$$-\ln\Gamma(x)+\frac{1}{12(x+a)}]\}$$
$$=\lim_{x\to\infty}\{x^2[\frac{1}{12(x+a)}-\frac{1}{12x}+O(\frac{1}{x^2})]\}$$
$$=-\frac{a}{12}.$$

In virtue of (11) and the above limit, we obtain

$$x^2f_a(x)=x^2[\frac{1}{2}\ln(2\pi)-x+(x-\frac{1}{2})\ln x$$
$$-\ln\Gamma(x)+\frac{1}{12(x+a)}]$$
$$+\frac{x^2}{12}[\psi'(x+a)-\frac{1}{x+a}]$$
$$\to-\frac{a}{12}+\frac{1}{12}\cdot\frac{1}{2}$$

as x tends to ∞. So the necessary condition for $-f_a(x)$ to be completely monotonic on $(0,\infty)$ is $a\ge\frac{1}{2}$.

If $f_a(x)$ for $a>0$ is completely monotonic on $(0,\infty)$, then $f_a(x)$ should be decreasing and positive on $(0,\infty)$, but utilizing (7) leads to

$$\lim_{x\to 0^+}f_a(x)=\lim_{x\to 0^+}[\frac{1}{2}\ln(2\pi)$$
$$-x+(x-\frac{1}{2})\ln x$$
$$-\ln\Gamma(x)]+\frac{1}{12}\psi'(a)$$

$$= \frac{1}{12}\psi'(a) - 2\lim_{x\to 0^+}\int_0^\infty \frac{\arctan(t/x)}{e^{2\pi t}-1}dt$$
$$= \frac{1}{12}\psi'(a) - \pi\int_0^\infty \frac{1}{e^{2\pi t}-1}dt = -\infty$$

which leads to a contradiction. So the necessary condition for $f_a(x)$ to be completely monotonic on $(0,\infty)$ is $a=0$. The first proof of Theorem 2 is complete.

Proof. The second proof. The famous Binet's first formula of $\ln\Gamma(x)$ for $x>0$ is given by

$$\ln\Gamma(x) = \left(x-\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \theta(x),$$

where

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-xt}}{t}dt$$

for $x>0$ is called the remainder of Binet's first formula for the logarithm of the gamma function $\Gamma(x)$. (Magnus *et al.* 1966; Qi & Guo, 2010c). Combining this with the integral representation

$$\psi^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}}e^{-xt}dt$$

for $x>0$ and $k\in\mathrm{N}$, (Abramowitz & Stegun, 1972), yields

$$f_a(x) = \frac{1}{12}\psi'(x+a) - \theta(x)$$
$$= \int_0^\infty \left[\frac{te^{(1-a)t}}{12(e^t-1)} - \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right]e^{-xt}dt. \quad (12)$$

It is not difficult to see that the positivity and negativity are equivalent to

$$a \text{ £ } -\frac{1}{t}\ln\left[\frac{12(e^t-1)}{t^2e^t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\right] = -\varphi(t),$$

where $\varphi(t)$ is defined by (10). The rest proof is the same as in the first proof of Theorem 2. The second proof of Theorem 2 is complete.

4. Remarks

In this section, we list more results in the form of remarks.

Remark 1. From proofs of Theorem 1 and Theorem 2, we can abstract a general and much useful conclusion below.

Theorem 3 *A function $f(x)$ defined on an infinite interval I tending to ∞ is completely monotonic if and only if*

1. there exist positive numbers α_i such that the differences $(-1)^i[f(x)-f(x+\alpha_i)]^{(i)}$ are nonnegative for all integers $i\ge 0$ on I;
2. the limits $\lim_{x\to\infty}[(-1)^i f^{(i)}(x)] = a_i \ge 0$ exist for all integers $i\ge 0$.

This is essentially a generalization of Guo *et al.* (2006); Guo & Qi (2011a); Guo & Qi (2010b); Guo & Qi (2010c); Qi (2007); Qi & Guo (2009) and Qi *et al.* (2010) which was also implicitly applied in Guo *et al.* (2010) and Qi & Guo (2010a), for example.

Remark 2. *Because* $f_a(x) = \frac{1}{2}\ln(2\pi) - x - F_a(x)$ and $f_{a''}(x) = -F_{a''}(x)$ on $(0,\infty)$, the concavity of $F_0(x)$ and the convexity of $F_a(x)$ for $a\ge\frac{1}{2}$ obtained in Merkle (1998) can be concluded readily from the complete monotonicity of $f_\alpha(x)$ and $-f_\beta(x)$ on $(0,\infty)$ established in Theorems 1 and 2.

Remark 3. We also recall from Atanassov & Tsoukrovski (1988); Qi & Guo (2004); Qi *et al.* (2006); Qi *et al.* (2004) that a function $f(x)$ is said to be logarithmically completely monotonic on an interval I if it has derivatives of all orders on I and its logarithm $f(x)$ satisfies $0\le(-1)^k[\ln f(x)]^{(k)}<\infty$ for all integers $k\ge 1$ on I. It was proved once again in Berg (2004); Guo & Qi (2010a); Qi & Chen (2004); Qi & Guo (2004); Qi *et al.* (2006) that logarithmically completely monotonic functions on an interval I must be completely monotonic on I, but not conversely. For more information on the history and properties of logarithmically completely monotonic functions, please refer to Atanassov & Tsoukrovski (1988); Berg (2004); Guo & Qi (2010a); Qi (2010); Qi *et al.* (2010) and closely related references therein.

For $a\ge 0$ and $x>0$, let

$$g_a(x) = -\ln\Gamma(x) + \left(x-\frac{1}{2}\right)\ln x - x + \frac{1}{12}\psi'(x+a).$$

It is obvious that

$$f_a(x) = \frac{1}{2}\ln(2\pi) + g_a(x)$$

on $(0,\infty)$ for $a\ge 0$, with the limit (8). It is not difficult to see that Theorem 1 in Alzer (1993) may be reworded as follows: For $0<s<1$ the function $\exp[g_a(x+s)-g_a(x+1)]$ is logarithmically completely monotonic on $(0,\infty)$ if and only if $a\ge\frac{1}{2}$, and so is the function $\exp[g_a(x+1)-g_a(x+s)]$ if and only if $a=0$. This was reviewed in Qi (2010).

In virtue of complete monotonicity of $f_a(x)$ and

Remark 1, it follows that the difference

$$f_a(x+s)-f_a(x+t)=g_a(x+s)-g_a(x+t)$$

for $t>s$ and $a\geq 0$ is completely monotonic with respect to $x\in(-s,\infty)$ if and only if $a=0$, and so is its negative if and only if $\alpha\geq\frac{1}{2}$. Therefore, by the second item of Theorem 5 in Qi & Guo (2010b), it follows that the function $\exp[f_a(x+s)-f_a(x+t)]$ for $t>s$ and $a\geq 0$ is logarithmically completely monotonic with respect to x on $(-s,\infty)$ if and only if $a\geq\frac{1}{2}$, and so is the function $\exp[f_a(x+t)-f_a(x+s)]$ if and only if $a=0$. In other words, the function

$$\frac{\Gamma(x+s)}{\Gamma(x+t)}\cdot\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\exp[s-t+\frac{\psi'(x+t+\alpha)-\psi'(x+s+\alpha)}{12}] \tag{13}$$

for $s<t$ and $\alpha\geq 0$ is logarithmically completely monotonic with respect to $x\in(-s,\infty)$ if and only if $\alpha\geq\frac{1}{2}$, and so is the reciprocal of (13) if and only if $\alpha=0$.

The monotonicity of (13) and its reciprocal implies that the double inequality

$$\exp[t-s+\frac{\psi'(x+s+\beta)-\psi'(x+t+\beta)}{12}]\leq\frac{\Gamma(x+s)}{\Gamma(x+t)}\cdot\frac{(x+t)^{x+t-1/2}}{(x+s)^{x+s-1/2}}\leq\exp[t-s+\frac{\psi'(x+s+\alpha)-\psi'(x+t+\alpha)}{12}]$$

for $\alpha>\beta\geq 0$, $s<t$ and $x\in(-s,\infty)$ is valid if and only if $\beta=0$ and $\alpha\geq\frac{1}{2}$.

5. Conclusion

In conclusion, from complete monotonicity in Theorem 2, Theorem 1 and Corollary 1 in Alzer (1993) together with Theorem 3 and its corollary in Li *et al.* (2006) may be deduced and extended straightforwardly. This means that Theorem 2 is stronger not only than Alzer (1993) but also than Li *et al.* (2006).

Remark 4. As a consequence of Theorem 2, the following double inequality is easily obtained: for $x>0$, the double inequality

$$\sqrt{2\pi}\,x^{x-1/2}\exp[\frac{\psi'(x+\beta)}{12}-x]<\Gamma(x)<\sqrt{2\pi}\,x^{x-1/2}\exp[\frac{\psi'(x+\alpha)}{12}-x]$$

is valid if and only if $\alpha=0$ and $\beta\geq\frac{1}{2}$.

For more inequalities for bounding the gamma function $\Gamma(x)$, please refer to Guo & Qi (2010c); Guo *et al.* (2008); Zhao *et al.* (2012a) and closely related references therein.

Remark 5. *The equation (12) in the second proof of Theorem 2 tells us the integral representations of the completely monotonic functions* $f_0(x)$ *and* $-f_a(x)$ *for* $a\geq\frac{1}{2}$.

Remark 6. For the history, background, motivations, and recent developments of this topic, please refer to the survey and expository papers (Qi (2010); Qi (2014); Qi & Luo (2012); Qi & Luo (2013)) and plenty of references therein.

Remark 7. This paper is a revised version of the preprint (Qi, 2013).

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خلاصة

نقدم في هذا البحث شروط ضرورية و كافية على الثابت a حتى تكون الدالة

$$\frac{1}{2}\ln(2\pi)-x+(x-\frac{1}{2})\ln x-\ln\Gamma(x)+\frac{1}{12}\psi'(x+a)$$

ويكون سالبها أيضاً تام الرتابة على $(0,\infty)$ حيث أن $a\ge 0$ هو عدد حقيقي، $\Gamma(x)$ هي دالة غاما الكلاسيكية و $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ هي دالة داي غاما. و كتطبيق لما وجدناه، نستخرج بعض النتائج المعروفة، و بعض المتباينات الجديدة.