Change of nullity of a graph under two operations

Gohdar H. Mohiaddin[1,*], Khidir R. Sharaf [2]

[1] *Dept. of Mathematics Faculty of Education University of Zakho, Zakho, Iraq*

[2] *Dept. of Mathematics Faculty of Science University of Zakho, Zakho, Iraq*

[*] Corresponding author: gohdar.mohiaddin@uoz.edu.krd

Abstract

The nullity of a graph G is the multiplicity of zero as an eigenvalue of its adjacency matrix. An assignment of weights to the vertices of a graph, that satisfies a zero sum condition over the neighbors of each vertex, and uses maximum number of independent variables is denoted by a high zero sum weighting of the graph. This applicable tool is used to determine the nullity of the graph. Two types of graphs are defined, and the change of their nullities is studied, namely, the graph G+ab constructed from G by adding a new vertex ab which is joint to all neighbors of both vertices a and b of G, and G•ab which is obtained from G+ab by removing both vertices a and b.

Keywords: Adjacency matrix; graph theory; high zero sum weighting; nullity; vertex identification.

1. Introduction

The adjacency matrix A(G) of a labeled simple graph G with vertex set V, $V = \{v_1, v_2, \cdots, v_n\}$ is an n×n matrix in which $a_{ij} = 1$ if v_i and v_j are adjacent, and 0 if they are not. The neighborhood N(v) (or $N_G(v)$) of a vertex v in a graph G is the set of all vertices adjacent with v in G. A graph in which every two vertices are adjacent is a complete graph K_n. A bipartite graph is a graph whose vertices can be partitioned into two sets, such that each edge joins a vertex of the first set to a vertex of the second set. A complete bipartite graph $K_{m,n}$ is a bipartite graph where each vertex of the first set is adjacent to every vertex of the second. A graph of order n, which is a path, is called a path graph P_n. A cycle graph G is a path $u_0, u_1, u_2, \cdots, u_n$ with an edge joining vertices u_0 and u_n, $n > 2$, see (Ali *et al.*, 2016a; Sciriha, 2007). The complement G^c of a graph G is a graph whose vertex set is the same as that of G and two vertices are adjacent in G^c if and only if they are not adjacent in G. G-v and G-e are respectively the deletion of a vertex and an edge from G. A vertex weighting of a graph G is a function $f : V(G) \rightarrow R$ where R is the set of real numbers, which assigns real numbers (weights) to each vertex. The weighting of G is said to be non-trivial if there is at least one vertex $v \in V(G)$ for which $f(v) \neq 0$. A non-trivial vertex weighting of a graph G is called a zero sum weighting provided that for each $v \in V(G)$, $\Sigma_{u \in N(v)} f(u) = 0$, that is the summation is taken over all $u \in N(v)$, See (Brown *et al.,* 1993). Out of all zero-sum weightings of a graph G, a high zero sum weighting (hzsw) of G, is one that uses a maximum number of non-zero independent variables, see (Sharaf & Ali, 2013). The nullity, η(G of a graph G) is the multiplicity of zero as an eigenvalue of its adjacency matrix, see (Cheng & Liu, 2007)., So a graph is singular if its nullity is at least one. Also, a graph is singular if it possesses a non-trivial zero sum weighting. A graph with nullity η contains η cores (vectors) determines a basis for the null space of A. Any non-zero vector X that satisfies AX=0, and uses only one variable in its components, is in the null space of the graph, also any linear combination of such vectors. A singular graph, on at least two vertices, with a kernel eigenvector X such that AX=0 having no zero entries, is said to be a core graph. Core graphs of nullity one are called nut graphs. see (Sciriha & Gutman, 1998). In (da Fonseca, 2019) the eigenvalues of some anti-tridiagonal Hankel matrices are defined.

The nullity of most known graphs can be easily determined using the hzsw tool, since number of independent variables used in a hzsw of the graph is exactly $\eta(G)$. Thus, $\eta(K_n)=0$, for $n\geq 2$, $\eta(K_{m,n})=m+n-2$, $\eta(P_n)=1$ if n is odd and zero where n is even, $\eta(C_n)=2$ if $n=0$ mod4 and it is non-singular otherwise, see (Cheng, & Liu, 2007) and (Sharaf & Ali, 2013). Maximum nullity of graphs was studied in (Gutman & Sciriha, 1996).

1.1. Theorem. (Interlacing Theorem).

Let G be a graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let the eigenvalues of $G-v$ be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{(n-1)}$. Then the eigenvalues of $G-v$ interlace eigenvalues of G that is, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{(n-1)} \geq \lambda_n$, see (Omidi, 2009). □

The degree deg(v) of a vertex v in a graph is the number of edges incident to v. In (Valencia *et al.*, 2019) the inverse degree index is studied. A pendant or an end vertex v is a vertex of a graph of degree one, it has a unique neighbor w and the edge vw is a pendant edge.

1.2. Lemma. (End Vertex Lemma).

If G is a graph with a pendant vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G)=\eta(H)$, see (Brown et al., 1993; Sciriha, 2007).

2. Change in nullity of a graph by adding a new vertex

Two vertices of a graph G, are said to be identified if they are combined to form one vertex whose neighbors are the union of their neighbors, ignoring loops and multiple edges. We define the vertex operation G + ab to be the graph obtained from G by adding a new vertex ab which is adjacent to all vertices in $N(a)\cup N(b)$. Also, the graph G•ab is defined to be G+ab−a−b. Two vertices a and b of a graph G are said to be duplicate (co-neighbor) vertices if they are non-adjacent and have the same set of neighbors.

Change in the nullity of a coalescence Fiedler and core vertices of graphs, in which a cut vertex is removed was studied in (Ali *et al.,* 2016a). While, in the other paper of (Ali *et al.*, 2016b) change in nullity of graphs and those derived from them under geometrical operations, including, deletion, contraction, and insertion of an edge are determined. In this paper, we study the change in the nullity of a graph due to each of the above operation.

A basic inequality about the bounds for the nullity of the graph G+ab, is given in the next result.

2.1. Proposition.

For any two vertices a and b of a graph G,

$$\eta(G)-1\leq \eta(G+ab) \leq \eta(G)+1.$$

Proof: This is a version of Interlacing Theorem, for the graph H=G+ab.

Equality holds in the left side where the graph $G+ab=P_3+ab$ is obtained from the path P_3 and a, b are adjacent vertices, while in the right holds where G+ab= P3+ab and a,b are end vertices.

2.2. Theorem.

For any two non-adjacent vertices a and b of a graph G, if $N(a) \cap N(b) = \phi$ then,

$\eta(G+ab)=\eta(G)+1$

Proof*:* In the adjacency matrix of the graph G +ab, the rows R_a, R_b and R_{ab} corresponding to the vertices a, b and ab of the adjacency matrix of G+ab, satisfy the row equation

$R_a + R_b = R_{ab}$

This is only true if the neighbors of a and b are disjoint sets. So, the rank of the graph G + ab is the same as that of G (the order is the sum of the rank and the nullity). Then $\eta(G+ab) = \eta(G)+1$. In other words, any high zero sum weighting of G+ab will contains an extra distinct independent variable say x, which is given to the vertex ab. For more details replace the weights of a and b in the high zero sum weighting of G+ab by w(a) −x and w(b)−x. Thus, in the new weighting, the summation over all neighbors of each vertex v is zero for all v in G+ab, especially those adjacent to a or to b, and more precisely over the neighborhood of the vertex ab, because $N(a) \cap N(b)=\phi$, this implies that $\eta(G+ab) \geq \eta(G)+1$. But by Interlacing Theorem $\eta(G+ab) \leq \eta(G)+1$. Thus, $\eta(G+ab)=\eta(G)+1$.

Moreover, if the vertices a and b are adjacent then, the above row equation holds, as long as the open neighborhoods of a and b are disjoint.

We construct two graphs, to explain the statement of Th. 2.2, by definition of G+ab, the added vertex ab dominates (it is adjacent

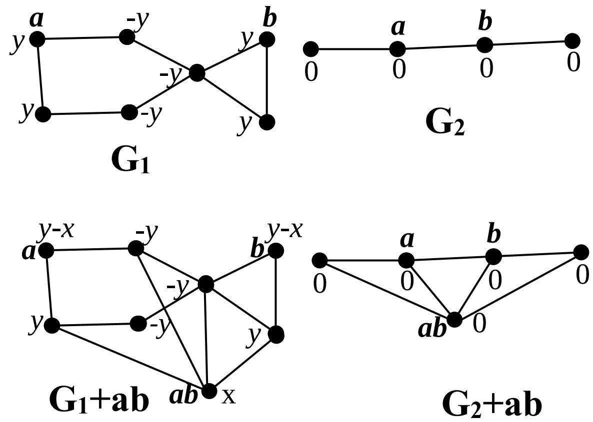

Fig. 1. hzsws of the graphs G_1, G_1+ab, *G_2 and G_2+ab.*

to) all the neighbors of the vertices a and b. We see that the nullity of the graph G_1 in the above figure is increased by 1 when the vertex ab is added, while the nullity of the graph G_2 remains unchanged where ab is added, because a and b are adjacent, and in both cases $N(a) \cap N(b) = \phi$.

2.3. Theorem.

For any two duplicate vertices a and b of a graph G, $\eta(G+ab)=\eta(G)+1$.

Proof: In this case, at least three duplicate vertices appear, namely a, b and ab, (that is N(a)=N(b)=N(ab) and all are pairwise non-adjacent), So, any new high zero sum weighting for G+ab will be of the form, w(ab) = x, w(a) is replaced by w(a)−x or w(b) is replaced by w(b) − x but not both, which uses an extra variable x. Furthermore, in this new hzsw, the sum of the weights over neighbors of each vertex of a, b and ab is zero. Applying Interlacing Theorem again, proves that $\eta(G+ab)=\eta(G)+1$.□

Clearly, where $N(a) \subset N(b)$ in G, then a pair of duplicate vertices is created, namely b and ab, and the nullity of the graph is increased.

2.4. Theorem.

For any two non-adjacent vertices a and b of a graph G, where $N(a)\cap N(b) \neq \phi$ and $\{N(a)\cup N(b)\}\backslash\{N(a)\cap N(b)\}\neq \phi$, and ab does not possess a duplicate vertex, then

$\eta(G+ab)\leq \eta(G)$.

Proof: Let G be a graph that contain two non-adjacent vertices a and b for which $N(a)\cap N(b) \neq\phi$ and $\{N(a)\cup N(b)\}\backslash\{N(a)\cap N(b)\}\neq\phi$. Without loss of generality, chose, $i\in N(a)\backslash\{N(a)\cap N(b)\}$, $j\in(N(a)\cap N(b))$ and $k\in N(b)\backslash\{N(a)\cap N(b)\}$. Let h be a hzsw of G, that uses $\eta(G)$ independent variables and embed h into [h,x] as a hzsw of G+ab where w(ab) = x.

Applying zero sum conditions for the weights over the neighbors of the above 3 vertices i, j and k, the relation $0 =\sum_{v\in N_{G+.ab}(i)} w_{G+ab}(v)$

$$=\sum_{v\in N_G(i)} w_G(v) - w_G(a)+w_{G+ab}(a)+x$$

implies $w_{G+ab}(a)= w_G(a)-x$, (1)

the relation $0 =\sum_{v\in N_{G+ab}(k)} w_{G+ab}(v)$

$$=\sum_{v\in N_G(k)} w_G(v) - w_G(b)+w_{G+ab}(b)+x$$

implies $w_{G+ab}(b)= w_G(b)-x$, (2)

the relation $0 =\sum_{v\in N_{G+ab}(j)} w_{G+ab}(v)$

$$=\sum_{v\in N_G(j)} w_G(v) - w_G(a)-w_G(b) +w_{G+ab}(a) +w_{G+ab}(b)+x$$

Implies

$w_{G+ab}(a)+ w_{G+ab}(b)= w_G(a)+ w_G(b)+x$. (3)

Solving equations (1), (2) and (3) for x, provided that, the vertex ab does not produce a duplicate vertex, gives, x = 0. This means that the number of independent variables in any hzsw of G+ab is at most as that of $\eta(G)$. That is, $\eta(G+ab) \leq \eta(G)$.

Note that, if the vertex ab produces a duplicate vertex, say s, then $x\neq 0$ and $w_{G+ab}(s)=w_G(s)-x$ which rebuild a hzsw for G+ab that uses $\eta(G)+1$ independent variables, and hence $\eta(G+ab)= \eta(G)+1$, and the summation of the weights over the neighbors of ab is zero. Note that, where x = 0 and summation of the weights of the neighbors of ab is not equal to zero, an independent variable in the hzsw of G is removed and $\eta(G+ab) = \eta(G)-1$. □

2.5. Theorem.

If a and b are two adjacent vertices of a graph G, then $\eta(G+ab)\leq\eta(G)$, provided that, the vertex ab does not possess a duplicate vertex.

Proof: It is clear that where a and b are adjacent in G, then $N(a)\neq N(b)$, so we prove the above result, only for two cases:

Case one: $N(a)\cap N(b)\neq\phi$, and ab didn't possess a duplicate vertex. Without loss of generality, assume that there exist vertices $a\in N(b)\backslash N(a)$, $b\in N(a)\backslash N(b)$ and $j\in(N(a)\cap N(b))$. Applying zero sum conditions for the weights over the neighbors of the above three vertices, a, b and j, it follows that:

The relation $0 =\sum_{v\in N_{G+ab}(a)} w_{G+ab}(v)$

implies $w_{G+ab}(b) = w_G(b)-x$,

the relation $0 =\sum_{v\in N_{G+ab}(b)} w_{G+ab}(v)$

implies $w_{G+ab}(a) = w_G(a)-x$,

the relation $0 =\sum_{v\in N_{G+ab}(j)} w_{G+ab}(v)$

implies $w_{G+ab}(a)+w_{G+ab}(b)+x = w_G(a) +w_G(b)+z$, for some real number z. Now, substituting the value of $w_{G+ab}(a)$ and $w_{G+ab}(b)$ in the last equation, gives $z = -x$. This is a contradiction for a duplicate vertex is constructed. Hence, the number of independent variables in the hzsw of G is not increased, which implies that $\eta(G+ ab)\leq \eta(G)$.

Case two*:* $N(a)\cap N(b)=\phi$. In this case, ab cannot possess a duplicate vertex, because any duplicate vertex must belong to the intersection of their neighbors, which is empty. If $N(a)=\{b\}$ and $N(b)=\{a\}$, then G+ ab constructs a cycle subgraph of length 3 and $\eta(G+ab)=\eta(G)=0$. Otherwise, by similar arguments as for case one for the vertices a, b and j, $j \neq b$, $j \in (N(a))$, we prove that weight of ab must be zero, so $\eta(G + ab) \leq \eta(G)$.□

3. Nullity of a graph in which two vertices are identified

Operations on graphs, is a most interesting subject in graph theory, for example Cartesian and Kronecker products and their related spectral. Vertex identification is an operation applied on any pair of vertices of a graph or of distinct graphs.

For some well-known graphs, we evaluate the following relations between the nullities of a vertex identified graph and its original graph.

3.1. Theorem.

For any two identified vertices of a well-known graph G of order n we have:

$$\eta(G\bullet ab) = \begin{cases} 1 & \text{if } G = K_2 = K_{1,1} \\ \eta(G) & \text{if } G \text{ is a complete graph} \\ \eta(G)-1 & \text{if } G \text{ is an empty graph} \\ \eta(G)-1 & \text{if } G = K_{m,n} \text{ where, } a \text{ and } b \text{ are non adjacent} \\ \eta(G)-2 & \text{if } G = K_{m,n} \text{ where, } a \text{ and } b \text{ are adjacent and cardinality of each Partition set is greater than } 1 \end{cases}$$

Proof: Straightforward.□

For identifying two vertices of path, we prove the next propositions.

3.2. Proposition.

Let $X = [x,0,-x,0,x,\cdots]$ be a vector of a high zero sum weighting of a path P_n for odd n, whose vertices are labeled consecutively from one end vertex to the other, then,

$$\eta(P_n\bullet ab) = \begin{cases} \eta(P_n)-1 & \text{if } w(a)\neq w(b) \\ \eta(P_n)+1 & \text{if } w(a) = w(b) \end{cases}$$

Proof: In any hzsw of the path P_n, if $w(a) \neq w(b)$, then either $w(a) = -w(b)$, or one of them is zero, not both. Now, for any vertex v, $v \in N(a)\{f\in N(b)\}$ the zero sum weighting condition of $P_n\bullet ab$ will vanish, because of v(f) will be in the neighborhood of ab and $\sum_{u\in N_{G\bullet ab}(v)} w_{G\bullet ab}(u) = w_{G\bullet ab}(ab)-w_G(a) = 0$ and $\sum_{u\in N_{G\bullet ab}(f)} w_{G\bullet ab}(u)= w_{G\bullet ab}(ab)-w_G(b) = 0$, gives $w_G(a) = w_G(b)$, which is a contradiction. Hence the variable which is non zero and weight of a (or b) will vanish. Therefore, $\eta(P_n\bullet ab) = \eta(P_n)-1$.

If $w_G(a)=w_G(b)=0$, then, either the distance $d_G(a,b)=4t$ (or $d_G(a,b)=4t+2$), and a cycle C_{4t} (C_{4t+2}) is constructed in P_n+ab. Applying End Vertex Lemma $\frac{n-1-4t}{2}$ times ($\frac{n-3-4t}{2}$ times), leaves two odd paths or a cycle C_{4t}.

If $w(a) =w(b)\neq 0$, then $d_G(a,b)=4t$, then a cycle C_{4t} is constructed in P_n+ab. Applying End Vertex Lemma $\frac{n-1-4t}{2}$ times, leaves 2 odd paths or a cycle C_{4t}, hence $\eta(P_n\bullet ab)=\eta(P_n)+1$.□

3.3. Proposition.

If the vertices of the even path P_n are labeled as, $v_1, v_2, \cdots, v_i,\cdots,v_{i+k},\cdots,v_n$ and the vertices v_i and v_{i+k} are identified, then the nullity of $P_n\bullet v_iv_{i+k}$ is

$$\eta(P_n\bullet v_iv_{i+k}) = \begin{cases} \eta(P_n) & \text{if } P_n\bullet v_iv_{i+k} \text{ is union of an odd cycle and 2 odd paths} \\ \eta(P_n)+1 & \text{otherwise} \end{cases}$$

Proof: If $k = 1$ or $k=2$, then the graph $P_n \bullet v_i v_{i+k}$ is isomorphic to P_{n-1} which is an odd path with nullity one.

If $k \geq 3$ then, the graph $P_n \bullet v_i v_{i+k}$ is a vertex identification of three graphs, namely, the path $v_1 - v_i v_{i+k}$ of order i, (for $i = 1$ this path does not exist), the cycle of order k and the path $v_i v_{i+k} - v_n$ of order $n + 1 - (k + i)$.

If the cycle is odd, then both paths must be even or both odd. If both are odd, applying End Vertex Lemma $\frac{n-1-k}{2}$ times, leaves the odd cycle. This proves part one.

Where both paths are even, applying End Vertex Lemma $\frac{n-1-k}{2}$ times, this leaves a single vertex and an even path.

If the cycle is even, then one of the paths must be odd and the other must be even, applying End Vertex Lemma $\frac{n-2-k}{2}$ times for the even path, leaves an odd and even paths. So in all other cases $\eta(P_n \bullet v_i v_{i+k}) = 1$. □

3.4. Theorem.

In a high zero sum weighting of a singular graph G, if $w(a)=w(b)\neq 0$ and $d_G(a,b) > 3$, then $\eta(G\bullet ab) = \eta(G)$, provided that the identification does not construct a cycle graph C_{4t}.

Proof: Label the vertices of the graph G by $v_1=a$, $v_2=b$, $v_3,\cdots,v_n$. Assume that the components of the vector $X=[w(v_1), w(v_2), \cdots, w(v_n)]^T$ are weights of a high zero sum weighting of G. Obtain $G\bullet ab$ by identifying a and b. Since $d_G(a,b) > 3$, then no new pair of vertices in the graph $G\bullet ab$ will be duplicated. And the components of the contract vector $X^* = [w(ab) = w(a), w(v_3), \cdots, w(v_n)]^T$ forms a high zero sum weighting of $G\bullet ab$, that uses the same number of distinct independent variables used in X. Moreover, the sum over the neighbors of ab remains zero, which is the sum over the neighbors of a and of b, therefore, $\eta(G\bullet ab) = \eta(G)$).□

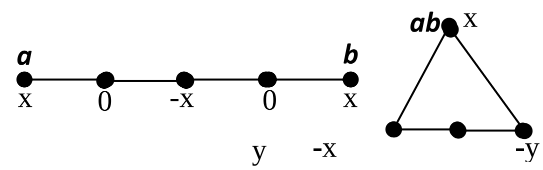

Fig. 2. hzsws of the graphs P_5 and $P_5\bullet ab$**.**

The condition $w(a) = w(b) \neq 0$ is necessary but not sufficient, for example in the above figure the nullity of P_5 is 1. But $\eta(P_5\bullet ab)=2$. This is true whenever $d(a,b) = 4t$, $t \geq 1$.

3.5. Lemma.

For non-adjacent vertices a and b of G, if $w(a) = w(b) \neq 0$ and $N(a)\cap N(b) = \{v\}$ then, $\eta(G\bullet ab) = \eta(G)-1$.

Proof: In a hzsw of $G\bullet ab$, which is obtained by contracting the vector X to X^*. For the vertex v, $v\in N(a)\cap N(b)$, $\sum_{u\in N_{G\bullet ab}(v)} w_{G\bullet ab}(u)=w(ab)\neq 0$. hence weight of ab will vanish, therefore, $\eta(G\bullet ab)=\eta(G)-1$.□

3.6. Proposition.

Let $w(a) \neq w(b)$ in a high zero sum weighting for a singular graph G, then $\eta(G\bullet ab) \leq \eta(G)$, strictly holds if a and b are co-neighbor vertices.

Proof: If a and b are co-neighbor vertices, then $\eta(G\bullet ab) = \eta(G)-1$, because the same high zero sum weighting is used with losing one variable say x. If a and b are non-co-neighbor vertices, then, there exists a vertex say $v \in N(a)\backslash N(b)$. So, in any high zero sum weighting of the graph $G\bullet ab$, the condition $\sum_{u\in N_{G\bullet ab}(v)} w_{G\bullet ab}(u)=0$, gives $w(a)=w(b)$. But we have assumed that $w(a) \neq w(b)$, then we lose a variable say x (which is not zero, provided that a duplicate vertex is not created) in a high zero sum weighting of $G\bullet ab$, therefore, $\eta(G\bullet ab) \leq \eta(G)$.□

Usually, if w(a) ≠ w(b) = 0, then η(G•ab) ≤ η(G) and if 0≠w(a) ≠w(b) ≠ 0 then η(G•ab) < η(G).

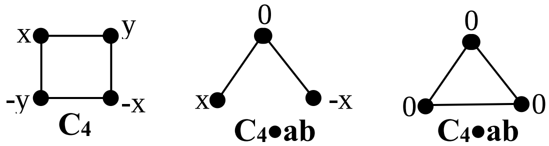

Fig. 3. Graphs C_4, C_4•ab and C_4•ab, where a and b are co-neighbors and non co-neighbors.

3.7. Proposition.

For any two vertices of a singular graph G, if w(a) = w(b) in a high zero sum weighting and $d_G(a,b) \geq 3$, then η(G•ab) ≥ η(G), with strict inequality if $d_G(a,b) = 4t$.

Proof: For any hzsw of the graph G•ab the sum condition over the neighborhoods of v ∈ N(ab) satisfies $\sum_{u \in N_{G\bullet ab}(f)} w_{G\bullet ab}(u)=0$, hence no old variable will vanish, and there is a possibility to introduce new variable. Moreover, if $d_G(a,b)=4t$ then a cycle C_{4t} is obtained and in a simple case where G is P_{4t+1} and a and b are vertices at distance 4t, then P_{4t+1}•ab is isomorphic to C_{4t} with nullity 2. □

3.8. Proposition.

In a high zero sum weighting of a graph G, if w(a) = kw(b), k ≠1 then,

η(G•ab) ≤ η(G)−1

Proof: Let w(a) = kw(b), k ≠ 1 in a (hzsw) of G. If the vertex v ∈ N(a), then in G•ab we have

$$\sum_{u \in N_{G\bullet ab}(v)} w_{G\bullet ab}(u)= (k-1)\, w(a)$$

and we vanish the variable w(a), similarly if v ∈ N(b), then in G•ab we have $\sum_{u \in N_{G\bullet ab}(v)} w_{G\bullet ab}(u)=(k-1)\, w(b)$ and we vanish w(b), therefore η(G•ab) < η(G). But in figure 4 we see that η(G•ab) = η(G) − 1, for any two vertices satisfying the properties of the statement, hence η(G•ab) ≤ η(G)−1. □

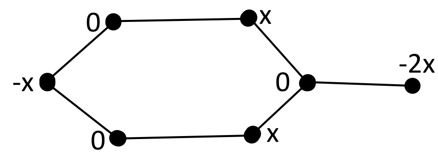

Fig. 4. A hzsw of the graph C_6 with a pendant vertex.

A main result of this section is proved in the next theorem.

3.9. Theorem.

For any two vertices a and b of a graph G, if N(a)∩N(b) = ϕ, then

η(G)−2 ≤ η(G•ab) ≤ η(G) + 2,

equality holds on the left when the graph G is C_4 and on the right when the graph G is C_5.

Proof: Let a and b be any two vertices in a G with a hzsw say h. In the graph G + ab, give the weight zero to the vertex ab, then the high zero sum weighting of the graph G union weight of ab is a zero sum weighting for G+ab which uses the same number of independent variables as η(G) because N(a) ∩ N(b) = ϕ. Hence η(G) ≤ η(G + ab). There are two cases:

i) If the zero sum weighting of G+ab is a high zero sum weighting, then

$$\eta(G) = \eta(G + ab) \qquad (4)$$

Hence, by Interlacing Theorem,

η(G+ab)−2≤ η(G+ab−a−b)≤ η(G+ab)+2 (5) and by equation (4), gives

η(G)−2≤ η(G•ab) ≤ η(G)+2.

ii) If the zero sum weighting of the graph G+ab is not a high zero sum weighting then, an extra independent variable can be introduced that is,

$$\eta(G + ab) = \eta(G) + 1 \qquad (6)$$

Again, the left inequality of equation (5) gives $\eta(G) - 1 \leq \eta(G\bullet ab)$. But on the right side of equation (5), we use the vector equation

$R_a + R_b = R_{ab}$, hence

$$\eta(G + ab - a - b) \leq \eta(G + ab) + 1 \qquad (7)$$

Combining equations (6) and (7) we get the result.□

Thus, disjoint neighbors of identified vertices play the main role to the change of the nullity of the vertex identified graphs. To illustrate this further, we construct two graphs. In the first the identification of two non-disjoint neighboring vertices increases the nullity by 3 while, in the second, decreases the nullity by 3.

a) We construct a graph G, Figure 5 which contains two vertices a and b, such that $N(a) \cap N(b) \neq \phi$. It is clear that $\eta(G) = 0$, $\eta(G + ab) = 1$, $\eta(G + ab - a) = 2$ and $\eta(G\bullet ab) = \eta(G+ab- a-b) = 3$ and the graph G•ab contains three pairs of co-neighbor vertices.

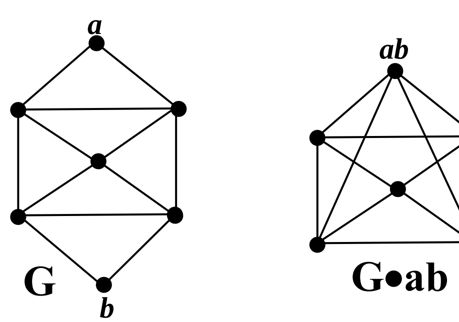

Fig. 5. Graphs G and G•ab, with $\eta(G) = 0$ and $\eta(G\bullet ab) = 3$.

b) We construct another graph H, Figure 6 which contains two vertices u and v, for which $N(u) \cap N(v) \neq \phi$. And in the graph H, $\eta(H)=3$, $\eta(H+uv)=2$, $\eta(H+uv-u)=1$ and $\eta(H\bullet uv) = \eta(H+uv-u-v)=0$.

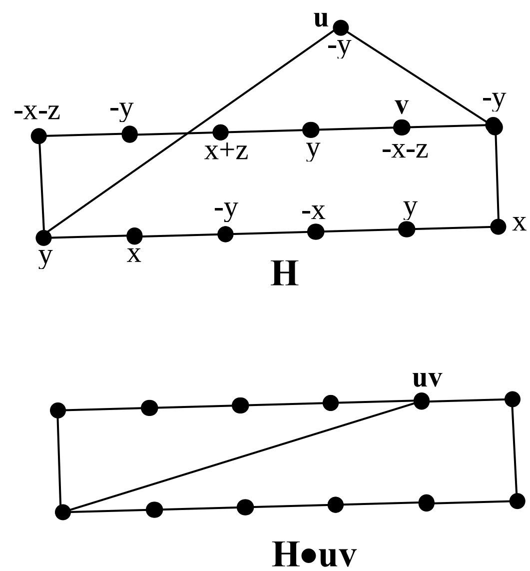

Fig. 6. Graphs H and H•uv with $\eta(H) = 3$ and $\eta(H\bullet uv) = 0$.

Finally, we conclude that for any two vertices a and b of a graph G, $|\, \eta(G\bullet ab) - \eta(G)\, | \leq 3$.

In other words, as an application of Interlacing Theorem this inequality holds for the dimension of the null space of any eigenvalue of the graph.

4. Constructing new nut graphs from old ones

In this section we start with nut graphs defined by (Sciriha & Gutman, 1998), and by identification of proper vertices we obtain nut graphs with extra properties, namely increasing the maximum degree. An extension of the smallest order nut graph $C_5\bullet C_3$ is the nut graph $C_{4n+1}\bullet C_{4n-1}$, $n \geq 1$ whose maximum degree is 4 which is unrelated to its order. In order to investigate nut graphs with maximum degree, consider the nut graph C_{2n+1}^{2} of order 6n + 3 obtained from an odd cycle C_{2n+1} where each vertex is identified with a vertex of a cycle C_3. The smallest such graph is illustrated in figure 7 where n=1.

4.1. Proposition.

For each n, there exist a nut graph of order 4n + 3 with a dominating vertex.

Proof: For any n, in the nut graph C^2_{2n+1} whose vertices are labeled as v_1, u_1,w_1, v_2, u_2, w_2, ⋯, v_{2n+1}, u_{2n+1}, w_{2n+1}, where the vertices w_1, w_2, ⋯, w_{2n+1} are the vertices of the cycle C_{2n+1}, identify the vertices v_1,v_2,⋯,v_{2n+1} to form a dominating vertex. Now, the resulting graph is a nut graph. To prove this, let the weight of the vertex with degree 4n+2 be -x, then all vertices of the cycle C_{2n+1} (each is of degree 3) must have weight x, while the remaining 2n+1 vertices will be weighted -x, Figure 7 and hence any hzsw of the resulting graph uses exactly one variable. Thus, all components of the core vector are non-zero and the graph has nullity one. Hence, the resulting graph is a nut graph.□

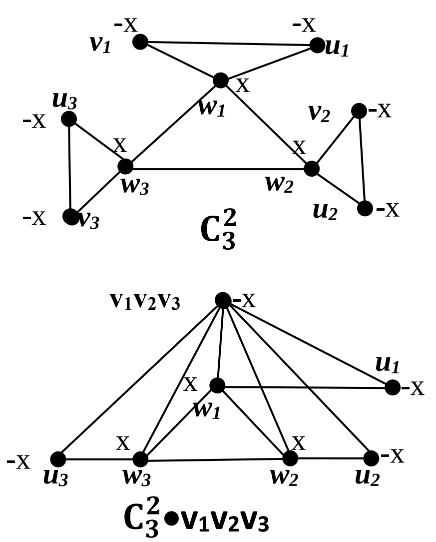

Fig. 7. A nutgraph of order 7 and maximum degree 6 is obtained from the nut graph C_3^2.

4.2. Proposition.

Let G be a nut graph, with V(G) = {a = v_1, b = v_2, v_3, ... , v_n } and a hzsw that uses one variable, say w(a), then,

1) η(G•ab) ≥ η(G) if N(a)∩N(b)=ϕ and w(a)=w(b)

2) η(G•ab) = 0 if w(a) ≠ w(b), provided that a duplicate vertex is not constructed

3) η(G•ab) = 0 if N(a)∩N(b) ≠ ϕ

Proof: 1) Obtain a weighting for G•ab from the hzsw of G by assuming w(ab) = w(a) and keep the remaining weights unchanged. Then, this weighting is a zsw of G•ab that uses at least one variable say w(a), because the sum over neighbors of all vertices of G•ab remains zero, so η(G•ab) ≥ η(G).

Moreover, if some duplicate vertex is created then the strict inequality holds. See Figure 8.

2) Assume that w(a) ≠ w(b) and w(a) = x, then w(b) must be −x. Now giving any non zero weight to ab yields a contradiction to the condition of the zero sum over the vertices in the neighbors of a and of b. Hence (where a duplicate vertex is not constructed), there exists no non trivial zsw for G•ab, which means η(G•ab) = 0.

3) Assume that N(a) ∩ N(b) ≠ ϕ. If w(a) ≠ w(b), then by part 2 the result holds. Let w(a) = w(b), chose w(ab) = 2w(a). Then, the zero sum condition will not be satisfied over any vertex not in the intersection of neighbors of a and b, and w(ab)=0, the unique variable in the zsw will vanish. If no such vertex exists, then a and b are co-neighbor vertices, which is a contradiction to the definition of nut graph. So η(G•ab)=0.□

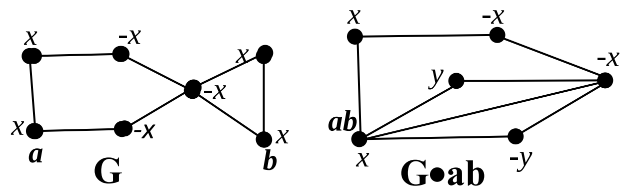

Fig. 8. Graphs G and G•ab, η(G) = 1 and η(G•ab) = η(G) + 1.

References

Ali, D. A., Gauci, J. B., Sciriha, I., & Sharaf, K. R. (2016a). Coalescing Fiedler and core vertices. Czechoslovak Mathematical Journal, **66**: 971–985.

Ali, D.A., Gauci, J.B., Sciriha, I., & Sharaf, K.R. (2016b). Nullity of a Graph with a cut-edge. Communication in Mathematical and in Computer Chemistry,**76**: 771–791.

Brown, M., Kennedy, J. W., & Servatius, B. (1993). Graph singularity. Graph Theory Notes of New York, **25**: 23–32.

Cheng, B., & Liu, B. (2007). On the nullity of graphs. J. of Linear Algebra, Volume, **16**: 6067.

Cvetkovic, D. M., Doob, M., & Sachs, H. (2009). Spectra of Graph Theory and Application. Academic Press, New York.

da Fonseca, C.M. (2019). The eigenvalues of some anti-tridiagonal Hankel matrices, Kuwait J. Sci. **45**(1):1-6.

Gutman, I. and Sciriha, I. (1996). Graphs with maximum singularity. Graph Theory Notes. New York, **30**: 17–20.

Omidi, G. R. (2009). On the nullity of bipartite graphs. Graphs and Combinatorics, **25**: 111-114

Sciriha, I. (2007). A characterization of singular graphs. Electronic Journal of Linear Algebra, **16**: 451-462.

Sciriha, I. and Gutman, I. (1998). Nut graphs: Maximally extending cores. J.Util. Math., **54**: 257-272

Sharaf, K. R., & Ali, D. A. (2013). Nullity and bounds to the nullity of dendrimer graphs. Al-Rafidain Journal of Computer Sciences Mathematics, **10**: 71-861.

Simic, S. K., Andelic, M., Carlos, M., da Fonseca, C. M. and Zvikovic, D. (2015). On the multiplicities of eigenvalues of graphs and their vertex deleted subgraphs: old and new results. Electronic Journal of Linear Algebra **30**: 85-105.

Valencia J. R., Gomez, J.H. and Hernandez, G. R. (2019). On the inverse degree index and decompositions in Graphs. Kuwait J. Sci. **46**(4): 14-22.

Submitted: 01/02/2020
Revised: 17/12/2020
Accepted: 20/12/2020
DOI: 10.48129/kjs.v48i4.9111