*Kuwait J. Sci.* **43 (3)** pp. 41-49, 2016

# Skew-symmetric matrices and integral curves in Lorentzian spaces

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## Abstract

We provide some existence results about integral curves of a linear vector field on 3-dimensional Lorentzian space $\mathbb{E}_1^3$. Moreover we have a classification of integral curves (flow lines) of a linear vector field and the normal forms of the matrix corresponding to a mapping **A** in Lorentzian spaces $\mathbb{E}_1^3$ and $\mathbb{E}_1^{2n+1}$.

**Key words:** Flow lines; integral curve; linear vector field; Lorentzian space; skew-symmetric matrix.

## 1. Introduction

A vector field is an assignment of a vector to each point in a subset of Euclidean space. As vector fields exist at all points of space, they can be specified along curves and surfaces as well. This is especially important because all laws of electricity and magnetism can be formulated through the behavior of vector fields along curves and surfaces. Vector fields are often used to model; for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point, (Galbis & Maestre, 2012).

We have a particular interest to vector fields on the Lorentzian space $\mathbb{E}_1^{2n+1}$ that are linear with respect to a chosen linear map $\mathrm{A}:\mathbb{E}_1^{2\mathrm{n}+1} \to \mathbb{E}_1^{2\mathrm{n}+1}$ (see Preliminaries section for details). Note that when $n=3$ and the index is zero, such vector fields can be specified by integral curves. We recall that an integral curve is a parametric curve that represents a specific solution to an ordinary differential equation or system of equations. If the differential equation is represented as a vector field, then the corresponding integral curves are tangent to the field at each point. Such curves could represent the histories of small text particles, in which case they would be geodesics, or they might represent the flow lines of a fluid (Hawking & Ellis, 1973). Integral curves are called by various other names, depending on the nature and interpretation of the differential equation or the vector field. In physics, such curves for an electric field or magnetic field are known as field lines, and for the velocity field of a fluid are known as streamlines. In dynamical systems, the integral curves for a differential equation that governs a system are referred to as trajectories or orbits (Lang, 1972).

A linear vector field $X$ on the Euclidean space $\mathbb{E}^3$ can be written as follows:

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

For every $\mathrm{P}=(\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathbb{E}^3$, where $A$ and $C$ are matrices.

Karger & Novak (1978) classified integral curves of a linear vector field according to the rank of matrix $[AC]$. They showed that the integral curves of a linear vector field $X$ on $\mathbb{E}^3$ are helixes, circles or parallel straight lines. Also, they expressed that the integral curves of a linear vector field on a sphere in $\mathbb{E}^3$ are circles lying in parallel planes.

Later on, Acratalishian (1989) extended the results of Karger and Novak to the general case $\mathbb{E}^{2n+1}$.

In this paper, we provide a sufficient and necessary condition on a linear vector field that determines a vector field of tangent vectors on the pseudo-sphere $\mathbb{S}_1^2$ in $\mathbb{E}_1^3$. Furthermore, we obtain the normal forms of the matrix corresponding to any linear map by considering the causal structure of the 3-dimensional Lorentzian space. As a result of that we analyze the non-zero solutions of the equation $A(x)=0$, $x \in \mathbb{E}_1^{2n+1}$, in the Lorentzian space $\mathbb{E}_1^{2n+1}$, where $A$ is the anti-symmetric matrix corresponding to the linear map **A**, and then we construct normal forms of the matrix $A$ depending on the causal characters of the vector $x$.

## 2. Preliminaries

The Lorentzian space $\left(\mathbb{E}^{2n+1}, \langle,\rangle\right) = \mathbb{E}_1^{2n+1}$ is the (2n+1)-dimensional vector space $\mathbb{E}_1^{2n+1}$ endowed with the pseudo scalar product

$$\langle \mathrm{v}, \mathrm{w} \rangle = -\mathrm{v}_1 \mathrm{w}_1 + \sum_{\mathrm{i}=2}^{2\mathrm{n}+1} \mathrm{v}_{\mathrm{i}} \mathrm{w}_{\mathrm{i}}$$

where $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_{2n+1})$, $\mathrm{w} = (\mathrm{w}_1, \mathrm{w}_2, ..., \mathrm{w}_{2n+1})$ in $\mathbb{E}_1^{2n+1}$. We say that the vector $v \in \mathbb{E}_1^{2n+1}$ is spacelike, lightlike or timelike if $\langle v, v \rangle > 0$ or $v = 0$, $\langle v, v \rangle = 0$ and $v \neq 0$, and $\langle v, v \rangle < 0$, respectively, (O'Neill, 1983). We define the signature of a vector $v$ as

$$\varepsilon = \begin{cases} 1, & v \text{ is spacelike} \\ 0, & v \text{ is lightlike} \\ -1, & v \text{ is timelike.} \end{cases}$$

The norm of a vector $v \in \mathbb{E}_1^{2n+1}$ is defined by $\|v\| = \sqrt{|\langle v, v \rangle|}$.

A frame field $\phi = \{u_1, ..., u_{2n}, u_{2n+1}\}$ on $\mathbb{E}_1^{2n+1}$ is called a pseudo orthonormal frame field, (Duggal & Bejancu, 1996), if

$$\begin{aligned} &\langle u_{2n}, u_{2n} \rangle = -\langle u_{2n+1}, u_{2n+1} \rangle = -1, \\ &\langle u_{2n}, u_{2n+1} \rangle = 0, \langle u_{2n}, u_i \rangle = \langle u_{2n+1}, u_i \rangle = 0, \\ &\langle u_i, u_j \rangle = \delta_{ij}, i, j = 1, ..., n. \end{aligned}$$

Definition 1. Let $\alpha(s)$, $s$ being the arclength parameter, be a non-null regular curve in semi-Euclidean space $\mathbb{E}_v^{2n+1}$. The changing of a pseudo orthonormal frame field $\phi = \{u_1, ..., u_n, ..., u_{2n}, u_{2n+1}\}$ of $\mathbb{E}_v^{2n+1}$ along $\alpha$ is given by

$$\begin{aligned} u_1'(s) &= \kappa_1(s) u_2(s) \\ u_i'(s) &= -\varepsilon_{i-1}\varepsilon_i \kappa_{i-1}(s) u_{i-1}(s) + \kappa_i(s) u_{i+1}(s), \\ u_{2n+1}'(s) &= -\varepsilon_{2n}\varepsilon_{2n+1}\kappa_{2n}(s) u_{2n}(s), \end{aligned}$$

where $2 \leq i \leq 2n$. These equations are called the Frenet-Serret type formulae for $\alpha(s)$, where $\kappa_i(s)$, $1 \leq i \leq 2n$, is the curvature function of $\alpha$, $\kappa_i(s) = \varepsilon_{i+1}\langle u_i'(s), u_{i+1}(s) \rangle$, and $\varepsilon_i$ is the signature of the vector $u_i$, $1 \leq i \leq 2n$, (Yücesan *et al.*, 2004).

The signature matrix $S$ in the Lorentzian space $\mathbb{E}_1^{2n+1}$ is the diagonal matrix whose diagonal entries are $s_1 = -1$ and $s_2 = s_3 = ... = s_{2n+1} = +1$. We call that $A$ is skew-symmetric matrix in (2n+1)-dimensional Lorentzian space if its transpose satisfies the equation $A^t = -SAS$, (O'Neill, 1983).

Let $X$ be a vector field on $\mathbb{E}_1^{2n+1}$. By an integral curve of the vector field $X$ we understand a curve $\alpha : (a,b) \to \mathbb{E}_1^{2n+1}$ such that its every tangent vector belongs to the vector field $X$. If $\frac{\mathrm{d}\alpha}{\mathrm{dt}} = \mathrm{X}(\alpha(\mathrm{t}))$, $\forall \mathrm{t} \in \mathrm{I}$, is satisfied, then the curve $\alpha$ is called an integral curve of the vector field $X$. A vector field $X$ on $\mathbb{E}_1^{2n+1}$ is called linear if $X_v = SA(v)S$ for all $v \in \mathbb{E}_1^{2n+1}$, where the $A$ is the representation matrix of a linear mapping **A** from $\mathbb{E}_1^{2n+1}$ into $\mathbb{E}_1^{2n+1}$.

## 3. Integral curves of a linear vector field on pseudo-sphere $\mathbb{S}_1^2$

The following theorem gives a way to construct a vector field on $\mathbb{S}_1^2$.

Theorem 1. Let $\mathbb{E}_1^3$ be a 3-dimensional Lorentzian space with the pseudo-sphere $\mathbb{S}_1^2$. Let an orthonormal basis $\{u_1, u_2, u_3\}$ be given in $\mathbb{E}_1^3$. A linear vector field $X_v = SA(v)S$ determines a vector field of tangent vectors on the pseudo-sphere $\mathbb{S}_1^2$ if and only if the representation of the mapping **A** in the base $\{u_1, u_2, u_3\}$ is given by a skew-symmetric matrix. Here, $S$ is a signature matrix in $\mathbb{E}_1^3$ and $\langle u_1, u_1 \rangle = -1$, $\langle u_2, u_2 \rangle = \langle u_3, u_3 \rangle = 1$.

We need some preparation for proving Theorem 1.

Let $\mathbf{A}: \mathbb{E}_1^3 \to \mathbb{E}_1^3$ be a linear mapping and $u, v \in \mathbb{E}_1^3$. Then, taking into consideration $\langle u, v \rangle = u^t S v$, we can write

$$< A^t u, v > = < u, SA(v)S > .$$

Then the function

$$\begin{aligned} X \;:\; \mathbb{S}_1^2 &\to \mathbb{E}_1^3 \\ v &\to X_v = SA(v)S \end{aligned}$$

is a tangent vector field on the pseudo-sphere $\mathbb{S}_1^2$ if and only if the scalar product $< v, SA(v)S > = 0$ for all $v \in \mathbb{S}_1^2$. This, in turn, is equivalent to the requirement that $A$ be skew-symmetric, that is

$$SAS + A^t = 0,$$

where $A^t$ is the transpose of the matrix $A$. Actually, suppose first that $< v, SA(v)S > = 0$ for all $v \in \mathbb{S}_1^2$. Then

$$\begin{aligned} <v,(SAS+A^t)(w)> &= <v,SA(w)S>+ \\ &\quad <v,A^t(w)> \\ &= <v+w,SA(v+w)S> \\ &= 0, \end{aligned}$$

for all $v,w \in \mathbb{E}_1^3$, and hence, $SAS+A^t=0$. So, we have $A^t=-SAS$.

Conversely, if $SAS+A^t=0$, then

$$\begin{aligned} <v,SAS(v)> &= \frac{1}{2}(<v,SA(v)S>+ \\ &\quad <v,SA(v)S>) \\ &= \frac{1}{2}<v,(SAS+A^t)(v)> \\ &= 0. \end{aligned}$$

Therefore, we will say that the vector field $X$ obtained in this way is a linear vector field. Now, we can give proof of the theorem.

Proof. Let $\{u_1,u_2,u_3\}$ be an ortho-normal basis in $\mathbb{E}_1^3$. First, we show that the representation of the mapping **A** in the basis $\{u_1,u_2,u_3\}$ is given by a skew-symmetric matrix. So, we take into consideration a mapping **A** as

$$\begin{aligned} \mathbf{A} \ : \ \mathbb{E}_1^3 &\rightarrow \mathbb{E}_1^3 \\ v &\rightarrow \mathbf{A}(v)=SA(v)S. \end{aligned}$$

Thus, the representation of the mapping **A** in the basis $\{u_1,u_2,u_3\}$ is given by

$$A=\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}.$$

This matrix is a skew-symmetric matrix in 3-dimensional Lorentzian space $\mathbb{E}_1^3$. Indeed, the matrix $A$ satisfies the equation $-SAS=A^t$.

Now, the representation of the mapping **A** in the basis $\{u_1,u_2,u_3\}$ be given by a skew-symmetric matrix. Then, in order to say that the function

$$X \ : \ \mathbb{S}_1^2 \rightarrow \mathbb{E}_1^3$$

defined by $X_v=SA(v)S$ is a tangent vector field on $\mathbb{S}_1^2$ for all $v \in \mathbb{S}_1^2$ we have to show that $<v,SA(v)S>=0$. Thus, the representation of the mapping **A** in the basis $\{u_1,u_2,u_3\}$ is given as $SAS+A^t=0$. So, from equation (1) we get

$$<v,SA(v)S>=0.$$

Now we state the following theorem which gives the normal forms of the mapping **A** in 3-dimensional Lorentzian space $\mathbb{E}_1^3$.

Theorem 2. Let **A** be a linear mapping in $\mathbb{E}_1^3$ given by a skew-symmetric matrix in an ortho-normal basis. Then it is possible to find an orthonormal basis in $\mathbb{E}_1^3$ such that the matrix of the mapping **A** assumes the forms

$$\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & -\lambda & 0 \end{bmatrix}$$

where $\lambda \in \mathbb{R}$.

Proof. Let the representation matrix of the mapping **A** be denoted by $A$. Then, since $A$ is a skew-symmetric matrix we have $\det A=0$. So, the equation $A(x)=0$ should have non-zero solutions. Assume that $x$ is spacelike vector and $x=u_3=(0, 0, 1)$ so we can write

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

hence we get $a_{13}=0$ and $a_{23}=0$. Also, if we choose $a_{12}=\lambda$, the representation matrix of the mapping **A** can be written as

$$\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$, (Yaylacı, 2006).

Now let us assume that $x$ is timelike vector and $x=u_1=(1, 0, 0)$. So we have

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

hence we get $a_{12}=0$ and $a_{13}=0$. Also, if we choose $a_{23}=\lambda$, the repre-sentation matrix of the mapping **A** can be written as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$.

And finally, we assume that $x$ is lightlike vector and $x=(1,1,0)$. Then, we have

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

hence we get $a_{12}=0$ and $a_{13}=-a_{23}$. Also, if we choose $a_{13}=\lambda$, the representation matrix of the mapping **A** can be written as

$$\begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & -\lambda & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$.

So, the results of this theorem can be extended to the general case $\mathbb{E}_1^{2n+1}$ as in the following:

Corollary 1. Let $\mathbb{E}_1^{2n+1}$ be a (2n+1)-dimensional Lorentzian vector space over $\mathbb{R}$ and **A** be a linear mapping in $\mathbb{E}_1^{2n+1}$ given by a skew-symmetric matrix $A$ with respect to a pseudo-orthonormal basis $\phi$. So, if there exists a non-zero solutions of the equation $A(x)=0$, then the normal forms of the matrix $A$ can be written as:

Case 1: If the vector $x$ is spacelike,

$$A=\begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

Case 2: If the vector $x$ is timelike,

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda_n & 0 \end{bmatrix},$$

Case 3: If the vector $x$ is lightlike,

$$A=\begin{bmatrix} 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{2n-2} & \lambda_{2n-1} \\ 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{2n-2} & \lambda_{2n-1} \\ \lambda_1 & -\lambda_1 & 0 & -\lambda_{2n} & -\lambda_{2n+1} & \cdots & -\lambda_{4n-4} & -\lambda_{4n-3} \\ \lambda_2 & -\lambda_2 & \lambda_{2n} & 0 & -\lambda_{4n-2} & \cdots & -\lambda_{6n-7} & -\lambda_{6n-6} \\ \lambda_3 & -\lambda_3 & \lambda_{2n+1} & \lambda_{4n-2} & 0 & \cdots & -\lambda_{8n-11} & -\lambda_{8n-10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{2n-2} & -\lambda_{2n-2} & \lambda_{4n-4} & \lambda_{6n-7} & \lambda_{8n-11} & \cdots & 0 & -\lambda_{n(2n-1)} \\ \lambda_{2n-1} & -\lambda_{2n-1} & \lambda_{4n-3} & \lambda_{6n-6} & \lambda_{8n-10} & \cdots & \lambda_{n(2n-1)} & 0 \end{bmatrix},$$

where $\lambda_i \in \mathbb{R}/\{0\}, 1 \le i \le n(2n-1)$.

Theorem 3. The integral curves of a linear vector field on pseudo-sphere $\mathbb{S}_1^2$ (or on hyperbolic space $\mathbb{H}_0^2$) are Lorentzian circles lying in parallel planes.

Proof. From Theorems 1 and 2, the value of the linear vector field $X$ for all points $P=(x,y,z)\in\mathbb{S}_1^2$ can be written as below

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 & a \\ \lambda & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

or

$$X(P)=(\lambda y+a, \lambda x+b, 0).$$

On the other hand, if a curve $\alpha : \mathrm{I}\subset\mathbb{R}\to\mathbb{S}_1^2$ is an integral curve of $X$, then we get the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= \lambda y+a \\ \frac{dy}{dt} &= \lambda x+b \\ \frac{dz}{dt} &= 0. \end{aligned}$$

For the sake of simplicity, we take $\lambda=1$. If this system is solved, the integral curves of $X$ are obtained as

$$\begin{aligned} \alpha(t)=(&B\sinh t+D\cosh t-b, B\cosh t \\ &+D\sinh t+a, d), \end{aligned}$$

where $B$ and $D$ are any constants. The curve $\alpha(t)$ is Lorentzian circle on $\mathbb{S}_1^2$.

## 4. Classification of integral curves of a linear vector field in $\mathbb{E}_1^{2n+1}$

Let $\mathbb{E}_1^{2n+1}$ be a (2n+1)-dimensional Lorentzian vector space over $\mathbb{R}$ and $X$ be a linear vector field on $\mathbb{E}_1^{2n+1}$. Let **A** be a linear mapping in $\mathbb{E}_1^{2n+1}$ given by a skew-symmetric matrix $A$ with respect to a pseudo-orthonormal base $\phi$. So, we have the following theorem.

Theorem 4. Let $X$ be a linear vector field in $\mathbb{E}_1^{2n+1}$ determined by the matrix

$$\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$$

with respect to a pseudo-orthonormal frame $\{0; u_1, u_2,...,u_{2n+1}\}$, whose $A$ is a normal formed skew-symmetric matrix and $C$ is a (2n+1)x1 column matrix such that

$$C = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \\ a_{2n} \\ a_{2n+1} \end{bmatrix}.$$

Then the integral curves of $X$ have the following properties:

i) If the rank of the matrix $[AC]$ is equal to 2k+1, $1 \le k \le n$, then the integral curves are the generalized helixes,

ii) If the rank of the matrix $[AC]$ is equal to 2k, $1 \le k \le n$, then the integral curves are Lorentzian circles in parallel planes whose centers lie on a same straight line perpendicular to those planes,

iii) If the rank of the matrix $[AC]$ is equal to 1, then the integral curves are the parallel straight lines.

Proof. i) If $rank[AC] = 2k+1$, for $k = n$. Then, suppose that the skew-symmetric matrix $A$ be as in Case 1. Let $X$ be a linear vector field in $\mathbb{E}_1^{2n+1}$. Then the value of the linear vector field $X$ for all points $P = (x_1, x_2,..., x_{2n+1}) \in E_1^{2n+1}$ can be written as

$$\begin{bmatrix} X(P) \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P \\ 1 \end{bmatrix}$$

or

$$X(P) = (\lambda_1 x_2 + a_1, \lambda_1 x_1 + a_2, \lambda_2 x_4 + a_3, -\lambda_2 x_3 + a_4, ..., \lambda_n x_{2n} + a_{2n-1}, -\lambda_n x_{2n-1} + a_{2n}, a_{2n+1}).$$

Here, for the sake of simplicity, we choose $\lambda_i = 1$, $1 \le i \le n$. If $\alpha$ is an integral curve of the linear vector field $X$, then by the definition of the integral curve we can write

$$\frac{d\alpha(t)}{dt} = X(\alpha(t)), \ \forall t \in I.$$

So, the integral curve with the initial condition $\alpha(t) = (x_1,...,x_{2n+1})$ is a solution of the differential equation

$$\frac{d\alpha(t)}{dt} = X(P).$$

Hence, we get the system of differential equations

$$\frac{d\alpha_2(t)}{dt} = x_1 + a_2, \ for \ i = 2,$$

and

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} x_{i+1} + a_i, & i = 2k-1, 1 \le k \le n, \\ -x_{i-1} + a_i, & i = 2k, 2 \le k \le n, \\ a_{2n+1}, & i = 2n+1 \end{cases}.$$

The general solutions of these equations are

$$\begin{array}{lcl} \alpha_1 & = & A_1 \sinh t + B_1 \cosh t - a_2, \\ \alpha_2 & = & A_1 \cosh t + B_1 \sinh t + a_1, \\ \vdots & \vdots & \vdots \\ \alpha_{2n-1} & = & A_n \sin t - B_n \cos t + a_{2n}, \\ \alpha_{2n} & = & A_n \cos t + B_n \sin t - a_{2n-1}, \\ \alpha_{2n+1} & = & ct + d. \end{array}$$

So, the integral curve of the linear vector field $X$ is obtained as

$$\begin{aligned} \alpha(t) = (&A_1 \sinh t + B_1 \cosh t - a_2, A_1 \cosh t + B_1 \sinh t \\ &+ a_1, A_2 \sin t - B_2 \cos t + a_4, A_2 \cos t + B_2 \sin t - a_3 \\ &,..., A_n \sin t - B_n \cos t + a_{2n}, A_n \cos t + B_n \sin t - a_{2n-1}, \\ &ct + d). \end{aligned}$$

Now, we can examine the character of the integral curve. If we take into consideration the derivations of $\alpha(t)$, we get linearly independent vectors $\alpha'$, $\alpha''$, $\alpha'''$, $\alpha^{(4)}$ and $\alpha^{(5)}$. The other higher order derivations are linear dependent. So, only Frenet quintette can be constructed on the curve $\alpha(t)$. Therefore, there exists four curvatures $k_1$, $k_2$, $k_3$ and $k_4$. In order to show that $\alpha(t)$ is a generalized helix in $\mathbb{E}_1^{2n+1}$, we must show that

$$\frac{k_1}{k_2} = const \ \text{and} \ \frac{k_3}{k_4} = const.$$

For this aim, firstly, let us calculate the velocity of $\alpha(t)$. The velocity of the curve is obtained as

$$\langle \alpha'(t), \alpha'(t) \rangle = B_1^2 - A_1^2 + B_2^2 + A_2^2 + ... + B_n^2 + A_n^2 + c^2.$$

Assume that $\alpha(t)$ is a timelike curve, that is $\langle \alpha'(t), \alpha'(t) \rangle = -1$. Thus, we have a pseudo-orthogonal system by the Gramm-Schmidt method:

$$
\begin{aligned}
u_1 &= (A_1\cosh t + B_1\sinh t, A_1\sinh t + B_1\cosh t,\\
&\quad A_2\cos t + B_2\sin t, -A_2\sin t + B_2\cos t, \dots,\\
&\quad A_n\cos t + B_n\sin t, -A_n\sin t + B_n\cos t, c),\\
u_2 &= (A_1\sinh t + B_1\cosh t, A_1\cosh t + B_1\sinh t,\\
&\quad -A_2\sin t + B_2\cos t, -A_2\cos t - B_2\sin t, \dots,\\
&\quad -A_n\sin t + B_n\cos t, -A_n\cos t - B_n\sin t, 0),\\
u_3 &= (\gamma-1)\left(\left(\frac{\gamma+1}{\gamma-1}\right)(A_1\cosh t + B_1\sinh t),\right.\\
&\quad \left(\frac{\gamma+1}{\gamma-1}\right)(A_1\sinh t + B_1\cosh t), A_2\cos t +\\
&\quad B_2\sin t, -A_2\sin t + B_2\cos t, \dots, A_n\cos t +\\
&\quad \left. B_n\sin t, -A_n\sin t + B_n\cos t, \frac{\gamma}{\gamma-1}c\right),\\
u_4 &= \left(\frac{\gamma-\theta}{\gamma}\right)\left(\left(\frac{\gamma+\theta}{\gamma-\theta}\right)(A_1\sinh t + B_1\cosh t,\right.\\
&\quad A_1\cosh t + B_1\sinh t), A_2\sin t - B_2\cos t,\\
&\quad A_2\cos t + B_2\sin t, \dots, A_n\sin t - B_n\cos t,\\
&\quad A_n\cos t + B_n\sin t, 0),
\end{aligned}
$$

and

$$
\begin{aligned}
u_5 &= (\mu_1(A_1\cosh t + B_1\sinh t, A_1\sinh t + B_1\cosh t),\\
&\quad \mu_2(A_2\cos t + B_2\sin t, -A_2\sin t + B_2\cos t, \dots,\\
&\quad A_n\cos t + B_n\sin t, -A_n\sin t + B_n\cos t),\\
&\quad \left.\left(\frac{\gamma^2(1-\theta)}{(\theta-\gamma^2)(\gamma-1)} - \theta\right)c\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma &= -A_1^2 + B_1^2 - A_2^2 - B_2^2 - \dots - A_n^2 - B_n^2,\\
\theta &= A_1^2 - B_1^2 - A_2^2 - B_2^2 - \dots - A_n^2 - B_n^2
\end{aligned}
$$

and

$$
\mu_1 = \frac{(\gamma+\theta)(1-\theta)}{\theta-\gamma^2}, \ \mu_2 = \frac{(\theta-\gamma)(1-\theta)}{\theta-\gamma^2}.
$$

If we use

$$
k_i(s) = \varepsilon_{i+1} < u_i'(s), u_{i+1}(s) >
$$

for the curvature functions $k_i(s)$, $1 \le i \le 4$, we get

$$
k_1 = -\varepsilon_2\gamma,\ k_2 = \varepsilon_3(\gamma^2-\theta),\ k_3 = -\varepsilon_4\frac{\gamma^2-\theta^2}{\gamma}
$$

and $k_4 = \varepsilon_5\dfrac{(\theta^2-\gamma^2)(\theta-1)}{\theta-\gamma^2}$.

So, we obtain

$$
\frac{k_1}{k_2} = const \ \text{ and } \ \frac{k_3}{k_4} = const.
$$

This means that the curve $\alpha(t)$ is a generalized helix.

Assume that $\alpha(t)$ is a spacelike curve, that is $<\alpha'(t), \alpha'(t)>=1$. Thus, we have a pseudo-orthogonal system by the Gramm-Schmidt method:

$$
\begin{aligned}
u_1 &= (A_1\cosh t + B_1\sinh t, A_1\sinh t + B_1\cosh t,\\
&\quad A_2\cos t + B_2\sin t, -A_2\sin t + B_2\cos t, \dots,\\
&\quad A_n\cos t + B_n\sin t, -A_n\sin t + B_n\cos t, c),\\
u_2 &= (A_1\sinh t + B_1\cosh t, A_1\cosh t + B_1\sinh t,\\
&\quad -A_2\sin t + B_2\cos t, -A_2\cos t - B_2\sin t, \dots,\\
&\quad -A_n\sin t + B_n\cos t, -A_n\cos t - B_n\sin t, 0),\\
u_3 &= (\gamma+1)\left(\left(\frac{1-\gamma}{\gamma+1}\right)(A_1\cosh t + B_1\sinh t),\right.\\
&\quad \left(\frac{1-\gamma}{\gamma+1}\right)(A_1\sinh t + B_1\cosh t), A_2\cos t +\\
&\quad B_2\sin t, -A_2\sin t + B_2\cos t, \dots, A_n\cos t +\\
&\quad \left. B_n\sin t, -A_n\sin t + B_n\cos t, \frac{-\gamma}{\gamma+1}c\right),\\
u_4 &= \left(\frac{\gamma-\theta}{\gamma}\right)\left(\left(\frac{\gamma+\theta}{\gamma-\theta}\right)(A_1\sinh t + B_1\cosh t,\right.\\
&\quad A_1\cosh t + B_1\sinh t), A_2\sin t - B_2\cos t,\\
&\quad A_2\cos t + B_2\sin t, \dots, A_n\sin t - B_n\cos t,\\
&\quad \left. A_n\cos t + B_n\sin t, 0\right),\\
u_5 &= (\mu_1(A_1\cosh t + B_1\sinh t, A_1\sinh t +\\
&\quad B_1\cosh t), \mu_2(A_2\cos t + B_2\sin t, -A_2\sin t\\
&\quad + B_2\cos t, \dots, A_n\cos t + B_n\sin t, -A_n\sin t\\
&\quad \left. + B_n\cos t), \left(\theta - \frac{\gamma^2(1-\theta)}{(\theta+\gamma^2)(\gamma+1)}\right)c\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma &= -A_1^2 + B_1^2 - A_2^2 - B_2^2 - \dots - A_n^2 - B_n^2,\\
\theta &= A_1^2 - B_1^2 - A_2^2 - B_2^2 - \dots - A_n^2 - B_n^2
\end{aligned}
$$

and

$$
\mu_1 = \frac{(\gamma+\theta)(1+\theta)}{\theta+\gamma^2}, \ \mu_2 = \frac{(\theta-\gamma)(1+\theta)}{\theta+\gamma^2}
$$

Therefore, the curvature functions $k_i(s)$, $1 \le i \le 4$, are found as

$$k_1 = \varepsilon_2 \gamma,\ k_2 = \varepsilon_3(\gamma^2 + \theta),\ k_3 = \varepsilon_4 \frac{\gamma^2 + \theta^2}{\gamma}$$

and $k_4 = -\varepsilon_5 \dfrac{(\gamma^2 - \theta^2)(1+\theta)}{\theta + \gamma^2}$.

So, we obtain

$$\frac{k_1}{k_2} = const \text{ and } \frac{k_3}{k_4} = const.$$

This means that the curve $\alpha(t)$ is a generalized helix.

Now, let us assume that the skew-symmetric matrix $A$ be as in Case 2. Then the value of the linear vector field $X$ for all points $P = (x_1, x_2, ..., x_{2n+1}) \in E_1^{2n+1}$ can be written as

$$X(P) = (a_1, -\lambda_1 x_3 + a,\ \lambda_1 x_2 + a_3, \lambda_2 x_5 + a_4, -\lambda_2 x_4 + a_5, ..., \lambda_n x_{2n+1} + a_{2n}, -\lambda_n x_{2n} + a_{2n+1}).$$

Here, for the sake of simplicity, we choose $\lambda_i = 1,\ 1 \le i \le n$. If $\alpha$ is an integral curve of the linear vector field $X$, then by the definition of the integral curve, we get the system of differential equations

$$\frac{d\alpha_1(t)}{dt} = a_1,$$
$$\frac{d\alpha_2(t)}{dt} = -x_3 + a_2, \qquad \frac{d\alpha_3(t)}{dt} = x_2 + a_3,$$
$$\frac{d\alpha_4(t)}{dt} = x_5 + a_4, \qquad \frac{d\alpha_5(t)}{dt} = -x_4 + a_5,$$
$$\vdots$$
$$\frac{d\alpha_{2n}(t)}{dt} = x_{2n+1} + a_{2n}, \qquad \frac{d\alpha_{2n+1}(t)}{dt} = -x_{2n} + a_{2n+1}.$$

If we solve this system of differential equations, the integral curve of the linear vector field $X$ is obtained as

$$\begin{aligned}\alpha(t) = (&a_1 t + d, A_1 \cos t + B_1 \sin t - a_3, A_1 \sin t \\ &- B_1 \cos t + a_2, A_2 \cos t + B_2 \sin t + a_5, \\ &- A_2 \sin t + B_2 \cos t - a_4, ..., A_n \cos t + \\ &B_n \sin t + a_{2n+1}, - A_n \sin t + B_n \cos t - a_{2n}).\end{aligned}$$

The curve $\alpha(t)$ is a generalized helix. We can prove this idea with the same method which was used above.

If $rank[AC] = 2k+1 = r+1,\ r = 2,4,...,2n-2,$ then the linear first order system of differential equation for $\alpha_i(t)$, $1 \le i \le 2n+1$, becomes

$$\frac{d\alpha_2(t)}{dt} = x_1 + a_2,\ for\ i = 2,$$

and

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} x_{i+1} + a_i, & i = 2k-1, 1 \le k \le \frac{r}{2}, \\ -x_{i-1} + a_i, & i = 2k,\ 2 \le k \le \frac{r}{2}, \\ a_{r+1}, & i = r+1, \\ 0, & r+2 \le i \le 2n+1 \end{cases}.$$

This system of differential equations has the solution

$$\begin{aligned}\alpha(t) \ = \ (&A_1 \sinh t + B_1 \cosh t - a_2, A_1 \cosh t + \\ &B_1 \sinh t + a_1, A_2 \sin t - B_2 \cos t + a_4, \\ &A_2 \cos t + B_2 \sin t - a_3, ..., A_{r/2} \sin t - \\ &B_{r/2} \cos t + a_r, A_{r/2} \cos t + B_{r/2} \sin t - a_{r-1}, \\ &a_{r+1} t + d, d_{r+2}, ..., d_{2n+1}).\end{aligned}$$

It is easy to show that the curve $\alpha(t)$ is a generalized helix.

ii) Let $rank[AC] = 2k, 1 \le k \le n.$ Then:

a) If $rank[AC] = 2n,\ k = n,$ then the linear first order system of differential equation for $\alpha_i(t)$, $1 \le i \le 2n+1$, becomes

$$\frac{d\alpha_2(t)}{dt} = x_1 + a_2,\ for\ i = 2,$$

and

$$\frac{d\alpha_i(t)}{dt} = \begin{cases} x_{i+1} + a_i, & i = 2k-1, 1 \le k \le n, \\ -x_{i-1} + a_i, & i = 2k,\ 2 \le k \le n, \\ 0, & i = 2n+1 \end{cases}.$$

Hence, the solutions of this system is

$$\begin{aligned}\alpha(t) = (&A_1 \sinh t + B_1 \cosh t - a_2, A_1 \cosh t \\ &+ B_1 \sinh t + a_1, A_2 \sin t - B_2 \cos t + a_4, \\ &A_2 \cos t + B_2 \sin t - a_3, ..., A_n \sin t - B_n \cos t \\ &+ a_{2n}, A_n \cos t + B_n \sin t - a_{2n-1}, d).\end{aligned}$$

It is easy to show that the curve $\alpha(t)$ is a Lorentzian circle.

b) If $rank[AC] = r,\ r = 2,4,...,2n-2.$ Then, the linear first order system of differential equation for $\alpha_i(t)$, $1 \le i \le 2n+1$, becomes

$$\frac{d\alpha_2(t)}{dt} = x_1 + a_2,\ for\ i = 2,$$

and

$$\frac{d\alpha_i(t)}{dt}=\begin{cases} x_{i+1}+a_i, & i=2k-1, 1\le k\le \frac{r}{2},\\ -x_{i-1}+a_i, & i=2k, 2\le k\le \frac{r}{2},\\ 0, & r+1\le i\le 2n+1\end{cases}.$$

If we solve this system, we get

$$\alpha(t)=(A_1\sinh t+B_1\cosh t-a_2, A_1\cosh t+ B_1\sinh t+a_1, A_2\sin t-B_2\cos t+a_4, A_2\cos t + B_2\sin t-a_3,\ldots, A_{r/2}\sin t-B_{r/2}\cos t+a_r, A_{r/2}\cos t+B_{r/2}\sin t-a_{r-1}, d_{r+2},\ldots,d_{2n+1}).$$

So, the curve $\alpha(t)$ is a Lorentzian circle.

iii) If $rank[AC]=1$, then $\lambda_i=0$ which gives us a linear first order system of the differential equations. This system has the solutions $\alpha(t)$ which are the parallel straight lines in $\mathbb{E}_1^{2n+1}$.

If the vector x satisfying the equation $A(x)=0$ is lightlike, then taking into consideration the skew-symmetric matrix $A$ as in Case 3, the value of the linear vector field $X$ for all points $P=(x_1,x_2,\ldots,x_{2n+1})\in E_1^{2n+1}$ can be written as

$$\begin{aligned}X(P)=(&\lambda_1x_3+\lambda_2x_4+\ldots+\lambda_{2n-1}x_{2n+1}+a_1,\\ &\lambda_1x_3+\lambda_2x_4+\ldots+\lambda_{2n-1}x_{2n+1}+a_2,\ldots,\\ &\lambda_{2n-2}x_1-\lambda_{2n-2}x_2+\lambda_{4n-4}x_3+\ldots\\ &-\lambda_{n(2n-1)}x_{2n+1}+a_{2n}, \lambda_{2n-1}x_1-\lambda_{2n-1}x_2+\\ &\lambda_{4n-3}x_3+\ldots-\lambda_{n(2n-1)}x_{2n}+a_{2n+1}).\end{aligned}$$

Here, we can choose $\lambda_i=1, 1\le i\le n(2n-1)$. If $\alpha$ is an integral curve of the linear vector field $X$, then by the definition of the integral curve, we get the system of differential equations

$$\begin{aligned}\frac{d\alpha_1(t)}{dt}&=x_3+x_4+\ldots+x_{2n+1}+a_1,\\ \frac{d\alpha_2(t)}{dt}&=x_3+x_4+\ldots+x_{2n+1}+a_2,\\ &\vdots\\ \frac{d\alpha_{2n}(t)}{dt}&=x_1-x_2+x_3+\ldots-x_{2n+1}+a_{2n},\\ \frac{d\alpha_{2n+1}(t)}{dt}&=x_1-x_2+x_3+\ldots+x_{2n}+a_{2n+1}.\end{aligned}$$

We can not find the solutions of this system for the general case. On the other hand, the integral curves of the linear vector field $X$ for special case $n=1$ are as in the following

$$\alpha(t)=(c_1t^3+c_2t^2+c_3t+k_1, c_1t^3+c_2t^2 +c_3t+k_2, c_4t^2+c_5t+k_3),$$

where $c_i=1,\ 1\le i\le 5$, and $k_j=1,\ 1\le j\le 3$, are constants.

Example 1. Consider the vector field $X:\mathbb{E}_1^3\to\mathbb{E}_1^3$ defined by $X(x,y,z)=(y,x,0)$. Then the value of the vector field $X$ for all points $P=(x,y,z)\in\mathbb{E}_1^3$ can be written as

$$\begin{bmatrix}X(P)\\1\end{bmatrix}=\begin{bmatrix}0&1&0&a\\1&0&0&b\\0&0&0&c\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}.$$

If $\alpha:I\subset\mathbb{R}\to\mathbb{E}_1^3$ is an integral curve of the vector field $X$, then by the definition of the integral curve, we get the system of differential equations as

$$\begin{aligned}\frac{dx}{dt}&=y+a,\\ \frac{dy}{dt}&=x+b,\\ \frac{dz}{dt}&=c.\end{aligned}$$

So, the solution of this system is

$$\alpha(t)=(B\sinh t+D\cosh t-b, B\cosh t+ D\sinh t+a, ct+d),$$

where $B$ and $D$ are any constants. The curve $\alpha(t)$ is a helix in $\mathbb{E}_1^3$.

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***Submitted*** **:** 07/01/2015

***Revised*** **:** 13/04/2015

***Accepted*** **:** 21/04/2015

# مصفوفات متناظرة تخالفياً ومنحنيات تكاملية في فضاءات لورنتزية

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## خلاصة

نقدم في هذا البحث بعض نتائج وجود حول منحنيات تكاملية لحقل متجهات خطي على فضاء لورنتزي ثلاثي البعدية $\mathbb{E}_1^3$ ، علاوة على ذلك ، نحصل على تصنيف للمنحنيات التكاملية ( انسياب خطوط ) لحقل متجهات خطي و الأشكال الطبيعية للمصفوفة الآتية من تطبيق $\mathbf{A}$ في فضاءات لورنتزية $\mathbb{E}_1^3$ و $\mathbb{E}_1^{2n+1}$