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Arithmetic properties of Ramanujan's general partition function for modulo 11

Belakavadi R. Srivatsa Kumar[1,*], Ramakrishna Narendra[2], Karpenahalli R. Rajanna[2]

[1]*Dept. of Mathematics, Manipal Institute of Technology*
Manipal Acedemy of Higher Education, Manipal - 576 104, India

[2]*Dept. of Mathematics, Acharya Institute of Technology*
Soladevanahalli, Bengaluru - 560 107, India

[*]*Corresponding author: sri_vatsabr@yahoo.com*

Abstract

In the present work, for the general partition function $p_r(n)$, we establish five new infinite families of congruences modulo 11. Our emphasis throughout this paper is to exhibit the use of q-identities to generate congruences of $p_r(n)$.

Keywords: Congruences; General partitions; q-identities; Theta-functions.

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1. Introduction

For $|xy| < 1$, Ramanujan's general theta function $f(x,y)$ is given by

$$f(x,y) := \sum_{k=-\infty}^{\infty} x^{\frac{k(k+1)}{2}} y^{\frac{k(k-1)}{2}}.$$

The function $f(a,b)$ enjoys the well-known Jacobi triple product identity (Berndt 1991, p.35),

$$f(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty,$$

where, here and throughout the paper, we will utilize the following q-shifted factorial and always assume $|q| < 1$.

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k).$$

One of the special cases of $f(x,y)$ as defined by S. Ramanujan (Berndt 1991) is as follows:

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty.$$

For convenience, we write $f_n = f(-q^n)$. Due to Euler, we have

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1},$$

where $p(n)$ is the number of partitions of n. S. Ramanujan initiated the general partition function $p_r(n)$ as

$$\sum_{n=0}^{\infty} p_r(n)q^n = f_1^r, \qquad (1)$$

for non-zero integer r. For partition function $p(n)$, Ramanujan's so called "most beautiful identity" is given by

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6},$$

which readily implies

$$p(5n+4) \equiv 0 \pmod 5.$$

The generalization of the congruences modulo powers of 5 and 7 for all $p_r(n)$ was proved by Ramanathan (1950). Later Atkin (1968) found that Ramanathan's proof is not correct. Further Newmann (1955, 1957a, 1957b) studied the function $p_r(n)$ and obtained several interesting congruences and identities involving $p_r(n)$. The functions $p_r(n)$ have been studied by many mathematicians. For the wonderful work one can see Atkin (1968), Baruah and Ojah (2011), Baruah and Sarmah (2013), Boylon (2004), Farkas and Kra (1999), Gandhi (1963), Gordon (1983), Kimming and Olsson (1992), and Saikia and Chetry (2018). Recently, Hammond and Lewis (2004) proved that

$$p_{-2}(5n+\ell) \equiv 0 \pmod 5,$$

where $\ell \in \{2,3,4\}$. Also Chen et al. (2014) proved

$$p_{-2}(25n+23) \equiv 0 \pmod{25}$$

by using modular forms. More recently, Tang (2018) proved some congruences modulo powers of 5 for $p_r(n)$ with $r \in \{2,6,7\}$. For example,

$$p_{-2}\left(5^{2\delta-1}n + \frac{7 \times 5^{2\delta-1}+1}{12}\right) \equiv p_{-6}\left(5^{2\delta}n + \frac{3 \times 5^{\delta}+1}{4}\right) \equiv p_{-7}\left(5^{2\delta-1}n + \frac{13 \times 5^{2\delta-1}+7}{24}\right) \equiv 0 \pmod{5^{\delta}}.$$

Motivated by the above work, we deduce new infinite families of congruences modulo 11 for $p_r(n)$ by using q-identities for any positive integer λ. The results in this paper are given below:

Theorem 1.1 If $\tau = 3, 6, 8, 9, 10$, then

$$p_{11\lambda+1}(11n+\tau) \equiv 0 \pmod{11}.$$

Theorem 1.2 If $\tau = 2, 4, 5, 7, 8, 9$, then

$$p_{11\lambda+3}(11n+\tau) \equiv 0 \pmod{11}.$$

Theorem 1.3 We have

$$p_{11\lambda+6}(11n+8) \equiv 0 \pmod{11}.$$

Theorem 1.4 If $1 \leq \tau \leq 10$, then

$$p_{121\lambda+1}(121n+11\tau+5) \equiv 0 \pmod{11}.$$

Theorem 1.5 If $1 \leq \tau \leq 10$, then

$$p_{121\lambda+2}(121n+11\tau+10) \equiv 0 \pmod{11}.$$

2. Proofs of Theorems 1.1 − 1.5

Proof of Theorem 1.1: Setting $r = 11\lambda+1$ in (1), we have

$$\sum_{n=0}^{\infty} p_{11\lambda+1}(n)q^n = f_1^{11\lambda+1} = f_1^{11\lambda} f_1. \tag{2}$$

From the binomial theorem, it follows that

$$f_1^{11} \equiv f_{11} \pmod{11}. \tag{3}$$

Substituting (3) into (2), we see that

$$\sum_{n=0}^{\infty} p_{11\lambda+1}(n)q^n \equiv f_{11}^{\lambda} f_1 \pmod{11}. \tag{4}$$

From Bernd (1991, p. 363, Entry 6(iii)), we have

$$f_1 = f_{121}(A(q^{11}) - qB(q^{11}) - q^2C(q^{11}) + q^5 + q^7D(q^{11}) - q^{15}E(q^{11})), \tag{5}$$

where

$$A := A(q^{11}) = \frac{f(-q^{44}, q^{77})}{f(-q^{22}, -q^{99})},$$

$$B := B(q^{11}) = \frac{f(-q^{22}, q^{99})}{f(-q^{11}, -q^{110})},$$

$$C := C(q^{11}) = \frac{f(-q^{55}, q^{66})}{f(-q^{33}, -q^{88})},$$

$$D := D(q^{11}) = \frac{f(-q^{33}, q^{88})}{f(-q^{44}, -q^{77})}$$

and

$$E := E(q^{11}) = \frac{f(-q^{11}, q^{110})}{f(-q^{55}, -q^{66})}.$$

Invoking (5) in (4), it is observed that

$$\sum_{n=0}^{\infty} p_{11\lambda+1}(n)q^n \equiv f_{11}^{\lambda} f_{121}(A - qB - q^2C + q^5 + q^7D - q^{15}E) \pmod{11}. \tag{6}$$

Selecting the terms containing $q^{11n+\tau}$ for $\tau = 3, 6, 8, 9, 10$ on both sides of (6), we obtain the required congruence.

Proof of Theorem 1.2: Setting $r = 11\lambda+3$ in (1) and then using (3), we obtain

$$\sum_{n=0}^{\infty} p_{11\lambda+3}(n)q^n = f_1^{11\lambda+3} = f_1^{11\lambda} f_1^3 \equiv f_{11}^{\lambda} f_1^3 \pmod{11}. \tag{7}$$

From Berndt (1991, p. 39, Entry 24(ii)), we have

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}, \tag{8}$$

and it follows that

$$f_1^3 = I_0(q^{11}) - 3qI_1(q^{11}) + 5q^3I_2(q^{11}) - 7q^6I_3(q^{11}) + 9q^{10}I_4(q^{11}) - 11q^{15}I_5(q^{11}) \equiv I_0(q^{11}) - 3qI_1(q^{11}) + 5q^3I_2(q^{11}) - 7q^6I_3(q^{11}) + 9q^{10}I_4(q^{11}) \pmod{11} \quad (9)$$

where I_0, I_1, I_2, I_3 and I_4 are the series with integral powers of q^{11}: Invoking (9) in (7), we have

$$\sum_{n=0}^{\infty} p_{11\lambda+3}(n)q^n \equiv f_{11}^{\lambda}(I_0 - 3qI_1 + 5q^3I_2 - 7q^6I_3 + 9q^{10}I_4) \pmod{11}. \quad (10)$$

Selecting the terms containing $q^{11n+\tau}$ for $\tau = 2$, 4, 5, 7, 8, 9 on both sides of (10), we obtain the required congruence.

Proof of Theorem 1.3: Setting $r = 11\lambda + 6$ in (1) and then using (3), we obtain

$$\sum_{n=0}^{\infty} p_{11\lambda+6}(n)q^n \equiv f_{11}^{\lambda} f_1^6 \pmod{11}. \quad (11)$$

On squaring (9) and then grouping, we deduce

$$f_1^6 \equiv I_0^2(q^{11}) + I_1(q^{11})I_4(q^{11}) + 5q(I_0(q^{11}) \times I_1(q^{11}) + I_3^2(q^{11})) + q^2(9I_1^2(q^{11}) + 2I_2(q^{11})I_4(q^{11})) + 10q^3I_0(q^{11})I_2(q^{11}) + 3q^4I_1(q^{11})I_2(q^{11}) + 6q^5I_3(q^{11})I_4(q^{11}) + q^6(8I_0(q^{11})I_3(q^{11}) + 3I_2^2(q^{11})) + 9q^7I_1(q^{11})I_3(q^{11}) + q^9(7I_2(q^{11})I_3(q^{11}) + 4I_4^2(q^{11})) + 7q^{10}I_0(q^{11})I_4(q^{11}) \pmod{11}, \quad (12)$$

Invoking (12) in (11), we have

$$\sum_{n=0}^{\infty} p_{11\lambda+6}(n)q^n \equiv f_{11}^{\lambda}(I_0^2 + I_1I_4 + 5q(I_0I_1 + I_3^2) + q^2(9I_1^2 + 2I_2I_4) + 10q^3I_0I_2 + 3q^4I_1I_2 + 6q^5I_3I_4 + q^6(8I_0I_3 + 3I_2^2) + 9q^7I_1I_3 + q^9(7I_2I_3 + 4I_4^2) + 7q^{10}I_0I_4) \pmod{11}. \quad (13)$$

Selecting the terms containing q^{11n+8} on both sides of (13), we obtain the required congruence.

Proof of Theorem 1.4: Setting $r = 121\lambda + 1$ in (1) and then using (3), we obtain

$$\sum_{n=0}^{\infty} p_{121\lambda+1}(n)q^n \equiv f_{121}^{\lambda} f_1 \pmod{11}. \quad (14)$$

Invoking (5) in (14), we have

$$\sum_{n=0}^{\infty} p_{121\lambda+1}(n)q^n \equiv f_{121}^{\lambda+1}(A - qB - q^2C + q^5 + q^7D - q^{15}E) \pmod{11}. \quad (15)$$

Selecting the terms containing q^{11n+5} on both sides of (15), dividing by q^5 and then letting q to $q^{1/11}$, we obtain

$$\sum_{n=0}^{\infty} p_{121\lambda+1}(11n+5)q^n \equiv f_{11}^{\lambda+1} \pmod{11}. \quad (16)$$

Selecting the terms containing $q^{11n+\tau}$ for $1 \le \tau \le 10$ on both sides of (16), we obtain the required congruence.

Proof of Theorem 1.5: Setting $r = 121\lambda + 2$ in (1) and then employing (3), we have

$$\sum_{n=0}^{\infty} p_{121\lambda+2}(n)q^n \equiv f_{121}^{\lambda} f_1^2 \pmod{11}. \quad (17)$$

Squaring (5), we find that

$$f_1^2 = f_{121}^2(A^2 - 2qAB + q^2(B^2 - 2AC) + 2q^3BC + q^4C^2 + 2q^5A - 2q^6B + q^7 \times 2(AD - C) - 2q^8BD - 2q^9CD + q^{10} + 2q^{12}D + q^{14}D^2 - 2q^{15}AE + 2q^{16}BE + 2q^{17}CE - 2q^{20}E - 2q^{22}DE + q^{30}E^2). \quad (18)$$

Invoking (18) in (17), it is observed that

$$\sum_{n=0}^{\infty} p_{121\lambda+2}(n)q^n \equiv f_{121}^{\lambda+2}(A^2 - 2qAB + q^2(B^2 - 2AC) + 2q^3BC + q^4C^2 + 2q^5A - 2q^6B + 2q^7(AD - C) - 2q^8 \times BD - 2q^9CD + q^{10} + 2q^{12}D + q^{14}D^2 - 2q^{15}AE + 2q^{16}BE + 2q^{17}CE - 2q^{20}E - 2q^{22}DE + q^{30}E^2) \pmod{11}. \quad (19)$$

Selecting the terms containing $q^{121\lambda+10}$ on both sides, dividing throughout by q^{10} and letting q by $q^{1/11}$, we deduce that

$$\sum_{n=0}^{\infty} p_{121\lambda+2}(121n+10)q^n \equiv f_{11}^{\lambda+2} \pmod{11}. \quad (20)$$

Selecting the terms containing $q^{121\lambda+10}$ for $1 \leq \tau \leq 10$ on both sides of (20), we obtain the required congruence.

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