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Scheduling stochastic jobs on a single machine to minimize weighted number of tardy jobs

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ABSTRACT

An important scheduling problem in manufacturing and service organizations is the on-time deliveries of products and services. This paper addresses a single machine scheduling problem wherein processing times and/or due-dates are random variables and fixed weights (penalties) are imposed on late jobs. The objective is to find the schedule that minimizes the expected weighted number of tardy jobs. The problem is NP-hard; however, we study the three resulting scenarios: the scenario with stochastic processing times and stochastic due-dates, the scenario with deterministic processing times and stochastic due-dates, and the scenario with stochastic processing times and deterministic due-dates. We prove that special cases of these scenarios are solvable optimally in polynomial time, and introduce heuristic methods for their general cases. Our computational results show that the proposed heuristics perform well in producing the optimal or near optimal solutions. The illustrative examples and computational results also demonstrate that the stochasticity of processing times and/or due-dates can affect scheduling decisions.

Keywords: Number of tardy jobs; scheduling; single machine; stochastic.

INTRODUCTION

Minimizing the weighted number of tardy jobs is an important real-world scheduling problem. It arises in most production and service environments where it is desirable to complete jobs with different weights (penalties) on or before their due-dates. In these environments, schedulers should frequently decide whether to schedule a job based on its processing time, due-date, and the penalty for tardy delivery to improve the system performance. For example, it is common to measure the weighted number of tardy jobs or the percentage of on-time shipments to evaluate the performance of a semiconductor manufacturing facility or an automobile assembly line.

A majority of the literature on scheduling to minimize the (weighted) number of tardy jobs deals with the deterministic single machine problem wherein job attributes such as processing times, due-dates, and tardy weights are fixed known quantities (Moore, 1968; Baptiste, 1999; Erel & Ghosh, 2007; Lee *et al.*, 2011; Steiner & Zhang, 2011; Lee & Kim, 2012; and Jafari & Moslehi, 2012). There are also many studies on deterministic single machine scheduling involving multiple criteria with one of the criteria being the (weighted) number of tardy jobs (Jolai *et al.*, 2007; Wan & Yen, 2009; Erenay *et al.*, 2010; Molaee *et al.*, 2011; Reisi & Moslehi, 2011; Hoogeveen & T'kindt, 2012; and Rasti-Barzoki *et al.*, 2013).

The problem of minimizing the (weighted) number of tardy jobs has been also investigated in other deterministic scheduling environments. For example, Chiang & Fu (2009) examine a job shop scheduling problem with sequence-dependent setup times. Desprez *et al.* (2009) consider a hybrid flowshop. Mosheiov & Sarig (2010) and Aldowaisan & Allahverdi (2012) study *m*-machine flowshop problems. Mosheiov & Oron (2012) address a proportionate flowshop, and Panwalkar & Koulamas (2012) investigate a two-machine flowshop. Varmazyar & Salmasi (2012) and Dhouib *et al.* (2013) examine, respectively, a flowshop and a permutation flowshop with sequence-dependent setup times.

In contrast to the deterministic scheduling problems with the objective of minimizing the (weighted) number of tardy jobs, there is a limited amount of literature on their stochastic counterparts. It is important to incorporate variations of job attributes into scheduling decisions because schedulers encounter such deviations in real-world settings. The significance of research in stochastic scheduling is also emphasized by the interest in synchronous manufacturing, which recognizes that variations in job attributes disrupt schedules (e.g., Umble & Srikanth, 1995).

Most of the existing research on minimizing the (weighted) number of tardy jobs deals with some special cases of the stochastic single machine scheduling problem. For example, Balut (1973) presents a chance-constrained formulation for a case where jobs have a common tardiness weight, a common due-date, and normally distributed processing times. Boxma & Forst (1986) study cases where processing times or due-dates have independent and identical exponential distributions. De *et al.* (1991) examine a case with random processing times and an exponentially distributed common due-date. Assuming jobs have a common due-date and a common tardiness penalty, Pinedo (1983) analyzes a case with exponential processing times, while Seo *et al.* (2005) examine a case with normal processing times. Jang & Klein (2002) study a case with normally distributed processing times and a common due-date. Cai & Zhou (2005) consider a case with exponential processing times and random due-dates. Li & Alfa (2005) address a case with random processing times where the machine is subject to stochastic breakdown and the repair time is a random variable. Trietsch &

Baker (2008) solve cases with stochastically ordered processing times. Under some agreeable conditions, Tang *et al.* (2010) investigate a case with preemptive-resume scheduling with disruption where the starting time and the disruption duration are random variables. Elyasi & Salmasi (2013a) use chance-constrained programming to solve cases with normal and gamma processing times. Soroush (2007) extensively studies the related problem of minimizing the expected weighted number of early and tardy jobs on a single machine when jobs have deterministic due-dates and random processing times whose cumulative probability distributions for every pair of jobs cross in at most one point. He optimally solves special cases of this problem and presents a heuristic method for the general case. Soroush (2006) examines the same problem when processing times and due-dates are independently and identically distributed random variables. Elyasi & Salmasi (2013b) minimize the expected number of late jobs in a stochastic flowshop problem with normally distributed due-dates.

In this paper, we study a stochastic single machine scheduling problem in which processing times and/or due-dates are random variables and fixed weights (penalties) are imposed on the late jobs. The objective is to find the optimal sequence that minimizes the expected weighted number of tardy jobs. The problem is NP-hard to solve since its deterministic counterpart is NP-hard (Karp, 1972). However, we explore the three resulting scenarios: the scenario with stochastic processing times and stochastic due-dates, the scenario with deterministic processing times and stochastic due-dates, and the scenario with stochastic processing times and deterministic due-dates. (The last scenario is a special case of Soroush's (2007) problem of minimizing the expected weighted number of early and tardy jobs on a single machine with random processing times and deterministic due-dates.) We show that various special cases of these scenarios can be solved optimally in polynomial time, and introduce efficient heuristic methods for their general cases. Some computational results on the performance of these heuristics are also provided.

The organization of the rest of the paper is as follows. The problem formulation is given in the next section. This is followed by the investigation of three scenarios of the problem and the development of exact solution methods for various special cases. The heuristic methods and some computational results for the general cases are provided. Finally, we give a summary and some concluding remarks.

PROBLEM FORMULATION

A set N of n jobs is available at time zero to be processed sequentially, without preemption and idle time insertions, on a continuously available single machine. Let p_k, ξ_k, and ω_k denote, respectively, the processing time, due-date, and tardy weight (penalty) for each job k, $k \in N$, where p_k and ξ_k are independent random variables with arbitrary probability distributions while ω_k are known constants. The probability density function (*pdf*) and the cumulative distribution function (*cdf*) of p_k are respectively denoted by $f_k(.)$ and $F_k(.)$, and those of ξ_k are represented by $g_k(.)$ and $G_k(.)$.Let us denote the expected value and variance of p_k by m_k and ν_k, respectively, and those of ξ_k by μ_k and σ_k^2. In addition, let $\psi = ([1],...,[k],...,[n]) \in \Psi$ be a job sequence where $[k]$, $[k]=1,...,n$, represents the job in the k^{th} position, $k = 1, \ldots, n$, of ψ with Ψ being the set of all sequences. The completion time $t_{[k]}$ of job $[k]$ in ψ is given by

$$t_{[k]} = \sum_{\ell=1}^{k} p_{[\ell]} \,. \tag{1}$$

Let $U_{[k]}$ be the tardiness indicator variable for job $[k]$ defined as $U_{[k]} = 1$ if the job is tardy with probability $Pr(t_{[k]} > \xi_{[k]})$, and 0 otherwise. Then, the expected weighted number of tardy jobs in ψ, denoted by $WNT(\psi)$, is defined as

$$WNT(\psi) = E[\sum_{k=1}^{n} \omega_{[k]}\, U_{[k]}] = \sum_{k=1}^{n} \omega_{[k]}\, E[U_{[k]}] = \sum_{k=1}^{n} \omega_{[k]} Pr(t_{[k]} > \xi_{[k]}). \tag{2}$$

The goal is to find the *optimal* sequence $\psi* = arg\ min_{\psi \in \Psi}\ \{WNT(\psi)\}$ where $WNT(\psi)$ is given by (2). Using the three-field notation of Graham *et al.* (1979), our stochastic scheduling problem, represented by $1//\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, is NP-hard since its deterministic counterpart is NP-hard (Karp, 1972).

Note that if $Pr(t_{[k]} > \xi_{[k]}) = 1$, $k=1,...,n$ (i.e., all jobs are tardy with certainty), using (2), any $\psi \in \Psi$ is an optimal sequence. Observe also that the problem $1//\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ is general since its special cases reduce to some classical single machine scheduling problems. For example, it reduces to the problem of Boxma & Forst (1986) if $p_{[k]}$ and $\xi_{[k]}$ are independently and identically distributed (*iid*) exponential random variables; De *et al.* (1991) if $\xi_{[k]}$ are *iid* and exponential; Cai & Zhou (2005) if $p_{[k]}$ are exponential; Pinedo (1983) if $\xi_{[k]}$ and $\omega_{[k]}$ have common deterministic values and $p_{[k]}$ are exponential; Seo *et al.* (2005) if $\xi_{[k]}$ and $\omega_{[k]}$ have common deterministic values and $p_{[k]}$ are normally distributed; Jang & Klein (2002) if $\xi_{[k]}$ have a common deterministic value and p_k are normal; Elyasi & Salmasi (2013a) if $\xi_{[k]}$ are deterministic given by $d_{[k]}$, and $p_{[k]}$ have normal and gamma distributions; and Moore (1968) and Baptiste (1999) if both $p_{[k]}$ and $\xi_{[k]}$ are fixed known quantities given by $\pi_{[k]}$ and $d_{[k]}$, respectively. The latter constitutes the deterministic problem $1//\sum_{k=1}^{n} \omega_{[k]} U_{[k]}$ in which $U_{[k]} = 1$ if $t_{[k]} > d_{[k]}$ where $t_{[k]} = \sum_{\ell=1}^{k} \pi_{[\ell]}$. and 0 otherwise.

In what follows, we analyze the problem $1//\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where p_k and/or ξ_k have arbitrary probability distributions or some other distributions than the ones used in the literature.

ANALYSIS AND SOLUTION STRATEGIES

Consider a sequence $\psi = (B,i,j,A)$ in which jobs i and j, $i \neq j \in N$, are adjacent in sequence appearing at the positions q and $q+1$, $q \in \{1,...,n\text{-}1\}$, of ψ, and B and A are partial sequences containing, respectively, disjoint subsets of $(q-1)$ and $(n-q\text{-}1)$ jobs (i.e., $B = ([1],\ldots,[q\text{-}1])$ and $A = ([q+2],\ldots,[n])$). Also, consider the sequence $\psi' = (B,j,i,A)$ that is identical to ψ except that jobs i and j are interchanged. The expected weighted number of tardy jobs in ψ and ψ', using (1) and (2), are computed as

$$\begin{aligned} WNT(\psi) &= \sum_{k=1}^{q-1}\omega_{[k]}\ Pr(\sum_{\ell=1}^{k}p_{[\ell]} > \xi_{[k]}) + \omega_i\ Pr(p_B + p_i > \xi_i) \\ &+\omega_j\ Pr(p_B + p_i + p_j > \xi_j) + \sum_{k=q+2}^{n}\omega_{[k]}\,Pr(p_B + p_i + p_j + \sum_{\ell=q+2}^{k}p_{[\ell]} > \xi_{[k]}), \end{aligned} \tag{3}$$

and

$$\begin{aligned} WNT(\psi') &= \sum_{k=1}^{q-1}\omega_{[k]}\ Pr(\sum_{\ell=1}^{k}p_{[\ell]} > \xi_{[k]}) + \omega_j\ Pr(p_B + p_j > \xi_j) \\ &+\omega_i\ Pr(p_B + p_j + p_i > \xi_i) + \sum_{k=q+2}^{n}\omega_{[k]}\,Pr(p_B + p_j + p_i + \sum_{\ell=q+2}^{k}p_{[\ell]} > \xi_{[k]}), \end{aligned} \tag{4}$$

where $p_B = \sum_{k=1}^{q-1}p_{[k]}$.(i.e., the sum of processing times of jobs in B). Sequence ψ is *preferred to* sequence ψ' (denoted by $\psi \succ \psi'$) if $WNT(\psi) \le WNT(\psi')$ for any pair of jobs $i \neq j \in N$ and any B and A in ψ and ψ'. That is, we have $\psi \succ \psi'$, using (3) and (4), if

$$\begin{aligned} WNT(\psi) - WNT(\psi') &= \omega_i[Pr(p_B + p_i > \xi_i) - Pr(p_B + p_j + p_i > \xi_i)] \\ &-\omega_j[Pr(p_B + p_j > \xi_j) - Pr(p_B + p_i + p_j > \xi_j)] \le 0, \end{aligned}$$

or

$$\begin{aligned} WNT(\psi) - WNT(\psi') &= \omega_i[Pr(p_B + p_j + p_i < \xi i - Pr(p_B + p_i < \xi_i)] \\ &-\omega_j[Pr(p_B + p_i + p_j < \xi_j) - Pr(p_B + p_j < \xi_j)] \le 0, \end{aligned} \tag{5}$$

where $Pr(p_B + p_j + p_i < \xi_i) \le Pr(p_B + p_i < \xi_i)$ and $Pr(p_B + p_i + p_j < \xi_j) \le$

$Pr(p_B + p_j < \xi_j)$ since p_k and ξ_k, $k = 1,\ldots, n$, are non-negative random variables. From (5) we see that $WNT(\psi)$-$WNT(\psi')$ depends on the jobs i and j and the jobs (but not their ordering) in B.

Inequality (5) is too general to allow the development of useful statements. However, in the following subsections, we prove that special cases of three resulting scenarios of $1//\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ are solvable optimally in polynomial time. Later, we introduce heuristic solutions for the general cases of these scenarios.

Scenario with stochastic processing times and stochastic due-dates

Consider the problem scenario $1/p_k{\sim}f_k(.),\xi_k{\sim}g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ in which both p_k and ξ_k, $k = 1,\ldots,n$, have arbitrary distributions $f_k(.)$ and $g_k(.)$ (i.e., $p_k{\sim}f_k(.)$ and $\xi_k{\sim}g_k(.)$). Then $\psi \succ \psi'$, using (5), if

$$\begin{aligned}&\omega_i \int_0^{\infty}[\tilde{F}_B * F_j * F_i(x) - \tilde{F}_B * F_i(x)]dG_i(x)\\ &\qquad -\omega_j \int_0^{\infty}[\tilde{F}_B * F_i * F_j(x) - \tilde{F}_B * F_j(x)]dG_j(x) \le 0,\end{aligned} \tag{6}$$

where $\tilde{F}_B(x) = F_{[1]} *\ldots* F_{[q-1]}(x)$, is the convolution of the *cdf*s of processing times for the jobs in B (i.e., jobs $[k]$, $k = 1,\ldots,q-1$, in ψ and ψ'), $\tilde{F}_B * F_i(x)$ is that for the jobs in B and the job i, $\tilde{F}_B * F_j(x)$ is that for the jobs in B and the job j, and $\tilde{F}_B * F_i * F_j(x)$ is that for the jobs in B and the jobs i and j, where $\tilde{F}_B * F_j * F_i(x) \le \min\{\tilde{F}_B * F_i(x), \tilde{F}_B * F_j(x)\}$.

Due to difficulty in analyzing (6), we first study $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where p_k have exponential distributions (i.e., $p_k{\sim}exp(\alpha_k)$) with *cdf*s $F_k(x) = 1\text{-}exp(\text{-}\alpha_k x)$. (The use of exponential processing times in scheduling is justified by, e.g., Boxma & Forst, 1986; Cai & Zhou, 2005; Pinedo, 1983; Soroush, 2006, 2007, 2010; Baker & Trietsch, 2009a).

The convolution $F_i * F_j(x)$ of the *cdf*s of the exponential processing times for jobs i and j is defined by

$$\begin{aligned}F_i * F_j(x) &= \int_0^{x} F_i(x-y)f_j(y)dy\\ &= \alpha_j \int_0^{x}[1 - exp(-\alpha_i(x-y))]exp(-\alpha_j y)dy\\ &= 1 - exp(-\alpha_j x) - \alpha_j\left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right], \quad \alpha_i \ne \alpha_j,\end{aligned} \tag{7}$$

where $[exp(\text{-}\alpha_i x)\text{-}exp(\text{-}\alpha_j x)]/(\alpha_i\text{-}\alpha_j) \leq 0$ for all $x \geq 0$. Utilizing (7) in (6), we obtain

$$\omega_i\alpha_i \int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right]\tilde{F}_B(x)dG_i(x)$$
$$-\omega_j\alpha_j \int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right]\tilde{F}_B(x)dG_j(x) \leq 0, \quad \alpha_i \neq \alpha_j.$$

Thus, $\psi \succ \psi'$ if

$$\frac{\omega_i\alpha_i}{\omega_j\alpha_j} \geq \frac{\int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right]\tilde{F}_B(x)dG_j(x)}{\int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right]\tilde{F}_B(x)dG_i(x)}, \quad \alpha_i \neq \alpha_j. \tag{8}$$

We investigate below the problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g_k(.)/\sum_{k=1}^n \omega_{[k]}E[U_{[k]}]$ by assuming ξ_k, $k=1,\ldots,n$, have exponential, uniform, normal as well as *iid* distributions. (These distributions have been utilized in scheduling by, e.g., Aydilek & Allahverdi, 2010; Baker & Trietsch, 2009b; Cai & Zhou, 2005; Jang & Klein, 2002; Ruiz & Allahverdi, 2009; Sarin *et al.*, 1991; Soroush, 1999, 2007, 2010; Soroush & Allahverdi, 2005; and Zhang *et al.*, 2012.)

The next lemma solves the problems $1/p_k \sim exp(\alpha_k), \xi_k \sim exp(\gamma_k)/\sum_{k=1}^n \omega_{[k]}E[U_{[k]}]$ and $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^n \omega_{[k]}E[U_{[k]}]$ where $p_k{\sim}exp(\alpha_k)$ with expected values $m_k = 1/\alpha_k$ and variances $\nu_k = 1/\alpha_k^2$, and ξ_k have, respectively, exponential distributions (i.e., $\xi_k{\sim}exp(\gamma_k)$) with $G_k(x) = 1\text{-}exp(\text{-}\gamma_k x)$, expected values $\mu_k = 1/\gamma_k$ and variances $\sigma_k^2 = 1/\gamma_k^2$ (i.e., $\mu_k = \sigma_k$) and uniform distributions with lower and upper bounds ℓ_k and u_k (i.e., $\xi_k{\sim}U[\ell_k,u_k]$), $G_k(x) = (x\text{-}\ell_k)/(u_k\text{-}\ell_k)$, $\ell_k < x < u_k$, $\mu_k = (\ell_k + u_k)/2$, and $\sigma_k^2 = (u_k\text{-}\ell_k)^2/12$. Note that $\xi_k{\sim}U[\ell_k,u_k]$ provides a time window during which the due-date for each job k can occur with equal probability.

Lemma 1. *For any pair of jobs* $i{\neq}j{\in}N$, *suppose that due-date standard deviations and the weighted products of expected processing times and due-date standard deviations are agreeable, i.e.,* $\sigma_i \geq \sigma_j$ *implies* $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$. *Then, the optimal sequences for the problems* $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}exp(\gamma_k)/\sum_{k=1}^n \omega_{[k]}E[U_{[k]}]$ *and* $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^n \omega_{[k]}E[U_{[k]}]$ *can be obtained, in O(n log n) time, by arranging jobs in non-decreasing order of* $m_k\sigma_k/\omega_k$, *i.e., the shortest weighted product of expected processing time and due-date standard deviation (SWEPTDSD) rule:* $(m_{[1]}\sigma_{[1]}/\omega_{[1]} \leq \ldots \leq m_{[n]}\sigma_{[n]}/\omega_{[n]})$ *with tie broken by placing first the job with larger* σ_k.

Proof. For $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}exp(\gamma_k)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where $G_k(x) = 1\text{-}exp(\text{-}\gamma_k x)$, using (8), we have $\psi{\succ}\psi'$ if

$$\frac{\omega_i\alpha_i\gamma_i}{\omega_j\alpha_j\gamma_j} \geq \frac{\int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right] exp(-\gamma_j x)\tilde{F}_B(x)dx}{\int_0^\infty \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right] exp(-\gamma_i x)\tilde{F}_B(x)dx}, \quad \alpha_i \neq \alpha_j \qquad (9)$$

If $\gamma_i \leq \gamma_j$ (i.e., $\sigma_i \geq \sigma_j$) implies $\omega_i\alpha_i\gamma_i \geq \omega_j\alpha_j\gamma_j$ (i.e.,$m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$),$i \neq j \in N$, the left hand side (LHS) of (9) is greater than or equal to one, and its right hand side (RHS) is less than or equal to one; thus, (9) holds. As these conditions are transitive, it follows that jobs have to be sequenced in non-increasing order of $\omega_k\alpha_k\gamma_k$ (or non-decreasing order of $m_k\sigma_k/\omega_k$, i.e., the SWEPTDSD rule) with tie broken by placing first the job with smaller γ_k (or larger σ_k). This ordering requires a time of polynomial order $O(n \log n)$ (Horowitz & Sahni, 1983). Similarly, for $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where $G_k(x) = (x\text{-}\ell_k)/(u_k\text{-}\ell_k)$, $\ell_k < x < u_k$, using (8), we have $\psi{\succ}\psi'$ if

$$\frac{\omega_i\alpha_i/(u_i - \ell_i)}{\omega_j\alpha_j(u_j - \ell_j)} \geq \frac{\int_{\ell_j}^{u_j} \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right] \tilde{F}_B(x)dx}{\int_{\ell_i}^{u_i} \left[\frac{exp(-\alpha_i x) - exp(-\alpha_j x)}{\alpha_i - \alpha_j}\right] \tilde{F}_B(x)dx}, \quad \alpha_i \neq \alpha_j. \qquad (10)$$

If $\ell_i \leq \ell_j$ and $u_i \geq u_j$ (i.e., $\sigma_i \geq \sigma_j$) imply $\omega_i\alpha_i/(u_i\text{-}\ell_i) \geq \omega_j\alpha_j/(u_j\text{-}\ell_j)$ (i.e., $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$), $i \neq j \in N$, the LHS of (10) is greater than or equal to one while its RHS is less than or equal to one; thus, (10) is satisfied. Since these conditions are transitive, the optimal sequence can be obtained, in $O(n \log n)$ time, by arranging jobs in non-increasing order of $\omega_k\alpha_k/(u_k\text{-}\ell_k)$ (or non-decreasing order of $m_k\sigma_k/\omega_k$, i.e., the SWEPTDSD rule) with tie broken by placing first the job with larger σ_k.

Example 1. Consider the 5-job problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ of Table 1. Based on Lemma 1, $\sigma_i \geq \sigma_j$ implies $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$, $i \neq j \in N$, where $m_k\sigma_k/\omega_k$ = 0.26, 0.28, 0.23, 0.24, and 0.3, $k = 1,\ldots,5$, respectively; thus ψ^* = (3,4,1,2,5). Note that if we use the expected processing times m_k and expected due-dates μ_k, $k = 1,\ldots,5$, as the deterministic processing times π_k and deterministic due-dates d_k, respectively, the optimal sequence for the deterministic problem, obtained by the method of Sadykov (2008), is (4,3,5,2,1) with the weighted number of tardy jobs 9.7. This shows that the optimal sequence for the stochastic problem with random processing times and random due-dates can differ from that for the deterministic problem.

Table 1. A 5-job problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$.

Job k	ω_k	α_k	m_k	ℓ_k	u_k	μ_k	σ_k
1	3.5	0.9	1.11	0.4	3.25	1.83	0.82
2	3.9	0.6	1.67	0.6	2.9	1.75	0.66
3	3.7	1.1	0.91	0.2	3.5	1.85	0.95
4	2.8	1.3	0.77	0.3	3.3	1.8	0.87
5	2.3	0.7	1.43	0.8	2.5	1.65	0.49

The next lemma solves the problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}N(\mu_k,\sigma_k^2)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ in which $p_k{\sim}exp(\alpha_k)$ and ξ_k have normal distributions with expected values μ_k and variances σ_k^2 (i.e., $\xi_k{\sim}N(\mu_k,\sigma_k^2)$) and

$$G_k(x)=\int_{-\infty}^{x}\frac{1}{\sigma_k\sqrt{2\pi}}\exp[-\frac{1}{2}(\frac{y-\mu_k}{\sigma_k})^2]dy=\int_{-\infty}^{\frac{x-\mu_k}{\sigma_k}}\frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})dz. \quad (11)$$

Since ξ_k are non-negative, we require σ_k to be small enough relative to μ_k so that, using (11), $G_k(0) = Pr(\xi_k < 0) = Pr(Z< -\mu_k/\sigma_k) \approx 0$ where Z is a standard normal random variable. To ensure $Pr(Z< -\mu_k/\sigma_k) \approx 0$, using the standard normal table, we need to have $cv_k = \sigma_k/\mu_k < 0.28$ where cv_k is the coefficient of variation of ξ_k.

Lemma 2. *For any pair of jobs $i\neq j\in N$, suppose that due-date standard deviations, due-date coefficients of variation, and the weighted products of expected processing times and due-date standard deviations are agreeable, i.e., $\sigma_i \geq \sigma_j$ and $cv_i\leq cv_j$ imply $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$. Then, the optimal sequence for the problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}N(\mu_k,\sigma_k^2)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ can be derived, in $O(n \log n)$ time, by arranging jobs in non-decreasing order of $m_k\sigma_k/\omega_k$ (i.e., the SWEPTDSD rule) with tie broken by placing first the job with smaller cv_k or larger σ_k.*

Proof. Using (11) in (8), we have $\psi\succ\psi'$ if

$$\frac{\omega_i\alpha_i\sigma_j}{\omega_j\alpha_j\sigma_i}\geq\frac{\int_0^\infty\left[\frac{exp(-\alpha_i x)-exp(-\alpha_j x)}{\alpha_i-\alpha_j}\right]exp\left[-\frac{1}{2}\left(\frac{x-\mu_j}{\sigma_j}\right)^2\right]\tilde{F}_B(x)dx}{\int_0^\infty\left[\frac{exp(-\alpha_i x)-exp(-\alpha_j x)}{\alpha_i-\alpha_j}\right]exp\left[-\frac{1}{2}\left(\frac{x-\mu_i}{\sigma_i}\right)^2\right]\tilde{F}_B(x)dx},\quad \alpha_i\neq\alpha_j. \quad (12)$$

If $\omega_i\alpha_i/\sigma_i \geq \omega_j\alpha_j/\sigma_j$ (i.e., $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$), the LHS of (12) is greater than or equal to one. If $(x\text{-}\mu_i)/\sigma_i \leq (x\text{-}\mu_j)/\sigma_j$ for $x \geq 0$ or if $\sigma_i \geq \sigma_j$ and $\mu_i/\sigma_i \geq \mu_j/\sigma_j$ (i.e., $cv_i \leq cv_j$), the RHS of (12) is less than or equal to one. Thus, if $\sigma_i \geq \sigma_j$ and $cv_i \leq cv_j$ imply $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$, $i \neq j \in N$, then (12) holds. As these conditions are transitive, the jobs must be scheduled in non-increasing order of $\omega_k\alpha_k/\sigma_k$ (or non-decreasing order of $m_k\sigma_k/\omega_k$, i.e., the SWEPTDSD rule) with tie broken by placing first the job with smaller cv_k or larger σ_k.

Remark 1. For the problem $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}N(\mu_k,\sigma^2)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where $\sigma_k^2=\sigma^2$ (i.e., jobs have equal due-date variances), utilizing Lemma 2, if $\mu_i \geq \mu_j$ implies $m_i/\omega_i \leq m_j/\omega_j$, $i \neq j \in N$, the optimal sequence is found by the shortest weighted expected processing time (SWEPT) rule: $(m_{[1]}/\omega_{[1]} \leq \ldots \leq m_{[n]}/\omega_{[n]})$ with tie broken by placing first the job with larger μ_k.

The following lemma solves $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ where $p_k{\sim}exp(\alpha_k)$ and $\xi_k{\sim}g(.)$ (i.e., ξ_k, $k=1,\ldots,n$, are *iid*).

Lemma 3. *For the problem* $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$,*the optimal sequence can be obtained, in O(n log n) time, by arranging jobs in non-decreasing order of* m_k/ω_k, *i.e., the SWEPT rule.*

Proof. Using $G_k(x) = G(x)$, $k=1,\ldots,n$, in (8), we have $\psi \succ \psi'$ if $\omega_i\alpha_i \geq \omega_j\alpha_j$ (i.e., $m_i/\omega_i \leq m_j/\omega_j$), $i \neq j \in N$. Since this condition is transitive, it follows that the jobs have to be scheduled in non-increasing order of $\omega_k\alpha_k$ (or non-decreasing order of m_k/ω_k, i.e., the SWEPT rule).

From the above analysis on $1/p_k{\sim}f_k(.),\xi_k{\sim}g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ wherein both processing times and due dates are stochastic, the following interpretations can be made. For the four problems $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}exp(\gamma_k)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}U[\ell_k,u_k]/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}N(\mu_k,\sigma_k^2),cv_k<0.28/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; and $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$, based on Lemmas 1-3, when $\sigma_i \geq \sigma_j$ implies $m_i\sigma_i/\omega_i \leq m_j\sigma_j/\omega_j$, $i \neq j \in N$, the optimal sequences can be derived by arranging jobs in non-decreasing order of $m_k\sigma_k/\omega_k$ with tie broken by placing first the job with larger σ_k. (Note that in $1/p_k{\sim}exp(\alpha_k),\xi_k{\sim}g(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ we have $\mu_k=\mu$ and $\sigma_k=\sigma$, $k=1,\ldots,n$.) This indicates that a job k, $k \in N$, with larger due-date standard deviation σ_k (i.e., larger due-date variation), smaller expected processing time m_k, and larger tardy weight ω_k should be scheduled earlier. This makes sense because in order to minimize the expected weighted number of tardy jobs, scheduling jobs with lower expected processing times earlier results in reducing the expected completion times for those jobs. In addition, since these jobs have higher due-date variances and higher tardy weights, their early completions diminish the chances of becoming tardy and thus paying heavy tardy penalties. Scheduling jobs in this manner pushes the remaining jobs with larger expected processing times (i.e., larger expected completion times), smaller due-date variations, and smaller tardy penalties to later positions in the schedule. This increases the chances of late completions of the jobs with low tardy penalties; therefore, reducing the expected weighted number of tardy jobs.

Scenario with deterministic processing times and stochastic

We now study the problem scenario $1/p_k = \pi_k, \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ in which p_k, $k = 1, \ldots, n$, are known constants given by π_k, while $\xi_k \sim g_k(.)$, $k = 1, \ldots, n$, have *pdfs* $g_k(.)$ and *cdfs* $G_k(.)$. Then $\psi \succ \psi'$, using (5), if

$$\omega_i[G_i(\pi_B + \pi_i) - G_i(\pi_B + \pi_j + \pi_i)] - \omega_j[G_j(\pi_B + \pi_j) - G_j(\pi_B + \pi_i + \pi_j)] \leq 0, \quad (13)$$

where $\pi_B = \sum_{k=1}^{q-1} \pi_{[k]}$, $G_i(\pi_B + \pi_i) \leq G_i(\pi_B + \pi_i + \pi_j)$, and $G_j(\pi_B + \pi_j) \leq G_j(\pi_B + \pi_i + \pi_j)$.

The following three lemmas (Lemmas 4-6) solve the problem $1/p_k = \pi_k, \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ in which $\xi_k \sim g(.)$ (i.e., ξ_k are *iid*), $\xi_k \sim exp(\gamma_k)$, and $\xi_k \sim U[\ell_k, u_k]$, respectively, that is, the problems $1/p_k = \pi_k, \xi_k \sim g(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; $1/p_k = \pi_k, \xi_k \sim exp(\gamma_k)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; and $1/p_k = \pi_k, \xi_k \sim U[\ell_k, u_k]/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$.

Lemma 4. *For any pair of jobs $i \neq j \in N$, suppose that tardy weights and processing times are agreeable, i.e., $\omega_i \geq \omega_j$ implies $\pi_i \leq \pi_j$. Then, the optimal sequence for the problem $1/p_k = \pi_k$, $\xi_k \sim g(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ can be derived, in $O(n \log n)$ time, by the shortest processing time (SPT) rule: $(\pi_{[1]} \leq \ldots \leq \pi_{[n]})$ with tie broken by placing first the job with larger ω_k.*

Proof. Using $G_k(.) = G(.)$, $k = 1, \ldots, n$, in (13) if $\omega_i \geq \omega_j$ implies $G(\pi_B + \pi_i) \leq G(\pi_B + \pi_j)$ (i.e., $\pi_i \leq \pi_j$), $i \neq j \in N$, then (13) holds. Since these conditions are transitive, jobs are sequenced by the SPT rule with tie broken by placing first the job with larger ω_k.

Lemma 5. *For any pair of jobs $i \neq j \in N$, suppose that tardy weights, expected due-dates, and the products of processing times and expected due-dates are agreeable, i.e., $\omega_i \geq \omega_j$ and $\mu_i \geq \mu_j$ imply $\pi_i \mu_i \leq \pi_j \mu_j$. Then, the optimal sequence for the problem $1/p_k = \pi_k, \xi_k \sim exp(\gamma_k)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ can be obtained, in $O(n \log n)$ time, by the shortest product of processing time and expected due-date rule: $(\pi_{[1]}\mu_{[1]} \leq \ldots \leq \pi_{[n]}\mu_{[n]})$ with tie broken by placing first the job with larger ω_k or smaller μ_k.*

Proof. Using $G_k(x) = 1\text{-} exp(\text{-}\gamma_k x)$ in (13), this inequality holds if $\omega_i \geq \omega_j$ and

$$[1 - exp(-\gamma_i \pi_j)]exp[-\gamma_i(\pi_B + \pi_i)] \geq [1 - exp(-\gamma_j \pi_i)]exp[-\gamma_j(\pi_B + \pi_j)]. \quad (14)$$

If $\gamma_j \pi_i \leq \gamma_i \pi_j$ *(i.e.,* $\pi_i \mu_i \leq \pi_j \mu_j$), then $1\text{-} exp(\text{-}\gamma_i \pi_j) \geq 1\text{-} exp(\text{-}\gamma_j \pi_i)$. Also, if $\gamma_i \leq \gamma_j$ *(i.e.,* $\mu_i \geq \mu_j$) implies $\gamma_j \pi_i \leq \gamma_i \pi_j$ *(i.e.,* $\pi_i \mu_i \leq \pi_j \mu_j$), then $\pi_i \leq \pi_j$; thus, $\gamma_i \pi_i \leq \gamma_j \pi_j$ resulting in $exp[\text{-}\gamma_i(\pi_B + \pi_i)] \geq exp[\text{-}\gamma_j(\pi_B + \pi_j)]$. Therefore, if $\omega_i \geq \omega_j$ *and* $\mu_i \geq \mu_j$ imply $\pi_i \mu_i \leq \pi_j \mu_j$, then (14) holds. As these conditions are transitive, the optimal sequence can be obtained, in *O(n log n)* time, by arranging job in non-decreasing order of $\pi_k \mu_k$ with tie broken by placing first the job with larger ω_k or larger μ_k *(i.e., larger* σ_k).

Lemma 6. *For the problem* $1/p_k = \pi_k, \xi_k \sim U[\ell_k, u_k]/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$,*the optimal sequence can be derived, in O(n log n) time, by arranging jobs in non-decreasing order of* $\pi_k \sigma_k / \omega_k$, *i.e., the shortest weighted product of processing time and due-date standard deviation rule:* $(\pi_{[1]}\sigma_{[1]}/\omega_{[1]} \leq \ldots \leq \pi_{[n]}\sigma_{[n]}/\omega_{[n]})$.

Proof. Using $G_k(x) = (x\text{-}\ell_k)/(u_k\text{-}\ell_k)$, $\ell_k < x < u_k$, in (13), we obtain $\omega_i \pi_j/(u_i\text{-}\ell_i) \geq \omega_j \pi_i/(u_j\text{-}\ell_j)$ *or* $\pi_i \sigma_i/\omega_i \leq \pi_j \sigma_j/\omega_j$, $i \neq j \in N$. Since this condition is transitive, the optimal sequence can be obtained by arranging jobs in non-decreasing order of $\pi_k \sigma_k/\omega_k$.

Example 2. Consider the 5-job problem $1/p_k = \pi_k, \xi_k \sim U[\ell_k, u_k]/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ of Table 2. Based on Lemma 6, $\pi_k \sigma_k/\omega_k$ =1.65, 1.95, 1.42, 2.51, and 3.89, $k = 1, \ldots, 5$, respectively; thus, $\psi^* = (3,1,2,4,5)$. Using the expected due-dates μ_k as deterministic due-date d_k,the optimal sequence for the deterministic problem is (1,5,3,2,4) with the weighted number of tardy jobs 12.2. This shows that the stochasticity of only due-dates can affect the optimal sequences.

Table 2. A 5-job problem $1/p_k = \pi_k, \xi_k \sim U[\ell_k, u_k]/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$.

Job k	ω_k	π_k	ℓ_k	u_k	μ_k	σ_k
1	3.5	4.0	5.0	10.0	7.5	1.44
2	4.0	6.0	2.0	6.5	4.25	1.3
3	3.2	4.5	4.0	7.5	5.75	1.01
4	2.3	5.0	3.0	7.0	5.0	1.15
5	2.7	5.2	1.0	8.0	4.5	2.02

The above results for the three problems $1/p_k = \pi_k, \xi_k \sim g(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; $1/p_k = \pi_k$, $\xi_k \sim exp(\gamma_k)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; and $1/p_k = \pi_k, \xi_k \sim U[\ell_k, u_k]/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ in which processing times are known constants but due dates are stochastic indicate that when $\omega_i \geq \omega_j$ implies $\pi_i \mu_i \leq \pi_j \mu_j, i \neq j \in N$, the optimal sequences can be found by arranging jobs in non-decreasing order of $\pi_k \sigma_k/\omega_k$ with tie broken by placing first the job with larger ω_k (see Lemmas 3-6). (Note that in $1/p_k = \pi_k, \xi_k \sim g(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, we have $\mu_k = \mu$ and $\sigma_k = \sigma$, $k = 1,\ldots,n$, and in $1/p_k = \pi_k, \xi_k \sim exp(\gamma_k)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, $\mu_k = \sigma_k$, $k = 1,\ldots,n$.) Accordingly, a job k, $k \in N$, with larger tardy weight ω_k, smaller processing time π_k, and smaller due-date standard deviation σ_k should be scheduled earlier. Scheduling jobs in this manner yields lower completion times resulting in decreasing chances of these jobs becoming tardy and thus paying high tardy penalties. Similarly, jobs with larger due-date variations, larger processing times, and smaller tardy weights are scheduled later increasing the chances of late completions of the jobs with low tardy penalties. This scheduling practice can reduce the expected weighted number of tardy jobs.

Scenario with stochastic processing times and deterministic due-dates

We investigate here the problem scenario $1/p_k \sim f_k(.), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where ξ_k are deterministic given byd_k, and p_k, $k = 1,\ldots,n$, have arbitrary *pdf*s $f_k(.)$ and *cdf*s $F_k(.)$.As discussed in the introduction section, Soroush (2007) extensively studies the related problem of minimizing the expected weighted number of early and tardy jobs on a single machine when jobs have deterministic due-dates d_k and random processing times p_k with *cdf*s $F_k(.)$, $k = 1,\ldots,n$, where the $F_i(.)$ and $F_j(.)$ for every pair of jobs $i \neq j = 1,\ldots,n$, cross in at most one point with a finite value. He optimally solves special cases of this problem and presents a heuristic method for the general case. Accordingly, by setting the job earliness weights equal to zero in Soroush's (2007) problem, we can obtain the solutions to the respective cases of our problem of minimizing the expected weighted number tardy jobs.

For the problem $1/p_k \sim f_k(.), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where processing times have arbitrary distributions and deterministic due-dates are different, using (5), we have $\psi \succ \psi'$ if

$$\omega_i[\tilde{F}_B * F_j * F_i(d_i) - \tilde{F}_B * F_i(d_i)] - \omega_j[\tilde{F}_B * F_i * F_j(d_j) - \tilde{F}_B * F_j(d_j)] \leq 0, \quad (15)$$

where $\tilde{F}_B * F_j * F_i(d_i) \leq \tilde{F}_B * F_i(d_i)$ and $\tilde{F}_B * F_i * F_j(d_j) \leq \tilde{F}_B * F_j(d_j)$.

We now present the next two lemmas (Lemmas 7 and 8) that can be proved by using (15) and arguments similar to those of Soroush (2007).

Lemma 7. *For* $1/p_k \sim exp(\alpha), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, *the optimal sequence can be derived, in* $O(n \log n)$ *time, by*

(i) *the latest due-date (LDD) rule:* $(d_{[1]} \geq \ldots \geq d_{[n]})$ *if* $d_i \geq d_j$ *implies* $\omega_i d_i \exp(-\alpha d_i) \leq \omega_j d_j \exp(-\alpha d_j)$, $i \neq j \in N$, *and*

(ii) *the earliest due-date (EDD) rule:* $(d_{[1]} \leq \ldots \leq d_{[n]})$ *if* $d_i \leq d_j$ *implies* $\omega_i d_i^{n-1} \exp(-\alpha d_i) \geq \omega_j d_j^{n-1} exp(-\alpha d_j), i \neq j \in N$.

Lemma 8. *For any pair of jobs* $i \neq j \in N$*, suppose that tardy weights and the cdfs of processing times are agreeable, i.e.,* $\omega_i \geq \omega_j$ *implies* $F_i(y) \geq F_j(y)$, $0 \leq y \leq d$, $i \neq j \in N$. *Then, the optimal sequence for the problem* $1/p_k \sim f_k(.), \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ *can be obtained, in O(n log n) time, by arranging jobs in non-increasing order of* $F_k(y), 0 \leq y \leq d$*, with tie broken by placing first the job with larger* ω_k.

The following corollaries (Corollaries 1-3), induced from Lemma 8, respectively yield the optimal sequences for the problems $1/p_k \sim U[a_k, b_k], \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where $p_k \sim U[a_k, b_k]$; $1/p_k \sim W(\alpha_k, \beta_k), \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where p_k have weibull distributions with shape and scale parameters α_k and β_k (i.e., $p_k \sim W(\alpha_k, \beta_k)$) and cdfs $F_k(x) = 1 - \exp[-(\alpha_k x)^{\beta_k}]$; $1/p_k \sim exp(\alpha_k), \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where $p_k \sim exp(\alpha_k)$; and $1/p_k \sim N(m_k, \nu_k), \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ where $p_k \sim N(m_k, \nu_k)$ and $\sqrt{\nu_k}/m_k < 0.28$.

Corollary 1. *For any pair of jobs* $i \neq j \in N$*, suppose that tardy weights and probabilities that processing times are less than due-dates are agreeable, i.e.,* $\omega_i \geq \omega_j$ *implies* $F_i(d) \geq F_j(d)$ where $F_k(x) = (x\text{-}a_k)/(b_k - a_k), a_k < x < b_k$. *Then, the optimal sequence for the problem* $1/p_k \sim U[a_k, b_k], \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ *can be obtained, in O(n log n) time, by arranging jobs in non-increasing order of* $F_k(d)$ *with tie broken by placing first the job with larger* ω_k.

Proof. For $1/p_k \sim U[a_k, b_k], \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, using Lemma 8, $\omega_i \geq \omega_j$ implies $F_i(y) \geq F_j(y), 0 \leq y \leq d, i \neq j \in N$, where $F_k(x) = (x - a_k)/(b_k - a_k), a_k < x < b_k$. That is, $\omega_i \geq \omega_j$ implies $(y - a_i)/(b_i - a_i) \geq (y\text{-}a_j)/(b_j - a_j)$, $0 \leq y \leq d$, or

$$y[(b_j - a_j) - (b_i - a_i)] \geq a_i(b_j - a_j) - a_j(b_i - a_i), 0 \leq y \leq d. \tag{16}$$

If $b_i - a_i \leq b_j - a_j$ and $(b_i - a_i)/a_i \geq (b_j - a_j)/a_j$, then (16) holds for any $0 \leq y \leq d$; thus, $F_i(d) \geq F_j(d)$. If $b_i - a_i > b_j - a_j$, we can write (16) as $y \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$, $0 \leq y \leq d$, which is satisfied as long as $d \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$ or $F_i(d) \geq F_j(d)$. Thus, if $\omega_i \geq \omega_j$ implies $F_i(d) \geq F_j(d)$, $i \neq j \in N$, the optimal sequence can be found by arranging jobs in non-increasing order of $F_k(d)$ with tie broken by placing first the job with larger ω_k.

Example 3. Consider the 5-job problem $1/p_k \sim U[a_k, b_k], \xi_k = d / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ of Table 3 where $d = 3$. Since $\omega_i \geq \omega_j$ implies $F_i(d) \geq F_j(d)$, $i \neq j \in N$, where $F_k(3) = (3\text{-}a_k)/(b_k - a_k) = 0.8, 0.43, 1, 0.5$, and 0.25, $k = 1, \ldots, 5$, based on Corollary 1, $\psi^* = (3,1,4,2,5)$. Utilizing the expected processing times m_k as deterministic π_k, the optimal sequence for the deterministic problem is (3,5,2,4,1) with the weighted number of tardy jobs 8.5. This indicates that, when jobs have a common deterministic due-date, the stochasticity of only processing times can also affect the optimal sequence.

Table 3. A 5-job problem $1/p_k \sim U[a_k, b_k], \xi_k = d/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ *with* $d = 3$.

Job k	ω_k	a_k	b_k
1	3.0	1.0	3.5
2	2.0	1.5	5.0
3	3.5	1.0	2.5
4	2.5	2.0	4.0
5	1.0	2.5	4.5

Corollary 2. *For any pair of jobs $i \neq j \in N$, suppose that tardy weights, scale parameters of weibull distributions, and probabilities that processing times are less than due-dates are agreeable, i.e., if $\omega_i \geq \omega_j$ and $\beta_i \leq \beta_j$ imply $F_i(d) \geq F_j(d)$ where $F_k(x) = 1 - \exp[-(\alpha_k x)^{\beta_k}]$. Then, the optimal sequence for the problem $1/p_k \sim W(\alpha_k, \beta_k), \xi_k = d/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ can be obtained, in $O(n \log n)$ time, by arranging jobs in non-increasing order of $F_k(d)$ with tie broken by placing first the job with larger ω_k or smaller β_k.*

Proof. For $1/p_k \sim W(\alpha_k, \beta_k), \xi_k = d/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, using Lemma 8, $\omega_i \geq \omega_j$ implies $F_i(y) \geq F_j(y)$, $0 \leq y \leq d$, $i \neq j \in N$, where $F_k(y) = 1 - \exp[-(\alpha_k y)^{\beta_k}]$. That is, $\omega_i \geq \omega_j$ implies $exp[-(\alpha_i y)^{\beta_i}] \leq exp[-(\alpha_j y)^{\beta_j}]$ or $y^{\beta_i - \beta_j} \geq \alpha_j^{\beta_j} / \alpha_i^{\beta_i}$, $0 \leq y \leq d$. However, if $\beta_i \leq \beta_j$ and $(\alpha_i d)^{\beta_i} \geq (\alpha_j d)^{\beta_j}$ or $F_i(d) \geq F_j(d)$, then $y^{\beta_i - \beta_j} \geq \alpha_j^{\beta_j} / \alpha_i^{\beta_i}$, $0 \leq y \leq d$. Thus, if $\omega_i \geq \omega_j$ and $\beta_i \leq \beta_j$ imply $F_i(d) \geq F_j(d)$, $i \neq j \in N$, jobs have to be scheduled in non-increasing order of $F_k(d)$ with tie broken by placing first the job with larger ω_k or smaller β_k.

Corollary 3.

(i) *For the problem $1/p_k \sim exp(\alpha_k), \xi_k = d/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, the optimal sequence can be derived, in $O(n \log n))$ time, by arranging jobs in a non-decreasing order of m_k/ω_k, i.e., the SWEPT rule.*

(ii) *For any pair of jobs $i \neq j \in N$, suppose that tardy weights and probabilities that processing times are less than due-dates are agreeable, i.e., $\omega_i \geq \omega_j$ implies $F_i(d) \geq F_j(d)$ (i.e., either $\nu_i \leq \nu_j$ and $cv_i \geq cv_j$ where $cv_k = \sqrt{\nu_k}/m_k$, or $\nu_i \geq \nu_j$ and $z_i \geq z_j$ where $z_k = (d - m_k)/\sqrt{\nu_k}$)). Then, the optimal sequence for the problem $1/p_k \sim N(m_k, \nu_k), \xi_k = d/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ can be found, in $O(n \log n)$ time, by arranging jobs in non-increasing order of $F_k(d)$ with tie broken by placing first the job with larger ω_k.*

Proof. It follows by using Lemma 8 and arguments similar to those of Soroush (2007).

Remark 2. Based on Corollary 3(ii), for $1/p_k \sim N(m_k,\nu_k),\xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ when $cv_k = \tau,\ \tau < 0.28,\ k=1,\ldots,n,$ (i.e., processing time means and standard deviations are proportional), the optimal sequence can be obtained by the shortest processing time variance (SPTV) rule: $(\nu_{[1]} \leq \ldots \leq \nu_{[n]})$ if $\omega_i \geq \omega_j$ implies $\nu_i \leq \nu_j,\ i \neq j \in N$. When $\nu_k = \nu,\ k=1,\ldots,n,$ (i.e., jobs have equal processing time variances), the optimal sequence can be found by the shortest expected processing time (SEPT) rule: $(m_{[1]} \leq \ldots \leq m_{[n]})$ if $\omega_i \geq \omega_j$ implies $m_i \leq m_j,\ i \neq j \in N$. When $m_k = m,\ k=1,\ldots,n,$ (i.e., jobs have equal mean processing times), the optimal sequence is derived by (i) the SPTV rule if $\omega_i \geq \omega_j$ implies $\nu_i \leq \nu_j,\ i \neq j \in N,$ and $d \geq m$, and (ii) the longest processing time variance (LPTV) rule: $(\nu_{[1]} \geq \ldots \geq \nu_{[n]})$ if $\omega_i \geq \omega_j$ implies $\nu_i \geq \nu_j,\ i \neq j \in N,$ and $d < m$.

In problems $1/p_k \sim U[a_k,b_k],\xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; $1/p_k \sim W(\alpha_k,\beta_k),\ \xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; $1/p_k \sim exp(\alpha_k),\ \xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; and $1/p_k \sim N(m_k,\nu_k),\xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ with stochastic processing times and a common deterministic due-date, based on Lemma 8 and Corollaries 1-3, if $\omega_i \geq \omega_j$ implies $F_i(d) \geq F_i(d), i \neq j \in N,$ the optimal sequences can be found by sequencing jobs in non-increasing order of $F_k(d)$ with tie broken by placing first the job with larger ω_k. (Note that in $1/p_k \sim exp(\alpha_k),\xi_k = d/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$ we have $\alpha_i \geq \alpha_j$ (i.e., $m_i \leq m_j$) iff $F_i(d) \geq F_j(d), i \neq j \in N$). Therefore, a job $k,\ k \in N,$ with larger tardy weight ω_k and larger probability $F_k(d)$ that it's processing time is less than the due-date is scheduled earlier. Scheduling jobs in this way reduces the chances that such jobs become tardy; thus, avoiding high tardy penalties. This can reduce the expected number of such jobs being tardy. In addition, jobs with smaller probabilities that their processing times are less than the due-date have higher chances of becoming tardy but with smaller tardy weights. This can also reduce the expected weighted number of tardy jobs.

HEURISTIC METHODS AND COMPUTATIONAL RESULTS

In this section, we first introduce heuristic methods for the three problem scenarios $1/p_k \sim f_k(.),\ \xi_k \sim g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; $1/p_k = \pi_k,\xi_k \sim g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$; and $1/p_k \sim f_k(.),\xi_k = d_k/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$. Then, some computational results are presented to evaluate the performance of the three heuristics.

The first heuristic, referred to as *stoch-stoch*, solves $1/p_k \sim f_k(.),\xi_k \sim g_k(.)/\sum_{k=1}^{n}\omega_{[k]}E[U_{[k]}]$. This heuristic is developed based on Lemmas 1-3 and their subsequent discussions. In general, since the conditions of these lemmas do not hold simultaneously for all jobs $i \neq j \in N$, the heuristic derives, in $O(n \log n)$ time, a candidate $\tilde{\psi}$ for the optimal sequence ψ^* by arranging jobs in non-decreasing order of $m_k\sigma_k/\omega_k$ with tie broken by placing first the job with larger σ_k. The second heuristic, referred to as *det-stoch*, is for $1/p_k = \pi_k,\ \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$. This heuristic uses the conditions of Lemmas 4-6 to find, in *O(n log n)* time, a candidate $\tilde{\psi}$ for ψ* by arranging jobs in non-decreasing order of $\pi_k \sigma_k / \omega_k$ with tie broken by placing first the job with larger ω_k. The third heuristic, referred to as *stoch-det*, solves $1/p_k \sim f_k(.), \xi_k = d_k / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$. Based on Corollaries 1-3 and Soroush (2007), since conditions $\omega_i \geq \omega_j$ and $F_i(d) \geq F_j(d)$ imply $\omega_i F_i(d) \geq \omega_j F_j(d),\ i \neq j \in N$, the heuristic *stoch-det* extends the latter condition to find a candidate $\tilde{\psi}$ for ψ* by arranging jobs in non-increasing order of $\omega_k F_k(d_k)$with tie broken by placing first the job with larger ω_k.

Our computational experiments deal with the heuristics *stoch-stoch*, *det-stoch*, and *stoch- det* to solve, respectively, the three problem scenarios $1/p_k \sim f_k(.),\ \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}];\ 1\ /\ p_k = \pi_k,\ \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; and $1\ /\ p_k \sim f_k(.), \xi_k = d_k / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ of different sizes n = 7, 8, 9, 10, 11, and 12. Fifty instances for each problem scenario and size were randomly generated. We applied the appropriate proposed heuristic to obtain $\tilde{\psi}$ and the enumeration method (the only available exact method) to find ψ* for each of the 300 generated instances of every problem scenario. Due to the excessive time requirement of the enumeration method, we did not consider problems with more than twelve jobs. (Enumerating sequences in a 13-job problem took more than 5 CPU hours.) The heuristics and the enumeration method were coded in FORTRAN 77 and were run on a computer with Intel core 2 due, 2.24 GHz processor and with 1.99 GB RAM.

For the problem $1/p_k \sim f_k(.), \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, we used the four processing time distributions: $p_k \sim exp(\alpha_k), p_k \sim N(m_k, \nu_k),\ p_k \sim U(a_k, b_k)$, and $p_k \sim W(\alpha_k, \beta_k)$, and the three due-date distributions: $\xi_k \sim exp(\gamma_k),\ \xi_k \sim N(\mu_k, \sigma_k^2)$, and $\xi_k \sim U(\ell_k, u_k)$. The distributions for the $p_k$and ξ_k of each job k were randomly picked from the four distributions of p_k and the three distributions of ξ_k, respectively. For $p_k \sim exp(\alpha_k)$ and $\xi_k \sim exp(\gamma_k)$, we sampled α_k and γ_k from the intervals [0.05, 1] and [0.02, 0.2], respectively. For $p_k \sim N(m_k, \nu_k)$, m_k and ν_k were randomly sampled from [1, 20] such that $\sqrt{\nu_k}/m_k$ 0.28, and for $\xi_k \sim N(\mu_k, \sigma_k^2)$, μ_k and σ_k^2 were generated from [5, 50] such that $\sigma_k/\mu_k <$ 0.28. For $p_k \sim U(a_k, b_k)$ and $\xi_k \sim U(\ell_k, u_k)$, we respectively sampled a_k and b_k, $a_k < b_k$, from the intervals [0.1, 15] and [2, 25], and ℓ_k and u_k, $\ell_k < u_k$, from [5, 20] and [40, 60]. For $p_k \sim W(\alpha_k, \beta_k)$, the shape and scale parameters α_k *and* β_k were generated from [0.01, 1] and [0.02, 2]. The tardy weights ω_k for each job k was sampled from the interval [1, 10].

For the problem $1/p_k = \pi_k, \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, we randomly generated $\pi_k,\ k = 1, \ldots, n$, from the interval [1, 20], while $\xi_k \sim exp(\gamma_k),\ \xi_k \sim N(\mu_k, \sigma_k^2)$, and $\xi_k \sim U(\ell_k, u_k)$ were sampled as above. For the problem $1/p_k \sim f_k(.), \xi_k = d_k / \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$, we sampled d_k from [5, 50], while $p_k \sim exp(\alpha_k), p_k \sim N(m_k, \nu_k)$,

$p_k \sim U(a_k, b_k)$, and $p_k \sim W(\alpha_k, \beta_k)$ were generated as above.

Since p_k, $k=1,\ldots,n$, have different probability distributions, we could not derive the *cdf*s of the completion times $t_{[k]} = \sum_{\ell=1}^{k} p_\ell, k=1,\ldots,n$, in closed forms. In addition, it is very difficult to exactly compute $Pr(t_{[k]} > \xi_{[k]})$ in orderto find the expected weighted number of tardy jobs in a sequence. We therefore estimated $Pr(t_{[k]} > \xi_{[k]})$ utilizing random samples of sizes 10,000 from the *pdf*s of $p_{[k]}$ and $\xi_{[k]}$.

Table 4 displays our computational results on the heuristics *stoch-stoch*, *det-stoch*, and *stoch-det*. As shown in the table, the heuristics performed well in identifying the optimal sequences or near optimal solutions with low average percentages of relative absolute errors. The percentage of relative absolute error (Abs. Err(%)) is defined as $100[|WNT(\tilde{\psi})\text{-}WNT(\psi^*)|/min\ \{WNT(\tilde{\psi}), WNT(\psi^*)\}]$%. The heuristic *stoch-stoch* found at least 84% of the optimal sequences for the 300 instances of the problem $1/p_k \sim f_k(.), \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$. For the remaining instances, this heuristic obtained near optimal sequences with an average relative absolute error of at most 3.75%. The percentages of the optimal sequence and average relative absolute error of the heuristic *det-stoch* for the problem $1/p_k = \pi_k, \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ were, respectively, at least 82% and at most 3.43%, while of the heuristic *stoch-det* for the problem $1/p_k \sim f_k(.), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ were at least 80% and at most 3.96%. (Note that the heuristics *stoch-stoch, det-stoch*, and *stoch-det* could find the optimal sequences for the problems of Examples 1-3, respectively.)

In addition, we examined the effects of both stochastic processing times and stochastic due-dates on the optimal sequence. This was done by comparing the optimal sequences for the 300 generated instances of $1/p_k \sim f_k(.), \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ with their corresponding deterministic problems $1//\sum_{k=1}^{n} \omega_{[k]} U_{[k]}$. The results showed that 77% of the optimal sequences for the two problems were different indicating that the stochasticity of both processing times and due-dates can impact scheduling decisions. Investigating the effects of the stochastic due-dates only, it turned out that 75% of the optimal sequences for $1/p_k = \pi_k,\ \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ differed from those for $1//\sum_{k=1}^{n} \omega_{[k]} U_{[k]}$. In addition, 80% of the optimal sequences for $1/p_k \sim f_k(.), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$ were different from those for $1//\sum_{k=1}^{n} \omega_{[k]} U_{[k]}$. Therefore, the stochasticity of due-dates alone as well as the stochasticity of processing times alone can also affect scheduling decisions.

Table 4. Computational results on the heuristic *stoch-stoch* for $1/p_k \sim f_k(.), \xi_k \sim g_k(.)/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; heuristic *det-stoch* for $1/p_k = \pi_k, \xi_k \sim g_k(.)/ \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$; and heuristic *stoch-det* for $1/p_k \sim f_k(.), \xi_k = d_k/\sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$. The results, averages of 50 runs, show the percentage that the heuristic yields the optimal sequence and the percentage of relative absolute error resulted from the expected weighted number of tardy jobs of the heuristic and optimal sequences.

No. of Jobs	Heuristic *stoch-stoch for* $1/p_k \sim f_k(.), \xi_k \sim g_k(.)/ \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$		Heuristic *det-stoch* for $1/p_k = \pi_k, \xi_k \sim g_k(.)/ \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$		Heuristic *stoch-det* for $1/p_k \sim f_k(.), \xi_k = d_k/ \sum_{k=1}^{n} \omega_{[k]} E[U_{[k]}]$	
	Opt(%)	**Abs. Err(%)**	**Opt(%)**	**Abs. Err(%)**	**Opt(%)**	**Abs. Err(%)**
7	90.00	0.89	88.00	0.96	92.00	0.83
8	88.00	1.42	90.00	1.57	90.00	1.15
9	90.00	1.35	86.00	1.81	86.00	1.46
10	86.00	2.41	88.00	1.75	84.00	1.98
11	86.00	2.29	82.00	2.79	86.00	2.91
12	84.00	3.75	86.00	3.43	80.00	3.96

SUMMARY AND CONCLUSIONS

We have studied a stochastic single machine scheduling problem in which processing times and/or due-dates are random variables with arbitrary probability distributions, and fixed weights (penalties) are imposed on tardy jobs. The objective is to find the schedule that minimizes the expected weighted number of tardy jobs. The problem is NP-hard; however, based on the stochasticity of processing times and due-dates, we have explored the three resulting scenarios: the scenario with stochastic processing times and stochastic due-dates, the scenario with deterministic processing times and stochastic due-dates, and the scenario with stochastic processing times and deterministic due-dates. (The last scenario is a special case of Soroush's (2007) problem of minimizing the expected weighted number of early and tardy jobs on a single machine with random processing times and deterministic due-dates.) We have optimally solved various special cases of these scenarios in $O(n \log n)$ time, and have introduced polynomial time heuristic solutions for their general cases. Our computational results on problems with seven up to twelve jobs have indicated that the heuristics perform well in deriving either the optimal sequences (more than 80% of the time) or near optimal sequences with low percentages of relative absolute errors (less than 3.96%). (Due to the excessive CPU time of the enumeration method, we did not evaluate the heuristics' performance on larger problems.) The illustrative examples and computational results show that the stochasticity of processing times and/or due-dates can affect scheduling decisions. The problem studied here is general in the sense that its special cases reduce to some new and some classical stochastic single machine models. We are currently extending this research to incorporate sequence-dependent setup times as well as job learning/deterioration into scheduling decisions.

REFERENCES

Aldowaisan, T. A. & Allahverdi, A. 2012. No-wait flowshop scheduling problem to minimize the number of tardy jobs. International Journal of Advance Manufacturing Technology **6(1)**: 311--323.

Aydilek, H. & Allahverdi, A. 2010. Two-machine flowshop scheduling problem with bounded processing times to minimize total completion time. Computers & Mathematics with Applications **59**(2): 684-693.

Baker, K. R. & Trietsch, D. 2009a. Principles of Sequencing and Scheduling. John Wiley: New Jersey, USA.

Baker, K. R. & Trietsch, D. 2009b. Safe scheduling: Setting due dates in single-machine problems. European Journal of Operational Research **196**(1): 69-77.

Balut, S. J. 1973. Scheduling to minimize the number of late jobs when set-up and processing times are uncertain. Management Science **19**(11): 1283-1288.

Baptiste, P. 1999. Polynomial time algorithms for minimizing the weighted number of late jobs on a single machine with equal processing times. Journal of Scheduling **2**(5): 245-252.

Boxma, O. J. & Forst, F. G. 1986. Minimizing the expected weighted number of tardy jobs in stochastic flow shops. Operations Research Letters **5**(3): 119-126.

Cai, X. & Zhou, X. 2005. Single machine scheduling with exponential processing times and general stochastic cost functions. Journal of Global Optimization **31**(2): 317-332.

Chiang, T. C. & Fu, L. C. 2009. Using a family of critical ratio-based approaches to minimize the number of tardy jobs in the job shop with sequence dependent setup times. European Journal of Operational Research **196**(1): 78-92.

De, P., Ghosh, J. B. & Wells, C. E. 1991. On the minimization of the weighted number of tardy jobs with random processing times and deadline. Computers & Operations Research **18**(5): 457-463.

Desprez, C., Chu, F. & Chu, C. 2009. Minimising the weighted number of tardy jobs in a hybrid flow shop with genetic algorithm. International Journal of Computer Integrated Manufacturing **22**(8): 745-757.

Dhouib, E. Teghem, J. & Loukil, T. 2013. Minimizing the number of tardy jobs in a permutation flowshop scheduling problem with setup times and time lags constraints. Journal of Mathematical Modelling and Algorithms, **12(1)**: 85-99.

Elyasi, A. & Salmasi N. 2013a. Stochastic scheduling with minimizing the number of tardy jobs using chance constrained programming. Mathematical and Computer Modelling **57**(5-6): 1154-1164.

Elyasi, A. & Salmasi, N. 2013b. Stochastic flow-shop scheduling with minimizing the expected number of tardy jobs. International Journal of Advanced Manufacturing Technology, **66**(1-4): 337-346.

Erel, E. & Ghosh, J. B. 2007. Batch scheduling to minimize the weighted number of tardy jobs. Computers & Industrial Engineering **53**(3): 394-400.

Erenay, F. S., Sabuncuoglu, I. Toptal, A. & Tiwari, M. K. 2010.

Graham, R. L., Lawler, E. L., Lenstra, J. K. & Rinnooy Kan, A. H. G. 1979. Optimization and approximation in deterministic sequencing and scheduling: A survey. Annals of Discrete Mathematics **5**: 287-326.

Hoogeveen, H. & T'kindt, V. 2012. Minimizing the number of late jobs when the start time of the machine is variable. Operations Research Letters **40**(5): 353-355.

Horowitz, E. & Sahni, S. 1983. Fundamentals of Data Structures. Computer Science Press: Rockville, USA.

Jafari, A. & Moslehi, G. 2012. Scheduling linear deteriorating jobs to minimize the number of tardy jobs. Journal of Global Optimization **54**(2): 389-404.

Jang, W. & Klein, C. M. 2002. Minimizing the expected number of tardy jobs when processing times are normally distributed. Operations Research Letters **30**(2): 100-106.

Jolai, F. , Rabbani, M., Amalnick, S., Dabaghi, A., Dehghan, M. & Parast, M. Y. 2007. Genetic algorithm for bi-criteria single machine scheduling problem of minimizing maximum earliness and number of tardy jobs. Applied Mathematics and Computation **194**(2): 552-560.

Karp, R. M. 1972. Reducibility among combinatorial problems. In: Millerm R.E. and Thatcher, J.W. (Eds.). Complexity of Computer Computations. Pp 85-103. Plenum Press: New York.

Lee, W. C., Lin, J. B. & Shiau, Y. R. 2011. Deteriorating job scheduling to minimize the number of late jobs with setup times. Computers & Industrial Engineering **61**(3): 782-787.

Lee, J. Y. & Kim, Y. D. 2012. Minimizing the number of tardy jobs in a single-machine scheduling problem with periodic maintenance. Computers & Operations Research **39**(9): 2196--2205.

Li, W. & Alfa, A. S. 2005. Scheduling stochastic jobs on a repairable machine with general phase type uptime. Mathematical Methods of Operations Research **61(3):** 399-417.

Molaee, E., Moslehi, G. & Reisi, M. 2011. Minimizing maximum earliness and number of tardy jobs in the single machine scheduling problem with availability constraint. Computers and Mathematics with Applications **62**(8): 3622-3641.

Moore, J. M. 1968. An n job, one machine sequencing algorithm for minimizing the number of late jobs. Management Science **15**(1): 102-109.

Mosheiov, G. & Oron, D. 2012. Minimizing the number of tardy jobs on a proportionate flowshop with general position-dependent processing times. Computers & Operations Research **39**(7): 1601-1604.

Mosheiov, G. & Sarig, A. 2010. Minimum weighted number of tardy jobs on an m-machine flow-shop with a critical machine. European Journal of Operational Research **201**(2): 404-408.

Panwalkar, S. S. & Koulamas, C. 2012. An O(n2) algorithm for the variable common due date, minimal tardy jobs bicriteria two-machine flow shop problem with ordered machines. European Journal of Operational Research **221**(1): 7-13.

Pinedo, M. L. 1983. Stochastic scheduling with release dates and due dates. Operations Research **31**(3): 559-572.

Rasti-Barzoki, M., Hejazi, S. R. & Mazdeh, M. M. 2013. A branch and bound algorithm to minimize the total weighed number of tardy jobs and delivery costs. Applied Mathematical Modelling **37**(7): 4924-4937.

Reisi, M. & Moslehi, G. 2011. Minimizing the number of tardy jobs and maximum earliness in the single machine scheduling using an artificial immune system. International Journal of Advanced Manufacturing Technology **54**(5-8): 749-756.

Ruiz, R. & Allahverdi, A. 2009. Minimizing the bicriteria of makespan and maximum tardiness with an upper bound on maximum tardiness. Computers & Operations Research **36**(4): 1268-1283.

Sadykov, R. A. 2008. Branch-and-check algorithm for minimizing the weighted number of late jobs on a single machine with release dates. European Journal of Operational Research **189**(3): 1284-1304.

Sarin, S. C., Erel, E. & Steiner, G. 1991. Sequencing jobs on a single machine with a common due date and stochastic processing times. European Journal of Operational research **51**(2): 188-198.

Seo, D. K., Klein, C. M. & Jang, W. 2005. Single machine stochastic scheduling to minimize the expected number of tardy jobs using mathematical programming models. Computers & Industrial Engineering **48**(2): 153-161.

Soroush, H. M. 1999. Optimal sequencing and due-date determination in a stochastic single machine system with earliness and tardiness costs. European Journal of Operational Research **113**(2): 450-468.

Soroush, H. M. 2006. Single machine scheduling with stochastic processing times or stochastic due-dates to minimizing the number of early and tardy jobs. International Journal of Operations Research **3**(2): 90-108.

Soroush, H. M. 2007. Minimizing the weighted number of early and tardy jobs in a stochastic single machine scheduling problem. European Journal of Operational Research **181**(1): 266-277.

Soroush, H. M., 2010. Solving a stochastic single machine problem with initial idle time and quadratic objective. Computers & Operations Research **37**(7): 1328-1347.

Soroush, H. M. & Allahverdi, A. 2005. Stochastic two-machine flowshop scheduling problem with total completion time criterion. International Journal of Industrial Engineering: theory applications, and practice **12**(2): 157-170.

Steiner, G. & Zhang, Z. 2011. Minimizing the weighted number of tardy jobs with due date assignment and capacity-constrained deliveries. Annals of Operations Research **191**(1): 171-181.

Tang, H. Y., Tang, C. H. & Zhao, C. L. 2010. A preemptive-resume stochastic scheduling model with disruption. System Engineering Theory and Practice **30**(4): 751-757.

Trietsch, D. & Baker, K. R. 2008. Minimizing the number of tardy jobs with stochastically-ordered processing times. Journal of Scheduling **11**(1): 71-73.

Umble, M. M. and Srikanth, M. L. 1995. Synchronous manufacturing: Principles for world-class excellence. Spectrum Publication Company: Wallingford, USA.

Varmazyar, M. & Salmasi, N. 2012. Sequence-dependent flow shop scheduling problem minimizing the number of tardy jobs. International Journal of Production Research **50**(20): 5843-5858.

Wan, G. & Yen, B. P. C. 2009. Single machine scheduling to minimize total weighted earliness subject to minimal number of tardy jobs. European Journal of Operational Research **195**(1): 89-97.

Zhang, Q., Manier, H. & Manier, M. A. 2012. A genetic algorithm with tabu search procedure for flexible job shop scheduling with transportation constraints and bounded processing times. Computers & Operations Research **39**(7): 1713-1723.

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جدولة الأعمال العشوائية على المكينة الواحدة لتقليل عدد الأعمال المتأخرة

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خلاصة

أحد أهم مشاكل الجدولة وتنظيم الخدمة التي تعاني منها المنظمات الصناعية هي توصيل البضائع والخدمات في أوقاتها. تدرس ورقة البحث هذه مشاكل جدولة المكينة الواحدة، التي فيها وقت الإنجاز وتاريخه من المتغيرات العشوائية، وتطبيق عقوبات ثابتة على الأعمال المتأخرة. الهدف هو الحصول على جدولة تقلل من عدد العقوبات المتوقعة على الأعمال المتأخرة. المسألة هي (NP-hard)، ولكن أثبتنا أنه تحت بعض الظروف التوافقية هناك حالات خاصة لمسألة قابلة للحل في وقت سريع. وقمنا بتقديم طرق تقريبية للحالات العامة من هذه السيناريوهات. النتائج الحاصلة تشير إلى أن الطرق التقريبية تؤدي وظيفتها في تقديم الحل الأمثل أو الجيد. الأمثلة والنتائج تبين العشوائية في وقت الإنجاز وتاريخه، وهذا يمكن أن يؤثر على إتخاذ قرار الجدولة، بالإضافة إلى ذلك هناك حالات إستثنائية للمسألة يتقلص إلى احصاء جديد أو إلى بعض نماذج الماكينة الواحدة الموجودة.

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