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Numerical solution of chaotic Genesio system with multi-step Laplace Adomian decomposition method

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ABSTRACT

In this paper, a novel method for approximate analytic series solution called Multi-step Laplace Adomian Decomposition Method (MLADM) has been proposed for solving the chaotic Genesio system (CGS). The proposed method is a modification of the classical Laplace Adomian Decomposition Method (LADM) with multi-step approach. Fourth-order Runge-Kutta method (RK4) is used to evaluate the effectiveness of the proposed algorithm. Comparison of the results with RK4 confirms that MLADM performs very high accuracy. Results show that MLADM is a very promising method for obtaining approximate solutions to CGS. Moreover, it can be readily employed to solve other chaotic systems.

Keywords: Genesio system; multistep; Laplace-Adomian; nonlinear

INTRODUCTION

Generally mathematical models of the events in the universe are expressed by non-linear equations. Obtaining an analytical solution of a nonlinear system is not so easy. We generally have to use numerical integration, some particular transformations, linearization or discretization to obtain approximate solutions for nonlinear systems. Dynamical systems which demonstrate chaotic behavior are sensitive to initial conditions. The behavior of chaotic systems appears to be random because of its sensitivity to initial conditions even though they have deterministic structure. Chaotic phenomena can be found in many scientific and engineering fields such as biological systems, electronic circuits, power converters, chemical systems, and many other systems (Chen,1999). The Genesio-Tesi system, proposed by Genesio & Tesi (1992), is one of the chaos paradigms since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters. The Genesio-Tesi system is given in (Genesio & Tesi, 1992) as follows:

$$\begin{cases} \dfrac{dx_1}{dt} = x_{2,} \\ \dfrac{dx_2}{dt} = x_{3,} \\ \dfrac{dx_3}{dt} = a.x_1 + b.x_2 + c.x_3 + x_1^2, \end{cases} \tag{1}$$

where a, b, c are real parameters. The system given by (1) is chaotic when the parameters take the values of $a = -6$, $b = -2.92$ and $c = -1.2$.

Several methods have been utilized to study the CGS in literature. For example, Ghorbani & Saberi-Nadjafi (2011) have studied a piecewise-spectral parametric iteration method for solving the nonlinear chaotic Genesio system, Goh *et al*. (2009) have studied classical variational iteration method (VIM) and the multistage VIM (MVIM), Gökdoğan *et al*. (2012) have considered the modified algorithm for the differential transform method (MDTM) in their valuable study. Bataineh *et al*. (2008) have studied homotopy analysis method. The convergence region is very small in the mentioned works.

As known, LADM was successfully applied to find the exact solutions or approximations of high degree of accuracy of many different nonlinear differential equations (Khuri, 2001; Babolian *et al*., 2004; Syam & Hamdan, 2006; Yusufoğlu, 2006; Kiymaz, 2009; Wazwaz, 2010; Doğan, 2012 a; Doğan & Akin, 2011; Doğan & Akin, 2012). The main advantage of LADM is its capability of combining the two powerful methods for obtaining exact solutions for nonlinear equations. Although LADM gives sufficient results for small regions like VIM, MVIM and MDTM, it does not give a satisfactory approximation to solution of some differential equation for larger t.

Multi-step methods have been proposed to increase the accuracy by researchers so far. Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. The process continues with subsequent steps to map out the solution in multi-step methods.

Our inspiration point was to use a combination of Laplace transform and Adomian decomposition method (LADM) with multi-step approach for solution to CGS to improve the accuracy of the solution in a larger time interval. The proposed method called as the Multistep Laplace Adomian Decomposition Method (MLADM) by Doğan (b) (2012). He has implemented this method for the solution of HIV infection of CD4 + T cells (Doğan, 2012 b). Here, the MLADM is used to obtain an approximate solution of the chaotic Genesio-Tesi system.

Generally the RK4 method is used to evaluate the effectiveness of the proposed algorithms since it is widely accepted and used. Comparison with the RK4 method is a technique for similar methods because we don't know the exact solution. Furthermore, this new method has considerably good results. The convergence region is very small in the other methods (a piecewise-spectral parametric iteration method, classical variational iteration method (VIM) and the multistage VIM (MVIM), modified algorithm for the differential transform method (MDTM), homotopy analysis method-please see the references) for the same system. This new method increases convergence region for the series solution.

This paper is organized as follows: Section 2 gives the LADM solution, Section 3 deals with the MLADM and, lastly, Section 4 presents conclusions on the MLADM method.

LAPLACE ADOMIAN DECOMPOSITION METHOD

Application of the LADM to CGS is introduced in this section. The CGS given in (1) with initial conditions $x(0) = 0.2,\ y(0) = -0.3,\ z(0) = 0.1$ is considered. In order to solve (1) by using the LADM, we apply the LADM to CGS. We consider (1) with initial conditions $x(0) = 0.2,\ y(0) = -0.3,\ z(0) = 0.1$. To solve this model by using the LADM, the Laplace transform is used. As known, the Laplace transform of $x'$is defined as:

$$L\{x'\} = s.L\{x\} - x(0).$$

Similarly, the Laplace transforms of both sides of (1) are:

$$\begin{cases} s.L\{x(t)\} = & x(0) + L\{y(t)\}, \\ s.L\{y(t)\} = & y(0) + L\{z(t)\}, \\ s.L\{z(t)\} = & z(0) - a.L\{z(t)\} - b.L\{y(t)\} - c.L\{x(t)\} + L\{x^2(t)\}. \end{cases} \tag{2}$$

Moreover, the equation (2) can be rearranged as

$$\begin{cases} L\{x(t)\} = & \frac{x(0)}{s} + \frac{1}{s}L\{y(t)\}, \\ L\{y(t)\} = & \frac{y(0)}{s} + \frac{1}{s}L\{z(t)\}, \\ L\{z(t)\} = & \frac{z(0)}{s} - \frac{a}{s}.L\{z(t)\} - \frac{b}{s}.L\{y(t)\} - \frac{c}{s}.L\{x(t)\} + \frac{1}{s}L\{x^2(t)\}. \end{cases} \tag{3}$$

To address the nonlinear term,$F = x^2(t)$ in (3), the Adomian decomposition method and the Adomian polynomials can be used. Solutions in this method are represented by infinite series such as:

$$x = \sum_{k=0}^{\infty} x_k \ , \ \ y = \sum_{k=0}^{\infty} y_k \ , \ z = \sum_{k=0}^{\infty} z_k, \tag{4}$$

where the components x_k, y_k and z_k are recursively computed. However, the nonlinear term $F = x^2(t)$ at the right side of (3) will be represented by an infinite series of Adomian polynomials

$$F(t,\ x) = \sum_{k=0}^{\infty} A_k \tag{5}$$

where $A_k,\ k \geq 0$ are defined by

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(t,\ \sum_{j=0}^{k} \lambda^j x_j\right)\right],\ k = 0,\ 1,\ 2, \cdots \tag{6}$$

and so-called Adomian polynomials, A_k can be evaluated for all forms of nonlinearities. Substitution of (4) and (5) into (2) leads to:

$$\begin{cases} L\{\sum_{k=0}^{\infty} x_k\} = \frac{x(0)}{s} + \frac{1}{s} L\{\sum_{k=0}^{\infty} y_k\}, \\ L\{\sum_{k=0}^{\infty} y_k\} = \frac{y(0)}{s} + \frac{1}{s} L\{\sum_{k=0}^{\infty} z_k\}, \\ L\{\sum_{k=0}^{\infty} z_k\} = \frac{z(0)}{s} - \frac{a}{s}.L\{\sum_{k=0}^{\infty} z_k\} - \frac{b}{s}.L\{\sum_{k=0}^{\infty} y_k\} - \frac{c}{s}.L\{\sum_{k=0}^{\infty} x_k\} + \frac{1}{s} L\{\sum_{k=0}^{\infty} A_k\}. \end{cases} \tag{7}$$

An iterative approximation algorithm by means of both sides of (7) could be obtained as follows

$$\begin{cases} L\{x_0\} = \frac{x(0)}{s}, \quad L\{x_{k+1}\} = \frac{1}{s} L\{y_k\}, \\ L\{y_0\} = \frac{y(0)}{s}, \quad L\{y_{k+1}\} = \frac{1}{s} L\{z_k\}, \\ L\{z_0\} = \frac{z(0)}{s}, \quad L\{z_{k+1}\} = -\frac{a}{s}.L\{z_k\} - \frac{b}{s}.L\{y_k\} - c.L\{x_k\} + \frac{1}{s} L\{A_k\}. \end{cases} \tag{8}$$

The inverse Laplace transforms of the first part of (8) give the first terms of solutions $x_0,\ y_0$ and z_0 which are used to calculate, A_0. Consequently, the first term of Adomian polynomials A_0 is used to evaluate x_1, y_1 and z_1.

Subsequently, the determination of x_1, y_1 and z_1 leads to the determination of A_1, which is used to determine x_2, y_2 ,z_2 and so on. Finally, the components of x_k, y_k and z_k, $k \geq 0$ are determined by the second part of (8) and the series solutions of the equation (4) are obtained.

Five terms approximations for $x(t)$, $y(t)$ and $z(t)$ by using LADM are calculated as

$$\begin{cases} x(t) = & 0.2 - 0.3t + 0.05t^2 - 0.0673333t^3 + 0.0780333t^4 - 0.012064t^5, \\ y(t) = & -0.3 + 0.1t - 0.202t^2 + 0.312133t^3 - 0.06032t^4 - 0.0137413t^5, \\ z(t) = & 0.1 - 0.404t + 0.9364t^2 - 0.24128t^3 - 0.0687067t^4 - 0.0271009t^5. \end{cases} \tag{9}$$

Moreover, ten terms approximations for $x(t)$, $y(t)$ and $z(t)$ by using LADM are initially calculated, and then ten terms $x(t)$, $y(t)$ and $z(t)$ solutions are used to obtain the [5/5] Padé approximations. Mathematica 7 is utilized for the [5/5] Padé approximations. Accordingly, rational fraction approximations for $x(t)$, $y(t)$ and $z(t)$ are obtained as follows:

$$\begin{cases} px(t) = \dfrac{0.2 - 0.348797t + 0.120126t^2 - 0.0764055t^3 + 0.0942646t^4 - 0.0282743t^5}{1. - 0.243987t - 0.0153516t^2 - 0.00739134t^3 - 0.00823521t^4 - 0.00152925t^5}, \\ py(t) = \dfrac{-0.3 - 0.107086t - 0.114823t^2 + 0.155262t^3 + 0.169591t^4 - 0.080093t^5}{1. + 0.690286t - 0.0604958t^2 + 0.0379475t^3 + 0.00521748t^4 - 0.00437588t^5}, \\ pz(t) = \dfrac{0.1 - 0.34971t + 0.745268t^2 + 0.155859t^3 + 0.0564943t^4 - 0.118888t^5}{1. + 0.542897t + 0.281989t^2 + 0.0269327t^3 + 0.0301754t^4 + 0.00522833t^5}. \end{cases} \tag{10}$$

Subsequently, solutions range is extended by means of Padé approximations. For comparison, the numerical solution of equation (1) is obtained with Runge-Kutta fourth order method and the [5/5] Padé approximate using ten terms LADM approximations given in (10). The absolute errors obtained with both methods are illustrated in Table 1. As could be seen in Table 1, although Padé approximation was employed to get a better result in a larger region, good approximations only are achieved in a small region. It was observed from experiments that even if the number of terms of LADM approximations was increased, the approximation range was not extended as a desired manner. Figs. 1, 2 and 3 show comparisons between Padé-LADM and RK4. Obviously the Padé -LADM solutions diverges for $t > 2.5$.

Table 1. Absolute errors obtained by using Runge Kutta fourth order method and Padé-LADM

t_i	$\lvert px(t_i) - RKM\rvert$	$\lvert py(t_i) - RKM\rvert$	$\lvert pz(t_i) - RKM\rvert$
0	0	0	0
0.5	$1.802455787E-8$	$1.081288853E-7$	$1.472234361E-7$
1	$2.014281797E-5$	$4.016637420E-5$	$8.898250547E-7$
1.5	$1.979948894E-3$	$2.218777167E-3$	$4.222270838E-3$
2	$9.115657271E-2$	$3.299189443E-2$	$5.123111961E-2$
2.5	$5.744060938E-1$	$2.382888610E-1$	$2.849527474E-1$
3	$8.049168838E-1$	1.112112790	$9.506658412E-1$
3.5	1.378122707	4.070456537	2.214990045
4	2.124105521	14.68842487	3.952531329
4.5	2.870438941	166.2238315	5.697507633
5	3.454479029	46.82859652	6.856761121

MULTISTEP LAPLACE ADOMIAN DECOMPOSITION METHOD

In the previous section, a combination of Padé-LADM solutions was introduced and related results were given. It was shown that Padé-LADM solutions converge in a very small region and it had slow convergent rate or completely divergent in the wider region. To overcome this shortcoming, we introduced the MLADM in this section.

The multi-step approach for LADM proposed in this study is a novel idea for constructing the approximate solution. Let $[0, T]$ be the interval over which we want to find the solution of the initial value problem (1). The solution interval,$[0, T]$ is divided into M subintervals $[t_{m-1}, t_m]$, $m = 1, 2, \cdots, M$ of equal step size, $h = T/M$ by using the nodes, $t_m = mh$.

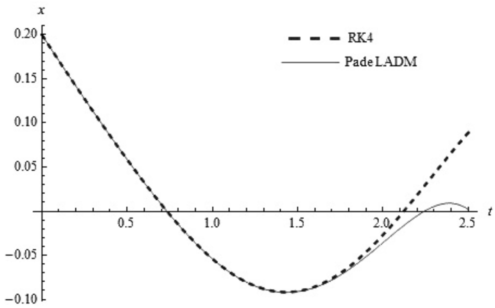

Fig.1. Comparison between Padé-LADM and RK4 solutions of $x(t)$.

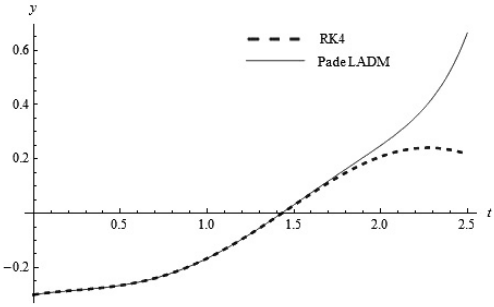

Fig.2. Comparison between Padé-LADM and RK4 solutions of $y(t)$.

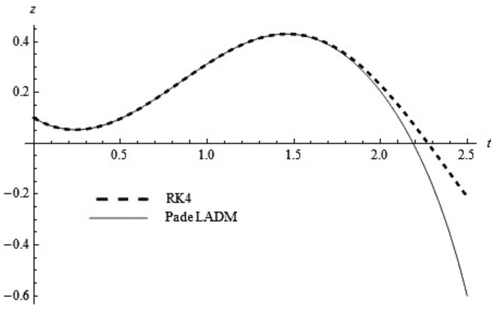

Fig.3. Comparison between Padé-LADM and RK4 solutions of $z(t)$.

The solution algorithm of the MLADM consists of the following steps. Initially, the LADM is applied to obtain the approximate solutions of $x_1(t), y_1(t), z_1(t)$ on the interval $[0, t_1]$ by using the initial conditions, $x(0) = 0.2, y(0) = -0.3, z(0) = 0.1$ respectively. For obtain the approximate solutions of Eq. (1) over the interval $[t_{m-1}, t_m]$, the LADM for $m > 2$ is used with the initial conditions $x_1(t_{m-1}), y_1(t_{m-1}), z_1(t_{m-1})$. The similar process is repeated to generate sequence of approximate solutions of $x_m(t), y_m(t), z_m(t), m = 1, 2, \cdots, M$. Consequently, final approximate MLADM solutions are obtained as follows (Doğan, 2012):

$$x(t = \begin{cases} x_1(t), & [0,\ t_1] \\ x_2(t), & [t_1,\ t_2] \\ \vdots & \vdots \\ x_M(t), & [t_{M-1},\ t_M] \end{cases},\quad y(t) = \begin{cases} y_1(t), & [0,\ t_1] \\ y_2(t), & [t_1,\ t_2] \\ \vdots & \vdots \\ y_M(t), & [t_{M-1},\ t_M] \end{cases} \quad and \quad z(t) = \begin{cases} z_1(t), & [0,\ t_1] \\ z_2(t), & [t_1,\ t_2] \\ \vdots & \vdots \\ z_M(t), & [t_{M-1},\ t_M] \end{cases}.$$

Table 2. Absolute errors obtained by uing Runge Kutta fourth order method and MLADM for M = 2000, T = 100. *n* = 5

t_i	$\lvert x(t_i) - RKM\rvert$	$\lvert y(t_i) - RKM\rvert$	$\lvert z(t_i) - RKM\rvert$
0	0	0	0
10	$1.922752104E-7$	$7.696088211E-8$	$6.104923682E-7$
20	$1.745417308E-6$	$2.165391176E-6$	$6.452932304E-7$
30	$1.112129596E-6$	$3.606388590E-6$	$7.522158440E-6$
40	$4.418757805E-6$	$3.83452017E-6$	$2.195148780E-5$
50	$6.935169329E-6$	$1.912673949E-6$	$2.821073544E-7$
60	$1.218487093E-6$	$6.082384957E-6$	$4.043107888E-6$
70	$4.879636380E-6$	$2.521005237E-6$	$2.066552070E-5$
80	$8.232163406E-6$	$1.820707663E-5$	$1.198184028E-6$
90	$2.938746882E-6$	$1.413213632E-5$	$1.416361888E-5$
100	$1.057194367E-5$	$3.421007639E-6$	$1.637102885E-5$

Application

To demonstrate the effectiveness of the proposed algorithm, the MLADM and RK4 are applied to the nonlinear CGS with the same initial conditions and parameters. MLADM results are obtained for *M* = *2000, T* = *100* and $n = 5$. Moreover, LADM results given in Table 1 are used for comparison. It is observed that the MLADM gives a much better performance in approximate solutions compared to LADM for larger time interval.

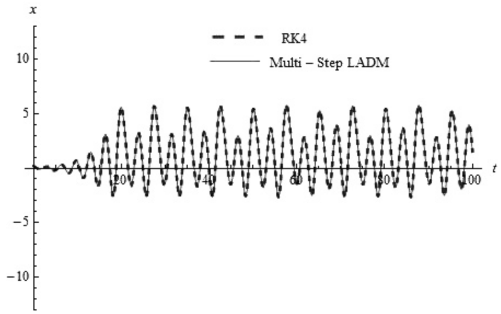

Fig. 4. Graphical comparison of $x(t)$ by using MLADM versus RK4.

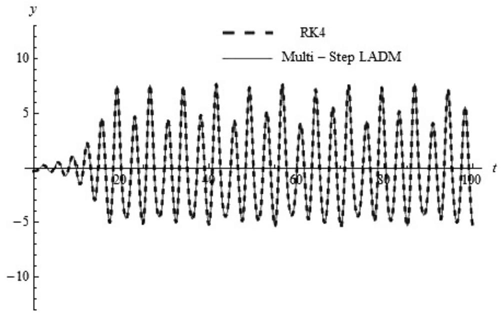

Fig. 5. Graphical comparison of $y(t)$ by using MLADM versus RK4.

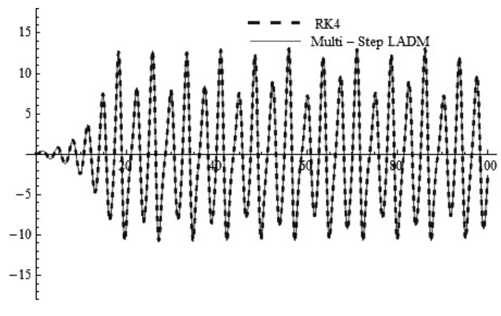

Fig. 6. Graphical comparison of $z(t)$ by using MLADM versus RK4.

Table 2 shows the numerical outputs for MLADM and RK4 for $t = 0$ to $t = 100$. As could be seen in Table 2, there is an extremely good agreement between MLADM and RK4 for given time frame in terms of errors. Figs. 4, 5 and 6 show that the multi-step LADM solutions are very closed to the Runge-Kutta solutions. In addition, phase portraits of solutions are given in Figs. 7, 8, 9 and 10. As could be seen, the system is highly chaotic.

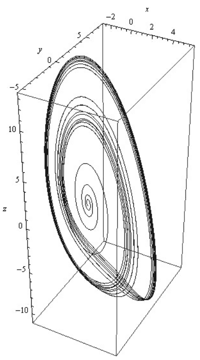

Fig. 7. Phase portrait for Genesio System (1) by using MLADM

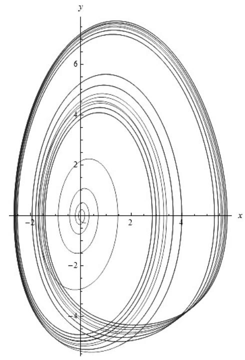

Fig. 8. $x(t) - y(t)$ Phase portrait for Genesio System (1) by using MLADM

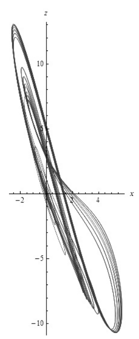

Fig. 9. $x(t) - z(t)$ Phase portrait for Genesio System (1) by using MLAD

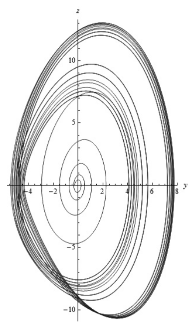

Fig. 10. $y(t) - z(t)$ Phase portrait for Genesio System (1) by using MLAD

CONCLUSIONS

In this study, a novel method called multi-step LADM for solution of CGS is introduced. The proposed method MLADM, which is a consistent modification of the LADM, can be directly used without linearization, perturbation or restrictive assumptions. Figs. 1, 2 and 3 show comparisons between Padé-LADM and RK4. We can conclude that the Padé-LADM is only reliable on a small domain of $t \in [0, 2.5]$. Figs. 4, 5 and 6 show that the MLADM approximate solutions for CGS are very closed to the Runge-Kutta approximate solutions. MLADM, on the other hand, gives considerably good results on a longer time span of $t \in [0, 100]$. Therefore, the proposed method is very efficient and accurate method that can be used to provide approximate analytical solutions for nonlinear systems of differential equations. This confirms that this new algorithm of the LADM increases the interval of convergence for the series solution. Of course the accuracy can be improved when the step size h becomes smaller and the number of terms in each subinterval n becomes larger. Figs. 7, 8, 9 and 10 indicate the phase diagram obtained from the MLADM solutions. We have shown that the proposed algorithm is a very accurate and efficient method for CGS and it can be applied to other chaotic systems.

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حل عددي لنظام جنسيو الشواشي بطريقة تفريق لابلاس - أدومين متعددة الخطوات

نورالدين دوجان

قسم هندسة الكمبيوتر - كلية التكنولوجيا - جامعة جازي - تركيا

خلاصة

نقترح في هذا البحث طريقة جديدة لحل تقريبي، تحليلي متسلسلي. ونسمي هذه الطريقة الجديدة بطريقة تفريق لابلاس - أدومين متعددة الخطوات ثم نقوم بإستخدامها لحل نظام جنسيو الشواشي. وتعتبر هذه الطريقة المقترحة تعديلاً لطريقة تفريق لابلاس - أدومين التقليدية. ونقوم بفحص فعالية طريقتنا بإستخدام طريقة رونغ - كوتا من المرتبة الرابعة. ويتبين من هذا أن طريقتنا تتمتع بدقة فائقة. ويدل ذلك على أن طريقتنا تعتبر طريقة واعدة للحصول على حلول تقريبية لنظام جنسيو الشواشي. بل أكثر من ذلك، يمكن استخدام هذه الطريقة لحل أنظمة شواشية أخرى.