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Some topological properties of the set of filter cluster functions

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Abstract

In Albayrak & Pehlivan (2013), we generalized the concepts of pointwise convergence, uniform convergence and α-convergence for sequences of functions on metric spaces by using the filters on N. In this work, we define the concepts of limit function, $\mathcal{F}$-limit function and $\mathcal{F}$-cluster function respectively for each of these three types of convergence, where $\mathcal{F}$ is a filter on N. We investigate some topological properties of the sets of $\mathcal{F}$-pointwise cluster functions, $\mathcal{F}$-α-cluster functions and $\mathcal{F}$-uniform cluster functions by using pointwise and uniform convergence topologies.

Keywords: $\mathcal{F}$-α-convergence; $\mathcal{F}$-cluster function; $\mathcal{F}$-exhaustiveness; $\mathcal{F}$-limit function; limit function.

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1. Introduction

The concepts of statistical limit points and statistical cluster points were defined for real sequences for the first time by Fridy (1993), and studied by many mathematicians (Connor & Kline, 1996; Di Maio & Kočinac, 2008; Kostyrko *et al*., 2001). For more applications of the set of statistical cluster points in R^n one can see Mamedov & Pehlivan (2000, 2001); Pehlivan & Mamedov (2000); Pehlivan *et al*. (2004); Zaslavski (2006), where many more references can be found. Then Kostyrko *et al*. (2001) defined the concepts of $\mathcal{I}$-limit point and $\mathcal{I}$-cluster point, which are the generalizations of statistical ones (Činčura *et al*., 2004; Kostyrko *et al*., 2005).

In recent years, some mathematicians (Albayrak & Pehlivan, 2013; Boccuto *et al*., 2011; Caserta & Kočinac, 2012; Das & Papanastassiou, 2004; Gregoriades & Papanastassiou, 2008) have concentrated on the concept of α-convergence, which was known as continuous convergence in the past (Carathéodory, 1929; Kelley, 1955; Stoilov, 1959) and its generalizations. In 2008, Gregoriades & Papanastassiou (2008) defined the concept of exhaustiveness for both families and sequences of functions, and using this notion they gave a generalization of the Ascoli theorem. In Albayrak & Pehlivan (2013), we obtained some results related to the concepts of $\mathcal{F}$-α-convergence, $\mathcal{F}$-pointwise convergence, $\mathcal{F}$-uniform convergence and $\mathcal{F}$-exhaustiveness for sequences of functions, where $\mathcal{F}$ is a filter on N.

In this paper, we give the definitions of limit, $\mathcal{F}$-cluster and $\mathcal{F}$-limit functions for pointwise convergence, uniform convergence and α-convergence, and examined some properties of the sets consisting of limit, $\mathcal{F}$-cluster or $\mathcal{F}$-limit functions. We investigate certain properties of the sets of $\mathcal{F}$-pointwise cluster functions, $\mathcal{F}$-α-cluster functions and $\mathcal{F}$-uniform cluster functions. Then we introduce the topological structures of these sets and examine some of its consequences. Finally we observe that these results can be further generalized for the set of cluster functions of sequence of special functions.

2. Preliminaries

In this section, we present some basic concepts and our definitions given in Albayrak & Pehlivan (2013). Throughout this work, the symbol $|.|$ denotes the cardinality for sets or the absolute value for real numbers. (X, ρ_X) and (Y, ρ_Y) denote two metric spaces. Let $D \subseteq X$.

The set of all functions and the set of all continuous functions from D to Y are denoted by Y^D and $C(D,Y)$, respectively. By $S(\xi,\delta)$, we denote the open ball with center ξ and radius δ.

Start with the concept of filter. $\mathcal{F} \subseteq \mathcal{P}(\mathrm{N})$ is a *filter* if $\mathcal{F}$ is closed under taking supersets and finite intersections (Engelking, 1989; Willard, 1970). A filter is said to be *free* if the intersection of all its members is empty, and *fixed* otherwise. If $\mathcal{F}$ is a filter on N, then the set $\mathcal{I}(\mathcal{F}) = \{\mathrm{N} - A \;:\; A \in \mathcal{F}\}$ forms an *ideal* on N.

Let $\mathcal{F}$ be a filter. A subset A of N is called $\mathcal{F}$*-stationary* if it has nonempty intersection with each member of the filter $\mathcal{F}$. We denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. In brief, for an $A \subseteq \mathrm{N}$ we have $A \in \mathcal{F}^*$ iff $A \notin \mathcal{I}(\mathcal{F})$.

Definition 2.1 (Aviles Lopez *et al*., 2007; Kadets *et al*., 2010; Katětov, 1968). A sequence $(x_n)_{n\in\mathrm{N}}$ in a metric space (X,ρ_X) is said to be $\mathcal{F}$*-convergent* to $\xi \in X$ if for every $\varepsilon > 0$ the set $\{n \in \mathrm{N} : \rho_X(x_n,\xi) < \varepsilon\}$ belongs to $\mathcal{F}$. In this case, we write $\mathcal{F} - \lim x_n = \xi$ or $x_n \xrightarrow{\mathcal{F}} \xi$.

Now we present some examples of filters.

1. Fréchet Filter. The family $\mathcal{F}_r = \{A \subseteq \mathrm{N} : \mathrm{N} - A \text{ is finite}\}$ is called the *Fréchet filter*. $\mathcal{F}_r$ is the minimum free filter with respect to the inclusion relation. Therefore, we can characterize free filters as follows: If $\mathcal{F} \supseteq \mathcal{F}_r$ then $\mathcal{F}$ is a free filter. $\mathcal{F}_r$-convergence coincides with the ordinary convergence.

2. Statistical Convergence Filter. If $d(A) = \lim_{n\to\infty} \frac{|A \cap [1,n]|}{n}$ exists, then the value of this limit is called the *asymptotic density* of the set A (Buck, 1953; Niven, 1951). The family $\mathcal{F}_{st} = \{A \subseteq \mathrm{N} : d(A) = 1\}$ is a free P-filter, and it is called the *statistical convergence filter*. $\mathcal{F}_{st}$-convergence is called the *statistical convergence* (Di Maio & Kočinac, 2008; Fast, 1951; Miller, 1995).

We firstly recall the concepts of equicontinuity and $\mathcal{F}$-exhaustiveness (Albayrak & Pehlivan, 2013; Boccuto *et al*., 2011) for the families or the sequences of functions.

Definition 2.2. Let (X,ρ_X) and (Y,ρ_Y) be two metric spaces and $\xi \in D \subseteq X$.

1. Let $\mathcal{K}$ be a family of continuous functions from D to Y. The family $\mathcal{K}$ is *equicontinuous* at ξ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\eta \in S(\xi,\delta) \cap D$ and every $f \in \mathcal{K}$ we have $\rho_Y(f(\eta), f(\xi)) < \varepsilon$.

2. Let $f_n : D \to Y$ $(n \in \mathrm{N})$ and $\mathcal{F}$ be a filter on N. The sequence $(f_n)_{n\in\mathrm{N}}$ is said to be $\mathcal{F}$*-exhaustive* at the point ξ provided that, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\{n \in \mathrm{N} \;:\; \rho_Y(f_n(\eta), f_n(\xi)) < \varepsilon, \forall \eta \in S(\xi,\delta)\} \in \mathcal{F}.$$

Now we introduce the concepts of $\mathcal{F}$-α-convergence, $\mathcal{F}$-pointwise convergence and $\mathcal{F}$-uniform convergence. In the following definitions, we assume that (X,ρ_X) and (Y,ρ_Y) are two metric spaces, $D \subseteq X$, $f, f_n : D \to Y$ $(n \in \mathrm{N})$, and $\mathcal{F}$ is a filter on N.

Definition 2.3 (Albayrak & Pehlivan, 2013; Boccuto *et al*., 2011). A sequence $(f_n)_{n\in\mathrm{N}}$ is called $\mathcal{F}$*-α-convergent* to f on D if for each $\xi \in D$ and for every sequence $(x_n)_{n\in\mathrm{N}}$ which is $\mathcal{F}$-convergent to ξ, the sequence $(f_n(x_n))_{n\in\mathrm{N}}$ is $\mathcal{F}$-convergent to $f(\xi)$ (i.e., $\mathcal{F} - \lim f_n(x_n) = f(\xi)$), and we write $f_n \xrightarrow{\mathcal{F}-\alpha} f$ (on D).

Definition 2.4 (Albayrak & Pehlivan, 2013; Balcerzak *et al*., 2007; Kostyrko *et al*., 2001). A sequence $(f_n)_{n\in\mathrm{N}}$ is called $\mathcal{F}$*-pointwise convergent* to f on D if $\mathcal{F} - \lim f_n(\xi) = f(\xi)$ for each $\xi \in D$, i.e., $\{n \in \mathrm{N} : \rho_Y(f_n(\xi), f(\xi)) < \varepsilon\} \in \mathcal{F}$ for every $\varepsilon > 0$. In this case, we write $f_n \xrightarrow{\mathcal{F}-pw} f$ (on D).

Definition 2.5 (Albayrak & Pehlivan, 2013; Balcerzak *et al*., 2007). A sequence $(f_n)_{n\in\mathrm{N}}$ is called $\mathcal{F}$*-uniformly convergent* to f on D provided that

$$\{n \in \mathrm{N} \;:\; \rho_Y(f_n(\xi), f(\xi)) < \varepsilon, \forall \xi \in D\} \in \mathcal{F}$$

for every $\varepsilon > 0$. In this case, we write $f_n \xrightarrow{\mathcal{F}-u} f$ (on D).

Now, we define the concepts of limit function, $\mathcal{F}$-limit function and $\mathcal{F}$-cluster function for above three types of convergence.

Definition 2.6. The function f is called an $\mathcal{F}$*-uniform limit function* of the sequence $(f_n)_{n\in\mathrm{N}}$ if there is a set $K = \{n_1 < n_2 < \ldots < n_k < \ldots\} \in \mathcal{F}^*$ such that the subsequence $(f_{n_k})_{k\in\mathrm{N}}$ is uniformly convergent to f on D. We denote the set of all $\mathcal{F}$-uniform limit functions of $(f_n)_{n\in\mathrm{N}}$ by $\Lambda^u_{f_n}(\mathcal{F})$.

Definition 2.7. The function f is called an $\mathcal{F}$*-uniform cluster function* of the sequence $(f_n)_{n\in\mathrm{N}}$ if for every $\varepsilon > 0$

$$\{n \in \mathrm{N} \;:\; \rho_Y(f_n(\xi), f(\xi)) < \varepsilon, \forall \xi \in D\} \in \mathcal{F}^*.$$

We denote the set of all $\mathcal{F}$-uniform cluster functions of $(f_n)_{n\in\mathrm{N}}$ by $\Gamma^u_{f_n}(\mathcal{F})$.

Definition 2.8. The function f is called an *uniform limit function* of the sequence $(f_n)_{n\in\mathrm{N}}$ if there is a infinite set

$K = \{n_1 < n_2 < ... < n_k < ...\}$ such that the subsequence $(f_{n_k})_{k\in N}$ is uniformly convergent to f on D. We denote the set of all uniform limit functions of (f_n) by $\mathcal{L}^u_{f_n}$.

Definition 2.9. The function f is called an *$\mathcal{F}$-pointwise limit function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^*$ such that $\lim_{k\to\infty} f_{n_k}(\xi_i) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. We denote the set of all $\mathcal{F}$-pointwise limit functions of (f_n) by $\Lambda^{pw}_{f_n}(\mathcal{F})$.

Definition 2.10. The function f is called an *$\mathcal{F}$-pointwise cluster function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each $\varepsilon > 0$

$$\{n \in N : \rho_Y(f_n(\xi_i), f(\xi_i)) < \varepsilon, \forall i \in \{1,...,m\}\} \in \mathcal{F}^*.$$

We denote the set of all $\mathcal{F}$-pointwise cluster functions of $(f_n)_{n\in N}$ by $\Gamma^{pw}_{f_n}(\mathcal{F})$.

Definition 2.11. The function f is called an *pointwise limit function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that $\lim_{k\to\infty} f_{n_k}(\xi_i) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. We denote the set of all pointwise limit functions of $(f_n)_{n\in N}$ by $\mathcal{L}^{pw}_{f_n}$.

Definition 2.12. The function f is called an *$\mathcal{F}$-α-limit function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each sequences $(x_{i,n})_{n\in N}$ in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ ($i \in \{1,2,...,m\}$) there is a set $K = \{n_1 < n_2 < ... < n_k < ...\} \in \mathcal{F}^*$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. We denote the set of all $\mathcal{F}$-α-limit functions of $(f_n)_{n\in N}$ by $\Lambda^{\alpha}_{f_n}(\mathcal{F})$.

Definition 2.13. The function f is called an *$\mathcal{F}$-α-cluster function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$, each sequences $(x_{i,n})_{n\in N}$ in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ ($i \in \{1,2,...,m\}$) and each $\varepsilon > 0$

$$\{n \in N : \rho_Y(f_n(x_{i,n}), f(\xi_i)) < \varepsilon, \forall i \in \{1,...,m\}\} \in \mathcal{F}^*.$$

We denote the set of all $\mathcal{F}$-α-cluster functions of $(f_n)_{n\in N}$ by $\Gamma^{\alpha}_{f_n}(\mathcal{F})$.

Definition 2.14. The function f is called an *α-limit function* of the sequence $(f_n)_{n\in N}$ if for each $\{\xi_1, \xi_2, ..., \xi_m\} \subseteq D$ and each sequences $(x_{i,n})_{n\in N}$ in D such that $\mathcal{F} - \lim x_{i,n} = \xi_i$ ($i \in \{1,2,...,m\}$) there is an infinite set $K = \{n_1 < n_2 < ... < n_k < ...\}$ such that $\lim_{k\to\infty} f_{n_k}(x_{i,n_k}) = f(\xi_i)$ for every $i \in \{1,2,...,m\}$. We denote the set of all α-limit functions of $(f_n)_{n\in N}$ by $\mathcal{L}^{\alpha}_{f_n}$.

We have the following inclusions among the above sets, where $\mathcal{F}$ is a free filter:

$$\Lambda^{\alpha}_{f_n}(\mathcal{F}) \subseteq \Gamma^{\alpha}_{f_n}(\mathcal{F}) \subseteq \mathcal{L}^{\alpha}_{f_n},$$
$$\Lambda^{u}_{f_n}(\mathcal{F}) \subseteq \Gamma^{u}_{f_n}(\mathcal{F}) \subseteq \mathcal{L}^{u}_{f_n},$$
$$\Lambda^{pw}_{f_n}(\mathcal{F}) \subseteq \Gamma^{pw}_{f_n}(\mathcal{F}) \subseteq \mathcal{L}^{pw}_{f_n}.$$

Lemma 2.1. Let $\mathcal{F}$ be a free filter on N. If the sequence $(f_n)_{n\in N}$ is $\mathcal{F}$-exhaustive then the following holds:

(i) $\Lambda^{u}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$ and $\Gamma^{u}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$

(ii) $\Lambda^{pw}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$ and $\Gamma^{pw}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$

(iii) $\Lambda^{\alpha}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$ and $\Gamma^{\alpha}_{f_n}(\mathcal{F}) \subseteq C(D,Y)$.

3. Main Results

In this section, we give some topological properties of the sets of cluster functions such as closeness and compactness. Firstly, we recall the following topologies.

Let f be a function from D to Y. The family of sets of the form

$$W(f, A, \varepsilon) = \{g \in Y^D : \rho_Y(f(\xi), g(\xi)) < \varepsilon, \forall \xi \in A\}$$

for $\varepsilon > 0$ and A a finite subset of D, forms a base at f for *the topology of pointwise convergence* τ_{pw}.

The family of sets of the form

$$W(f, \varepsilon) = \{g \in Y^D : \rho_Y(f(\xi), g(\xi)) < \varepsilon, \forall \xi \in D\}$$

for $\varepsilon > 0$, forms a base at f for *the topology of uniform convergence* τ_u.

Definition 3.1. Let (X, ρ_X), (Y, ρ_Y) be two metric spaces, $D \subseteq X$, and $\mathcal{F}$ be a filter on N.

1. Let $\mathcal{K}$ be a family of functions from D to Y. The set $\mathcal{K}$ is said to be *(uniform) bounded* on D if there exist a $\zeta \in Y$ and a $c \in R^+$ such that $\rho_Y(f(\xi), \zeta) \leq c$ for every $f \in \mathcal{K}$ and every $\xi \in D$.

2. Let $f_n : D \to Y$, $n \in N$. The sequence $(f_n)_{n\in N}$ is *$\mathcal{F}$-uniform bounded* on D *iff* there exist a $\zeta \in Y$ and a $c \in R^+$ such that

$$\{n \in N : \rho_Y(f_n(\xi), \zeta) \leq c, \forall \xi \in D\} \in \mathcal{F}.$$

Arzela-Ascoli's Theorem. Let (X, ρ_X) be a compact metric space and $\mathcal{K}$ be a subset of $C(X, R)$. Then $\mathcal{K}$ is compact iff $\mathcal{K}$ is closed, bounded and equicontinuous

(Engelking, 1989; Gregoriades & Papanastassiou, 2008; Kelley, 1955; Willard, 1970).

For the Fréchet filter $\mathcal{F}_r$ we have

$$\mathcal{L}_{f_n}^{u} = \Lambda_{f_n}^{u}(\mathcal{F}_r) = \Gamma_{f_n}^{u}(\mathcal{F}_r),$$
$$\mathcal{L}_{f_n}^{pw} = \Lambda_{f_n}^{pw}(\mathcal{F}_r) = \Gamma_{f_n}^{pw}(\mathcal{F}_r),$$
$$\mathcal{L}_{f_n}^{\alpha} = \Lambda_{f_n}^{\alpha}(\mathcal{F}_r) = \Gamma_{f_n}^{\alpha}(\mathcal{F}_r).$$

When we take $\mathcal{F}_r$ as the filter $\mathcal{F}$, the results given below are also satisfied for $\mathcal{L}_{f_n}^{u}$, $\mathcal{L}_{f_n}^{pw}$ and $\mathcal{L}_{f_n}^{\alpha}$.

Theorem 3.1. Let (X,ρ_X), (Y,ρ_Y) be two metric spaces, $D \subseteq X$, $(f_n)_{n\in\mathrm{N}}$ be a sequence of functions from D to Y, and $\mathcal{F}$ be a filter on N. Then

(i) $\Gamma_{f_n}^{u}(\mathcal{F})$ is a closed set in the space (Y^D,τ_u).

(ii) $\Gamma_{f_n}^{pw}(\mathcal{F})$ is a closed set in the space (Y^D,τ_{pw}).

(iii) $\Gamma_{f_n}^{\alpha}(\mathcal{F})$ is a closed set in the space (Y^D,τ_{pw}).

Proof. (i) Let $g \in Y^D$ be a (topological) cluster function of the set $\Gamma_{f_n}^{u}(\mathcal{F})$. Then we have $W(g,\varepsilon)\cap\Gamma_{f_n}^{u}(\mathcal{F}) \neq \varnothing$ for every $\varepsilon>0$. Let $\varepsilon>0$ and $f \in W\left(g,\frac{\varepsilon}{2}\right)\cap\Gamma_{f_n}^{u}(\mathcal{F})$. Since $f \in W\left(g,\frac{\varepsilon}{2}\right)$, for every $\xi \in D$

$$\rho_Y(f(\xi),g(\xi)) < \frac{\varepsilon}{2}$$

holds. Also since $f \in \Gamma_{f_n}^{u}(\mathcal{F})$ we have

$$K(\varepsilon) := \left\{n\in\mathrm{N} : \rho_Y(f_n(\xi),f(\xi)) < \tfrac{\varepsilon}{2}, \forall \xi \in D\right\} \in \mathcal{F}^*.$$

Then for each $n \in K(\varepsilon)$ and each $\xi \in D$ we have

$$\rho_Y(f_n(\xi),g(\xi)) \le \rho_Y(f_n(\xi),f(\xi)) + \rho_Y(f(\xi),g(\xi)) < \varepsilon.$$

Therefore $g \in \Gamma_{f_n}^{u}(\mathcal{F})$. Consequently, the set $\Gamma_{f_n}^{u}(\mathcal{F})$ is closed.

Since the proofs of (ii) and (iii) are similar, we only prove the item (iii).

(iii) Let $g \in Y^D$ be a cluster function of the set $\Gamma_{f_n}^{\alpha}(\mathcal{F})$ in the pointwise convergence topology. In this case, for every $\varepsilon>0$ and every finite set $A \subseteq D$ we have $W(g,A,\varepsilon)\cap\Gamma_{f_n}^{\alpha}(\mathcal{F}) \neq \varnothing$. Let $\varepsilon>0$, $A=\{\xi_1,\dots,\xi_m\}\subseteq D$ be a finite set, for each $i\in\{1,2,\dots,m\}$ $(x_{i,n})_{n\in\mathrm{N}}$ be a sequence in D such that $\mathcal{F}-\lim x_{i,n}=\xi_i$, and $f \in W\left(g,A,\frac{\varepsilon}{2}\right)\cap\Gamma_{f_n}^{\alpha}(\mathcal{F})$. From $f \in W\left(g,A,\frac{\varepsilon}{2}\right)$, for every $i \in \{1,2,\dots,m\}$

$$\rho_Y(f(\xi_i),g(\xi_i)) < \frac{\varepsilon}{2}$$

holds. Also from $f \in \Gamma_{f_n}^{\alpha}(\mathcal{F})$ we have
$K(\varepsilon,A,x_{i,n}) := \{n\in\mathrm{N} : \rho_Y(f_n(x_{i,n}),f(\xi_i)) < \frac{\varepsilon}{2}, \forall i \in \{1,2,\dots,m\}\} \in \mathcal{F}^*$.

Hence for each $n \in K(\varepsilon,A,x_{i,n})$ and each $i\in\{1,2,\dots,m\}$ we have

$$\rho_Y(f_n(x_{i,n}),g(\xi_i)) \le \rho_Y(f_n(x_{i,n}),f(\xi_i)) + \rho_Y(f(\xi_i),g(\xi_i)) < \varepsilon.$$

Therefore we get $g \in \Gamma_{f_n}^{\alpha}(\mathcal{F})$, and so the set $\Gamma_{f_n}^{\alpha}(\mathcal{F})$ is closed in the pointwise convergence topology. □

In next example, we observe that the set $\Lambda_{f_n}^{u}(\mathcal{F})$ is not closed.

Example 3.1. For each $p\in\mathrm{N}$ let us define the functions g_p from R to R as

$$g_p(\xi) = \begin{cases} \dfrac{1}{p} & \text{if } \xi > 0 \\ 0 & \text{if } \xi \le 0 \end{cases}.$$

Let $J_p = \{2^{p-1}(2q-1) : q \in \mathrm{N}\}$, and let the sequence of functions $(f_n)_{n\in\mathrm{N}}$ be defined by $f_n(\xi)=g_p(\xi)$ if $n\in J_p$. For each $p\in\mathrm{N}$ we have $d(J_p)=\frac{1}{2^p}$ and so $J_p \in \mathcal{F}_{st}^*$. For each $p\in\mathrm{N}$ the subsequences $(f_n)_{n\in J_p}$ are uniformly convergent to g_p on R. Therefore we have $g_p \in \Lambda_{f_n}^{u}(\mathcal{F}_{st})$ for each $p\in\mathrm{N}$. But $f \notin \Lambda_{f_n}^{u}(\mathcal{F}_{st})$ where $f(\xi)=0$ for every $\xi\in\mathrm{R}$. Since for each $\varepsilon>0$ there exists a $g_p \in W(f,\varepsilon)\cap\Lambda_{f_n}^{u}(\mathcal{F}_{st})$, f is a (topological) cluster function of the set $\Lambda_{f_n}^{u}(\mathcal{F}_{st})$. Due to $f \notin \Lambda_{f_n}^{u}(\mathcal{F}_{st})$, the set $\Lambda_{f_n}^{u}(\mathcal{F}_{st})$ is not closed. □

Theorem 3.2. Let (X,ρ_X), (Y,ρ_Y) be two metric spaces, $D\subseteq X$, $(f_n)_{n\in\mathrm{N}}$ be a sequence of functions from D to Y, and $\mathcal{F}$ be a filter on N. Let $\mathcal{K}\subseteq Y^D$ be a compact set in the uniform convergence topology. If $\{n\in\mathrm{N} : f_n \in \mathcal{K}\}\in\mathcal{F}^*$ then $\mathcal{K}\cap\Gamma_{f_n}^{u}(\mathcal{F})$ is nonempty.

Proof. Let us assume that $\mathcal{K}\cap\Gamma_{f_n}^{u}(\mathcal{F})=\varnothing$. In this case, for each $g\in\mathcal{K}$ there exists $\varepsilon_g>0$ such that

$$\{n\in\mathrm{N} : \rho_Y(f_n(\xi),g(\xi)) < \varepsilon_g, \forall \xi\in D\} \in \mathcal{I}(\mathcal{F}).$$

The family consisting of the sets

$$S(\varepsilon_g) = \{h\in Y^D : \rho_Y(h(\xi),g(\xi)) < \varepsilon_g, \forall \xi \in D\}$$

for each $g\in\mathcal{K}$ is an open cover of $\mathcal{K}$. Since $\mathcal{K}$ is compact, this cover has a finite subcover, i.e., $\mathcal{K} \subseteq \bigcup_{i=1}^{p} S(\varepsilon_{g_i})$. Then

$$\begin{aligned}\{n \in \mathrm{N} : f_n \in \mathcal{K}\} &\subseteq \{n \in \mathrm{N} : f_n \in \textstyle\bigcup_{i=1}^{p} S(\varepsilon_{g_i})\} \\ &\subseteq \textstyle\bigcup_{i=1}^{p} \{n \in \mathrm{N} : f_n \in S(\varepsilon_{g_i})\} \\ \subseteq \textstyle\bigcup_{i=1}^{p} \{n \in \mathrm{N} : \rho_Y(f_n(\xi), g_i(\xi)) < \varepsilon_{g_i}, \forall \xi \in D\} &\in I(\mathcal{F}),\end{aligned}$$

and so we get $\{n \in \mathrm{N} : f_n \in \mathcal{K}\} \in I(\mathcal{F})$. This is a contradiction. □

We now give examples to show that the above theorem is not satisfied without the condition of compactness of $\mathcal{K}$.

Example 3.2. If the set $\mathcal{K}$ is not closed, bounded or equicontinuous, Theorem 3.2 need not be true.

(1) We define the functions $g_n \in C([0,1], \mathrm{R})$ by $g_n(\xi) = \xi^n$ for each $n \in \mathrm{N}$. Let $\mathcal{K} = \{g_n : n \in \mathrm{N}\}$ and let us take the sequence $(f_n)_{n\in\mathrm{N}}$ defined by

$$f_n(\xi) = \begin{cases} g_n(\xi) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

The set $\mathcal{K}$ is closed, bounded, but not equicontinuous. Hence $\mathcal{K}$ is not compact. Then we have

$$\{n \in \mathrm{N} : f_n \in \mathcal{K}\} = \{2n-1 : n \in \mathrm{N}\} \in \mathcal{F}_{st}^*,$$

but $\Gamma_{f_n}^{u}(\mathcal{F}_{st}) = \{f\}$ where $f(\xi) = 0$ for every $\xi \in [0,1]$ and so $\mathcal{K} \cap \Gamma_{f_n}^{u}(\mathcal{F}_{st}) = \varnothing$.

(2) Let $\mathcal{F}$ be any free filter. Define the sequence $(f_n)_{n\in\mathrm{N}}$ by $f_n(\xi) = 1/n$ for every $\xi \in [0,1]$ and each $n \in \mathrm{N}$. Let $\mathcal{K} = \{f_n : n \in \mathrm{N}\}$. The set $\mathcal{K}$ is bounded, equicontinuous, but not closed. So it is not compact. We have

$$\{n \in \mathrm{N} : f_n \in \mathcal{K}\} = \mathrm{N} \in \mathcal{F},$$

but $\Gamma_{f_n}^{u}(\mathcal{F}) = \{f\}$ where $f(\xi) = 0$ for every $\xi \in [0,1]$ and so $\mathcal{K} \cap \Gamma_{f_n}^{u}(\mathcal{F}) = \varnothing$.

(3) Let $g_n(\xi) = n$ for every $\xi \in [0,1]$ and each $n \in \mathrm{N}$, and let us define the sequence $(f_n)_{n\in\mathrm{N}}$ in $C([0,1], \mathrm{R})$ by

$$f_n(\xi) = \begin{cases} g_n(\xi) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Let $\mathcal{K} = \{g_n : n \in \mathrm{N}\}$. The set $\mathcal{K}$ is closed, equicontinuous, but unbounded. Hence $\mathcal{K}$ is not compact. Then we have

$$\{n \in \mathrm{N} : f_n \in \mathcal{K}\} = \{2n-1 : n \in \mathrm{N}\} \in \mathcal{F}_{st}^*,$$

but $\Gamma_{f_n}^{u}(\mathcal{F}_{st}) = \{f\}$ where $f(\xi) = 0$ for every $\xi \in [0,1]$ and so $\mathcal{K} \cap \Gamma_{f_n}^{u}(\mathcal{F}_{st}) = \varnothing$. □

Theorem 3.3. Let D be a compact subset of a metric space (X, ρ_X), and (Y, ρ_Y) be a compact metric space. Let $(f_n)_{n\in\mathrm{N}}$ be a sequence of functions from D to Y, and $\mathcal{F}$ be a filter on N. If the sequence $(f_n)_{n\in\mathrm{N}}$ is $\mathcal{F}$-uniform bounded and equicontinuous on D then $\Gamma_{f_n}^{u}(\mathcal{F})$ is nonempty.

Proof. Since $(f_n)_{n\in\mathrm{N}}$ is $\mathcal{F}$-uniform bounded, there exist a $\zeta \in Y$ and a $c \in \mathrm{R}^+$ such that

$$L := \{n \in \mathrm{N} : \rho_Y(f_n(\xi), \zeta) \le c, \forall \xi \in D\} \in \mathcal{F}.$$

Then there exists a compact set $\mathcal{K}$ such that $f_n \in \mathcal{K}$ for every $n \in L$. If $\Gamma_{f_n}^{u}(\mathcal{F})$ is empty then $\mathcal{K} \cap \Gamma_{f_n}^{u}(\mathcal{F}) = \varnothing$. From Theorem 3.2 we have $\{n \in \mathrm{N} : f_n \in \mathcal{K}\} \in I(\mathcal{F})$. Then

$$L = \{n \in L : f_n \in \mathcal{K}\} \subseteq \{n \in \mathrm{N} : f_n \in \mathcal{K}\}$$

and we get $L \in I(\mathcal{F})$. This contradicts with $L \in \mathcal{F}$. □

If we remove one of the conditions in the above theorem, $\Gamma_{f_n}^{u}(\mathcal{F})$ can be empty. Let $\mathcal{F}$ be any free filter. The sequence $(g_n)_{n\in\mathrm{N}}$ in Example 3.2(1) is not equicontinuous, and $\Gamma_{g_n}^{u}(\mathcal{F}) = \varnothing$. Similarly, in Example 3.2(3) the sequence $(g_n)_{n\in\mathrm{N}}$ is not $\mathcal{F}$-uniform bounded, and $\Gamma_{g_n}^{u}(\mathcal{F}) = \varnothing$.

Theorem 3.4. Let (X, ρ_X), (Y, ρ_Y) be two metric spaces, $D \subseteq X$, and $\mathcal{F}$ be a filter on N. Let $(f_n)_{n\in\mathrm{N}}$ and $(g_n)_{n\in\mathrm{N}}$ be two sequences of functions on Y^D such that

$$\{n \in \mathrm{N} : f_n(\xi) = g_n(\xi) \text{ for all } \xi \in D\} \in \mathcal{F}.$$

Then we have

(i) $\Lambda_{f_n}^{u}(\mathcal{F}) = \Lambda_{g_n}^{u}(\mathcal{F})$ and $\Gamma_{f_n}^{u}(\mathcal{F}) = \Gamma_{g_n}^{u}(\mathcal{F})$,

(ii) $\Lambda_{f_n}^{pw}(\mathcal{F}) = \Lambda_{g_n}^{pw}(\mathcal{F})$ and $\Gamma_{f_n}^{pw}(\mathcal{F}) = \Gamma_{g_n}^{pw}(\mathcal{F})$,

(iii) $\Lambda_{f_n}^{\alpha}(\mathcal{F}) = \Lambda_{g_n}^{\alpha}(\mathcal{F})$ and $\Gamma_{f_n}^{\alpha}(\mathcal{F}) = \Gamma_{g_n}^{\alpha}(\mathcal{F})$.

Proof. We only give the proofs of (i) and (iii) due to the fact that the proof techniques are same. Let

$$L := \{n \in \mathrm{N} : f_n(\xi) = g_n(\xi) \text{ for all } \xi \in D\}.$$

(i) Let $f \in \Lambda_{f_n}^{u}(\mathcal{F})$. Then there is a set $K \in \mathcal{F}^*$ such that the subsequence $(f_n)_{n\in K}$ is uniformly convergent to the function f. Let $M := K \cap L$. Since $L \in \mathcal{F}$ and $K \in \mathcal{F}^*$, it is clear that $M \in \mathcal{F}^*$. Since $f_n = g_n$ for every $n \in M$, the subsequence

$(g_n)_{n\in M}$ is uniformly convergent to f. Therefore we get $f\in\Lambda^u_{g_n}(\mathcal{F})$, and so $\Lambda^u_{f_n}(\mathcal{F})\subseteq\Lambda^u_{g_n}(\mathcal{F})$. The reverse is also similar. Consequently, we have $\Lambda^u_{f_n}(\mathcal{F})=\Lambda^u_{g_n}(\mathcal{F})$. Now assume $f\in\Gamma^u_{f_n}(\mathcal{F})$. Let $\varepsilon>0$. Then we have

$$K(\varepsilon)=\{n\in\mathrm{N}:\rho_Y(f_n(\xi),f(\xi))<\varepsilon,\forall\xi\in D\}\in\mathcal{F}^*.$$

Let $M(\varepsilon):=K(\varepsilon)\cap L$. Then $M(\varepsilon)\in\mathcal{F}^*$, and for each $n\in M(\varepsilon)$ and each $\xi\in D$ we get

$$\rho_Y(f_n(\xi),f(\xi))<\varepsilon$$

and so

$$\rho_Y(g_n(\xi),f(\xi))<\varepsilon$$

from $f_n(\xi)=g_n(\xi)$. Hence the following holds

$$\{n\in\mathrm{N}:\rho_Y(g_n(\xi),f(\xi))<\varepsilon,\forall\xi\in D\}\supseteq M(\varepsilon).$$

From $M(\varepsilon)\in\mathcal{F}^*$, we obtain

$$\{n\in\mathrm{N}:\rho_Y(g_n(\xi),f(\xi))<\varepsilon,\forall\xi\in D\}\in\mathcal{F}^*.$$

Therefore we get $f\in\Gamma^u_{g_n}(\mathcal{F})$, and so $\Gamma^u_{f_n}(\mathcal{F})\subseteq\Gamma^u_{g_n}(\mathcal{F})$. The reverse can be similarly shown. Consequently, $\Gamma^u_{f_n}(\mathcal{F})=\Gamma^u_{g_n}(\mathcal{F})$ holds.

(iii) Assume $f\in\Lambda^\alpha_{f_n}(\mathcal{F})$. Let $\{\xi_1,\xi_2,...,\xi_m\}\subseteq D$ and for each $i\in\{1,2,...,m\}$ $(x_{i,n})_{n\in\mathrm{N}}$ be sequences in D such that $\mathcal{F}-\lim x_{i,n}=\xi_i$. Then there is a set $K=\{n_1<n_2<...<n_k<...\}\in\mathcal{F}^*$ such that $\lim_{k\to\infty}f_{n_k}(x_{i,n_k})=f(\xi_i)$ for every $i\in\{1,2,...,m\}$. Let $M:=K\cap L\in\mathcal{F}^*$. Since $f_n=g_n$ for every $n\in M$, we have $\lim_{n\to\infty,n\in M}g_n(x_{i,n})=f(\xi_i)$ for every $i\in\{1,2,...,m\}$. Therefore we get $f\in\Lambda^\alpha_{g_n}(\mathcal{F})$, and so $\Lambda^\alpha_{f_n}(\mathcal{F})\subseteq\Lambda^\alpha_{g_n}(\mathcal{F})$. The reverse is also similar.

Now assume $f\in\Gamma^\alpha_{f_n}(\mathcal{F})$. Let $\{\xi_1,\xi_2,...,\xi_m\}\subseteq D$, for each $i\in\{1,2,...,m\}$ $(x_{i,n})_{n\in\mathrm{N}}$ be sequences in D such that $\mathcal{F}-\lim x_{i,n}=\xi_i$, and $\varepsilon>0$. Then we have

$$K:=\{n\in\mathrm{N}:\rho_Y(f_n(x_{i,n}),f(\xi_i))<\varepsilon,\forall i\in\{1,...,m\}\}\in\mathcal{F}^*.$$

Let $M:=K\cap L\in\mathcal{F}^*$. Then for each $n\in M$ and each $i\in\{1,2,...,m\}$ we get

$$\rho_Y(f_n(x_{i,n}),f(\xi_i))<\varepsilon$$

and so

$$\rho_Y(g_n(x_{i,n}),f(\xi_i))<\varepsilon$$

from $f_n(x_{i,n})=g_n(x_{i,n})$. Hence the following holds

$$\{n\in\mathrm{N}:\rho_Y(g_n(x_{i,n}),f(\xi_i))<\varepsilon,\forall i\in\{1,2,...,m\}\}\supseteq M.$$

From $M\in\mathcal{F}^*$, we obtain

$$\{n\in\mathrm{N}:\rho_Y(g_n(x_{i,n}),f(\xi_i))<\varepsilon,\forall i\in\{1,2,...,m\}\}\in\mathcal{F}^*.$$

Therefore we get $f\in\Gamma^\alpha_{g_n}(\mathcal{F})$, and so $\Gamma^\alpha_{f_n}(\mathcal{F})\subseteq\Gamma^\alpha_{g_n}(\mathcal{F})$. The reverse can be similarly shown. □

Theorem 3.5. Let $\mathcal{F}$ be a free filter on N. Let D be a compact subset of a metric space (X,ρ_X), and $(f_n)_{n\in\mathrm{N}}$ be a sequence of functions from D to a compact metric space (Y,ρ_Y) which is $\mathcal{F}$-uniform bounded and $\mathcal{F}$-exhaustive. Then the set $\Gamma^u_{f_n}(\mathcal{F})$ is compact.

Proof. We show that the set $\Gamma^u_{f_n}(\mathcal{F})$ satisfies the conditions of Arzela-Ascoli's Theorem.

Closeness: It was shown in Theorem 3.1.

Boundedness: Since the sequence $(f_n)_{n\in\mathrm{N}}$ is $\mathcal{F}$-uniform bounded, there exist a $\zeta\in Y$ and a $c\in\mathrm{R}^+$ such that

$$K:=\{n\in\mathrm{N}:\rho_Y(f_n(\xi),\zeta)\le c,\forall\xi\in D\}\in\mathcal{F}.$$

If $f\in\Gamma^u_{f_n}(\mathcal{F})$ then for each $\varepsilon>0$

$$L(\varepsilon):=\{n\in\mathrm{N}:\rho_Y(f_n(\xi),f(\xi))<\varepsilon,\forall\xi\in D\}\in\mathcal{F}^*$$

holds. Then the intersection of the sets K and $L(\varepsilon)$ belongs to $\mathcal{F}^*$, and so is nonempty. Hence there is an $n_0\in K\cap L(\varepsilon)$, and we have

$$\rho_Y(f(\xi),\zeta)\le\rho_Y(f(\xi),f_{n_0}(\xi))+\rho_Y(f_{n_0}(\xi),\zeta)<\varepsilon+c$$

for every $\xi\in D$. Since for every $\varepsilon>0$ the above inequality is satisfied, we get $\rho_Y(f(\xi),\zeta)<c$ for every $\xi\in D$. Since the function f is arbitrary, the set $\Gamma^u_{f_n}(\mathcal{F})$ is bounded.

Equicontinuity: From Lemma 2.1 we can say that $\Gamma^u_{f_n}(\mathcal{F})\subseteq C(D,Y)$. Let $\xi\in D$ and $\varepsilon>0$. Since the sequence $(f_n)_{n\in\mathrm{N}}$ is $\mathcal{F}$-exhaustive at the point ξ, there is a $\delta>0$ such that

$$K(\varepsilon):=\{n\in\mathrm{N}:\rho_Y(f_n(\eta),f_n(\xi))<\varepsilon/3,\ \forall\eta\in S(\xi,\delta)\cap D\}\in\mathcal{F}.$$

If $f\in\Gamma^u_{f_n}(\mathcal{F})$ then the following holds

$$L(\varepsilon):=\{n\in\mathrm{N}:\rho_Y(f_n(\xi),f(\xi))<\varepsilon/3,\forall\xi\in D\}\in\mathcal{F}^*.$$

Then the intersection of the sets $K(\varepsilon)$ and $L(\varepsilon)$ belongs to $\mathcal{F}^*$, and so is nonempty. So there exists an $n_0\in K(\varepsilon)\cap L(\varepsilon)$ such that we have

$$\rho_Y(f(\eta),f(\xi))\leq\rho_Y(f(\eta),f_{n_0}(\eta))+\rho_Y(f_{n_0}(\eta),f_{n_0}(\xi))$$
$$+\rho_Y(f_{n_0}(\xi),f(\xi))<\varepsilon$$

for every $\eta\in S(\xi,\delta)\cap D$. Here δ is independent from choice of *f*. Hence the set $\Gamma^u_{f_n}(\mathcal{F})$ is equicontinuous at the point ξ.

According to Arzela-Ascoli's Theorem, $\Gamma^u_{f_n}(\mathcal{F})$ is compact. □

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References

Albayrak, H. & Pehlivan, S. (2013). Filter exhaustiveness and $\mathcal{F}$-convergence of function sequences. Filomat, **27**(8):1373-1383.

Aviles Lopez, A., Cascales Salinas, B., Kadets, V. & Leonov, A. (2007). The Schur theorem for filters. Journal of Mathematical Physics, Analysis, Geometry, **3**(4):383-398.

Balcerzak, M., Dems, K. & Komisarski, A. (2007). Statistical convergence and ideal convergence for sequences of functions. Journal of Mathematical Analysis and Applications, **328**:715-729.

Boccuto, A., Dimitriou, X., Papanastassiou, N. & Wilczyński, W. (2011). Ideal exhaustiveness, continuity and (α)-convergence for lattice group-valued functions. International Journal of Pure and Applied Mathematics, **70**(2):211-227.

Buck, R.C. (1953). Generalized asymptotic density. American Journal of Mathematics, **75**:335-346.

Carathéodory, C. (1929). Stetige konvergenz und normale Familien von Funktionen. Mathematische Annalen, **101**(1):515-533.

Caserta, A. & Kočinac, Lj.D.R. (2012). On statistical exhaustiveness. Applied Mathematics Letters, **25**(10):1447-1451.

Činčura, J., Šalát, T., Sleziak, M. & Toma, V. (2004). Sets of statistical cluster points and $\mathcal{I}$-cluster points. Real Analysis Exchange, **30**(2):565-580.

Connor, J.S. & Kline, J. (1996). On statistical limit points and consistency of statistical convergence. Journal of Mathematical Analysis and Applications, **197**(2):392-399.

Das, R. & Papanastassiou, N. (2004). Some types of convergence of sequences of real valued functions. Real Analysis Exchange, **29**(1):43-59.

Di Maio, G. & Kočinac, Lj.D.R. (2008). Statistical convergence in topology. Topology and its Applications, **156**(1):28-45.

Engelking, R. (1989). General Topology (Revised and completed edition). Heldermann Verlag, Berlin.

Fast, H. (1951). Sur la convergence statistique. Colloquium Mathematicum, **2**:241-244.

Fridy, J.A. (1993). Statistical limit points. Proceedings of the American Mathematical Society, **118**(4):1187-1192.

Gregoriades, V. & Papanastassiou, N. (2008). The notion of exhaustiveness and Ascoli-type theorems. Topology and its Applications, **155**(10):1111-1128.

Kadets, V., Leonov, A. & Orhan, C. (2010). Weak statistical convergence and weak filter convergence for unbounded sequences. Journal of Mathematical Analysis and Applications, **371**:414-424.

Katětov, M. (1968). Products of filters. Commentationes Mathematicae Universitatis Carolinae, **9**:173-189.

Kelley, J.L. (1955). General Topology. D. Van Nostrand Company, Toronto-New York-London.

Kostyrko, P., Šalát, T. & Wilczyński, W. (2001). $\mathcal{I}$-convergence. Real Analysis Exchange **26**(2):669-685.

Kostyrko, P., Mačaj, M., Šalát, T. & Strauch, O. (2001). On statistical limit points. Proceedings of the American Mathematical Society, **129**(9):2647-2654.

Kostyrko, P., Mačaj, M., Šalát, T. & Sleziak, M. (2005). $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. Mathematica Slovaca, **55**(4):443-464.

Mamedov, M.A. & Pehlivan, S. (2000). Statistical convergence of optimal paths. Mathematica Japonica, **52**(1):51-55.

Mamedov, M.A. & Pehlivan, S. (2001). Statistical cluster points and turnpike theorem in nonconvex problems. Journal of Mathematical Analysis and Applications, **256**(2):686-693.

Miller, H.I. (1995). A measure theoretical subsequence characterization of statistical convergence. Transactions of the American Mathematical Society, **347**:1811-1819.

Niven, I. (1951). The asymptotic density of sequences. Bulletin of the American Mathematical Society, **57**:420-434.

Pehlivan, S. & Mamedov, M.A. (2000). Statistical cluster points and turnpike. Optimization, **48**(1):93-106.

Pehlivan, S., Güncan, A. & Mamedov, M.A. (2004). Statistical cluster points of sequences infinite dimensional spaces. Czechoslovak Mathematical Journal, **54**(129):95-102.

Stoilov, S. (1959). Continuous convergence. Revue de Mathematiques Pures et Appliquees, **4**:341-344.

Willard, S. (1970). General Topology. Addison-Wesley Publishing Company, Massachusetts.

Zaslavski, A.J. (2006). Turnpike Properties in the Calculus of Variations and Optimal Control. Springer, New York.

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بعض الخصائص الطوبولوجية لمجموعة دوال المرشحات العنقودية

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خلاصة

كنا قد عممنا سنة ٢٠١٣ مفاهيم التقارب النقطي ، التقارب المنتظم ، و تقارب ∞ لمتتاليات دوال على فضاءات مترية و ذلك باستخدام مرشحات على N . ونعرف في بحثنا اليوم مفاهيم دالة النهاية ، دالة نهاية 6F ودالة F العنقودية على التوالي لكل أنواع التقارب الثلاثة سالفة الذكر ، حيث F هو مرشحة على N. و نقوم باستقصاء بعض الخصائص الطوبولوجية لمجموعات الدوال النقطية العنقودية من النوع F ، الدوال العنقودية من النوع ∞-F وكذلك الدوال العنقودية المنتظمة من النوع F ، و ذلك باستخدام طبولوجيات النقط و التقارب المنتظم.