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تقويم أسطح القواعد

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الملخص

نقوم في هذه الدراسة بتعريف التقويم العام لأسطح القواعد في الفضاء الإقليدي 3–space . نقدم في هذا البحث بعض توصيفات تقويم أسطح القواعد بتدارس تقوسات المُنحني الأساسي. حصلنا على تقوس جاوس وكذلك التقوس الوَسَطي ودرسنا شرط صغر السطح. علاوة على ذلك، نقدم توصيفًا لبعض المنحنيات الخاصة الموجودة على هذا السطح. أخيرًا درسنا العلاقات بين تقويم أسطح القواعد وأسطح القواعد المائلة.

On rectifying ruled surfaces

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Abstract

In this study, we define general rectifying ruled surfaces in the Euclidean 3-space E^3. We give some characterizations of rectifying ruled surfaces by considering the curvatures of base curve. We obtain the Gaussian curvature and the mean curvature and we investigate the condition for the surface to be minimal. Moreover, we give characterizations for some special curves lying on this surface. Finally, we obtain the relationships between rectifying ruled surfaces and slant ruled surfaces.

Keywords: Minimal surface; rectifying surface; slant helix; slant ruled surface.

1. Introduction

Izumiya & Takeuchi (2004) defined rectifying developable of a space curve α as a ruled surface whose generator line is modified Darboux vector of α defined by $\tilde{D}=(\tau/\kappa)T+B$, where T, B are unit tangent and unit binormal vectors and $\kappa\neq 0$, τ are curvatures of α and parametric form of this surface has been given by $F_{(\alpha,\tilde{D})}(s,u)=\alpha(s)+u\tilde{D}\,(s)$. In the same paper, they also defined a new kind of special curve called a slant helix. In addition, they studied the Darboux developable of a space curve whose singularities are given by the locus of the end points of the modified Darboux vector of the curve (Izumiya & Takeuchi, 2003; Izumiya & Takeuchi, 2003). Furthermore, Izumiya *et al*. (1999) studied singularities of the rectifying developable surface of a space curve and obtained that rectifying developable of a curve α is non-singular if and only if α is a cylindrical helix.

Later, Solimon *et al.* (2017) created a different definition for rectifying developable ruled surfaces. They defined this surface as the surface whose generator line is the unit Darboux vector of a space curve α given by $D=(\tau T+\kappa B)\big/\sqrt{\kappa^2+\tau^2}$. They obtained that this surface has pointwise a 1-type Gauss map of the first kind with a base plane curve if and only if the base curve is a circle or straight line.

Moreover, Önder (2018) defined some new kinds of ruled surfaces in the three-dimensional Euclidean space E^3 called slant-ruled surfaces. Önder & Kaya (2017) provided some characterizations for slant-ruled surfaces. They also defined two new types of slant ruled surfaces called a Darboux slant-ruled surface and slant-null scrolls (Önder & Kaya, 2015; Önder & Kaya, 2016). Furthermore, they studied the position vectors of developable slant-ruled surfaces (Kaya & Önder, 2017; Kaya & Önder, 2018).

This paper gives a general definition of rectifying ruled surfaces including the definitions given by Izumiya & Takeuchi (2004) and Solimon *et al*. (2017). It is obtained that the only developable rectifying surface of a space curve α with curvature $\kappa\neq 0$ is the surface $F_{(\alpha,D)}(s,u)$ defined by Izumiya & Takeuchi (2004). Moreover, we investigate the special curves on a rectifying surface. Finally, we give the relationships between rectifying surfaces and slant ruled surfaces and introduce some examples.

2. Preliminaries

A surface in R^3 is called a ruled (or regle) surface if it is constructed by a continuous moving of a line along a space curve $\alpha\,(s)$. Such a surface has the parametric form

$$F_{(\alpha,q)}(s,u): I\times R\rightarrow R^3,$$
$$F_{(\alpha,q)}(s,u)=\alpha(s)+uq(s),$$

where I is an open interval and $\alpha(s): I\rightarrow R^3$, $q(s): I\rightarrow R^3-\{0\}$ are smooth mappings and called the base curve and directrix curve, respectively. The straight lines $u\rightarrow\alpha(s)+uq(s)$ are called ruling. A ruled surface $F_{(\alpha,q)}(s,u)$ with $\|q\|=1$ is called cylindrical if $q'(s)=0$ and non-cylindrical if $q'(s)\neq 0$ (Karger & Novak, 1978).

A curve $c(s)$ lying on $F_{(\alpha,q)}(s,u)$ with the property that $\langle c',q'\rangle = 0$ is called the striction curve (or striction line) of $F_{(\alpha,q)}(s,u)$. A striction curve has an important geometric meaning such that if there exists a common perpendicular to two constructive rulings, then the foot of the common perpendicular on the main ruling is called a central point and the locus of the central points is a striction curve.

The unit normal U (or Gauss map) of the surface $F_{(\alpha,q)}(s,u)$ is defined by

$$U(s,u) = \frac{\left(\frac{\partial F_{(\alpha,q)}}{\partial s} \times \frac{\partial F_{(\alpha,q)}}{\partial u}\right)}{\left\| \frac{\partial F_{(\alpha,q)}}{\partial s} \times \frac{\partial F_{(\alpha,q)}}{\partial u} \right\|}. \tag{1}$$

The points (s_0, u_0) for which $\frac{\partial F_{(\alpha,q)}}{\partial s} \times \frac{\partial F_{(\alpha,q)}}{\partial u} = 0$ are called singular points of the surface $F_{(\alpha,q)}(s,u)$. Non-singular points of $F_{(\alpha,q)}(s,u)$ are called regular points. $F_{(\alpha,q)}(s,u)$ is called developable if the unit normal U along any ruling does not change its direction. Otherwise, $F_{(\alpha,q)}(s,u)$ is called non-developable or skew. A ruled surface $F_{(\alpha,q)}(s,u)$ is developable if and only if $\det(\alpha', q, q') = 0$.

Let $\|q(s)\| = 1$. Then, the unit vectors $h(s) = q'(s)/\|q'(s)\|$ and $a(s) = q(s) \times h(s)$ are called central normal and central tangent, respectively. Then, the orthonormal frame $\{q(s), h(s), a(s)\}$ is called the Frenet frame of ruled surface $F_{(\alpha,q)}(s,u)$ (Karger & Novak, 1978).

Definition 2.1. (Önder, 2018) *A ruled surface $F_{(\alpha,q)}(s,u)$ is called q-slant or a -slant (resp. h-slant) ruled surface if its ruling q(s) (resp. central normal h(s)) always makes a constant angle with a fixed direction.*

The first and second fundamental forms I and II of $F_{(\alpha,q)}(s,u)$ are defined by

$$\begin{cases} I = Eds^2 + 2Fdsdu + Gdu^2, \\ II = Lds^2 + 2Mdsdu + Ndu^2, \end{cases} \tag{2}$$

respectively, where

$$\begin{cases} E = \left\langle \frac{\partial F_{(\alpha,q)}}{\partial s}, \frac{\partial F_{(\alpha,q)}}{\partial s} \right\rangle, \\ F = \left\langle \frac{\partial F_{(\alpha,q)}}{\partial s}, \frac{\partial F_{(\alpha,q)}}{\partial u} \right\rangle, \\ G = \left\langle \frac{\partial F_{(\alpha,q)}}{\partial u}, \frac{\partial F_{(\alpha,q)}}{\partial u} \right\rangle, \end{cases} \tag{3}$$

$$\begin{cases} L = \left\langle \frac{\partial^2 F_{(\alpha,q)}}{\partial s^2}, U \right\rangle, \\ M = \left\langle \frac{\partial^2 F_{(\alpha,q)}}{\partial s \partial u}, U \right\rangle, \\ N = \left\langle \frac{\partial^2 F_{(\alpha,q)}}{\partial u^2}, U \right\rangle. \end{cases} \tag{4}$$

Then, the Gauss curvature K and mean curvature H are defined by

$$K = \frac{LN - M^2}{EG - F^2} \tag{5}$$

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} \tag{6}$$

respectively. An arbitrary surface is called minimal if $H = 0$ at all points of the surface (Do Carmo, 1976).

Theorem 2.2. (Catalan Theorem) (Fomenko & Tuzhilin, 2005) *Among all ruled surfaces except planes only the helicoid and fragments of it are minimal.*

Definition 2.3. (Izumiya & Takeuchi, 2004) *A curve $\alpha(s): I \to R^3$ is called a slant helix if its principal normal vector $N(s)$ always makes a constant angle with a fixed direction.*

3. General rectifying ruled surfaces

Izumiya & Takeuchi (2004) defined a rectifying developable surface by the parametric form $F_{(\gamma,\tilde{D})}(s,u) = \gamma(s) + u\tilde{D}(s)$ where $\gamma(s): I \to R^3$ is a differentiable unit speed curve with Frenet frame $\{T(s), N(s), B(s)\}$, and curvatures $\kappa(s) \neq 0$, $\tau(s)$ and $\tilde{D}(s)$ is the modified Darboux vector of $\gamma(s)$ defined by $\tilde{D}(s) = (\tau/\kappa)(s)T(s) + B(s)$. In this section, we define general rectifying ruled surfaces associated to a curve such that the ruling of the surface lies on the rectifying plane of the base curve. First, we define such surfaces as follows:

Definition 3.1. *Let $\alpha(s): I \to R^3$ be a unit speed smooth curve with arclength parameter s and Frenet apparatus $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$. The ruled surface $F_{(\alpha,q_r)}(s,u): I \times R \to R^3$ defined by*

$$\begin{cases} F_{(\alpha,q_r)}(s,u) = \alpha(s) + uq_r(s), \\ q_r(s) = a_1(s)T(s) + a_2(s)B(s) \end{cases} \tag{7}$$

is called the general rectifying surface of $\alpha(s)$, where $a_i(s), (i = 1,2)$ are smooth functions of arc-length *parameter s and the indices r is used to emphasize that $q_r(s)$ always lies on rectifying plane at the point $\alpha(s)$.*

It is clear that the rectifying developable surface $F_{(\alpha,\tilde{D})}(s,u)$ is an example of general rectifying surfaces with $a_1(s)=(\tau/\kappa)(s)$, $a_2(s)=1$. If we take $a_1(s)=1$, $a_2(s)=0$, then we have developable tangent surface $F_{(\alpha,T)}(s,u)$ of $\alpha(s)$. Similarly, taking $a_1(s)=0$, $a_2(s)=1$ gives the binormal surface $F_{(\alpha,B)}(s,u)$ of $\alpha(s)$. Both these surfaces are well-known examples of ruled surfaces associated to a space curve $\alpha(s)$. In this study, we will consider the general case. For this purpose, we will assume that $a_i(s)\neq 0$, $(i=1,2)$.

Proposition 3.2. *The locus of the singular points of $F_{(\alpha,q_r)}$ is the curve*

$$\gamma(s)=\alpha(s)+u(s)q_r(s),$$

where $a_1\kappa-a_2\tau=0$ *and* $u(s)=\dfrac{-1}{a_2\left(\dfrac{\tau}{\kappa}\right)'}$.

Proof. We can calculate that

$$\begin{cases}\dfrac{\partial F_{(\alpha,q_r)}}{\partial s}=(1+ua_1')T+u(a_1\kappa-a_2\tau)N\\ \qquad\qquad +ua_2'B,\\ \dfrac{\partial F_{(\alpha,q_r)}}{\partial u}=a_1T+a_2B.\end{cases} \tag{8}$$

Then we have

$$\begin{aligned}\frac{\partial F_{(\alpha,q_r)}}{\partial s}\times\frac{\partial F_{(\alpha,q_r)}}{\partial u}&=ua_2(a_1\kappa-a_2\tau)T\\ &-a_2\left(1+ua_2\left(\frac{a_1}{a_2}\right)'\right)N\\ &-ua_1(a_1\kappa-a_2\tau)B\end{aligned} \tag{9}$$

Therefore, a point (s_0,u_0) is a singular point of $F_{(\alpha,q_r)}$ if and only if

$$a_1\kappa-a_2\tau=0,\ 1+ua_2\left(\frac{a_1}{a_2}\right)'=0, \tag{10}$$

which gives $u(s)=\dfrac{-1}{a_2\left(\dfrac{\tau}{\kappa}\right)'}$.

Hereafter, for brevity, we will take

$$a_1\kappa-a_2\tau=x,\quad 1+ua_2\left(\frac{a_1}{a_2}\right)'=y.$$

From Equation (7), we have that $q_r'=a_1'T+xN+a_2'B$. Then, it is obtained that the surface $F_{(\alpha,q_r)}$ with $\|q\|=1$ is cylindrical if and only if a_i, $(i=1,2)$ are non-zero constants and $x=0$, which means that if the surface is cylindrical, then $\alpha(s)$ is a helix. The converse is not always true. Moreover, the striction parameter of $F_{(\alpha,q_r)}$ is obtained as

$$u(s)=-\frac{\langle\alpha',q_r'\rangle}{\langle q_r',q_r'\rangle}=-\frac{a_1'}{(a_1')^2+(a_2')^2+x^2} \tag{11}$$

From Equation (11), it follows that the base curve $\alpha(s)$ is striction line if and only if a_1 is constant. Under this condition, if we consider the function $x=0$ or equivalently, $a_1/a_2=\tau/\kappa$, we have that τ/κ is constant if and only if a_2 is constant. Then, we have the following proposition.

Proposition 3.3. *Let $\alpha(s)$ be a striction line of developable surface $F_{(\alpha,q_r)}$ with $\|q\|=1$. Then, $F_{(\alpha,q_r)}$ is cylindrical if and only if $\alpha(s)$ is a helix.*

Theorem 3.4. *If $\kappa\neq 0$, there exists no developable general rectifying surface $F_{(\alpha,q_r)}$ different from rectifying surface $F_{(\alpha,\tilde{D})}$.*

Proof. The surface $F_{(\alpha,q_r)}$ is developable if and only if $\det\left(\alpha'(s),q_r(s),q_r'(s)\right)=0$.

Then, $F_{(\alpha,q_r)}$ is developable if and only if $x=0$, which gives $a_1=a_2(\tau/\kappa)$. Thus, we get, $q_r(s)=a_2\left((\tau/\kappa)T+B\right)=a_2\tilde{D}(s)$, i.e., $F_{(\alpha,q_r)}=F_{(\alpha,\tilde{D})}$.

Corollary 3.5. *If $\kappa\neq 0$, then for the rectifying surface $F_{(\alpha,q_r)}$, the following are equivalent.*

(i) $F_{(\alpha,q_r)}$ is developable.

(ii) $F_{(\alpha,q_r)}=F_{(\alpha,\tilde{D})}$

Corollary 3.6. *For the general rectifying surface $F_{(\alpha,q_r)}$,*

(i) $\alpha(s)$ is a geodesic.

(ii) $\alpha(s)$ is not an asymptotic curve.

Proof. We know that $\alpha(s)$ is a geodesic (respectively, asymptotic curve) if and only if $U_\alpha\times N=0$ (respectively, $\langle U_\alpha,N\rangle=0$), where U_α is unit surface normal along $\alpha(s)$ and N is the principal normal of $\alpha(s)$. The unit normal of the surface $F_{(\alpha,q_r)}$ is

$$\begin{aligned}U&=\frac{\dfrac{\partial F_{(\alpha,q_r)}}{\partial s}\times\dfrac{\partial F_{(\alpha,q_r)}}{\partial u}}{A}\\ &=\frac{1}{A}\left[ua_2xT-a_2yN-ua_1xB\right]\end{aligned} \tag{12}$$

where $A=\left\|\frac{\partial F_{(\alpha,q_r)}}{\partial s}\times\frac{\partial F_{(\alpha,q_r)}}{\partial u}\right\|$. Then the surface normal along base curve $\alpha(s)$ is $U_\alpha=-N$. Thus, we have $\alpha(s)$ is a geodesic and not an asymptotic curve.

The fundamental coefficients of $F_{(\alpha,q_r)}$ are computed as

$$E=(1+ua_1')^2+u^2x^2+u^2(a_2')^2$$
$$F=a_1+u(a_1a_1'+a_2a_2')$$
$$G=a_1^2+a_2^2$$

$$\begin{aligned}L=\frac{1}{A}\Big[&-u^2x^2(a_2\,\kappa+a_1\tau)\\&+u^2x(a_2a_1''-a_1a_2'')\\&-a_2y\left(\kappa(1+ua_1')+ux'-ua_2'\tau\right)\Big]\end{aligned}\tag{13}$$

$$N=0,\ M=-\frac{a_2x}{A}$$

Then, from Equations (5) and (6), the Gauss curvature K and the mean curvature H of general rectifying surface $F_{(\alpha,q_r)}$ are computed as

$$K=-\frac{a_2^2x^2}{A^4},$$

$$H=\frac{1}{2A^3}\Big[x\{2a_1a_2+ua_2(a_1^2+a_2^2)'\tag{14}$$

$$-u^2(a_1^2+a_2^2)\left(x(a_2\kappa-a_1\tau)-a_2a_1''+a_1a_2''\right)\}$$
$$-ya_2(a_1^2+a_2^2)\left(\kappa(1+ua_1')+u(x'-a_2'\tau)\right)\Big]\tag{15}$$

respectively. We can easily see that the classical characterization for developable ruled surfaces given as "surface is developable if and only if the Gauss curvature K vanishes (Do Carmo, 1976)" holds. Then, from the above discussion, we can give the following corollary:

Corollary 3.7. *If $\kappa\neq0$, the only general rectifying surface with vanishing Gauss curvature K is $F_{(\alpha,\tilde{D})}$.*

Furthermore, a surface is called minimal if mean curvature H vanishes along the surface (Do Carmo, 1976). The point (s_0,u_0) of any surface in which $H(s_0,u_0)=0$ holds is called minimal point of the surface. Now, let us consider minimal points of our surface $F_{(\alpha,q_r)}$. Before, we obtained that $x=0,\ y=0$ hold along the singular points of surface $F_{(\alpha,q_r)}$. Moreover, we have $F_{(\alpha,q_r)}$ is developable if and only if $x=0$. Then, from Equation (15), we get the following.

Corollary 3.8. *Regular points of $F_{(\alpha,q_r)}$ are minimal if and only if*

$$\frac{x}{y}=\frac{-a_2(a_1^2+a_2^2)\left(\kappa(1+ua_1')+u(x'-a_2'\tau)\right)}{2a_1a_2+ua_2(a_1^2+a_2^2)'-u^2(a_1^2+a_2^2)\left(x(a_2\kappa-a_1\tau)-a_2a_1''+a_1a_2''\right)}\tag{16}$$

hold.

Theorem 3.9. *For developable rectifying surface $F_{(\alpha,q_r)}$,*

(i) If $\kappa=0$, then $F_{(\alpha,q_r)}$ is minimal.

(ii) If $\kappa\neq0$, then $F_{(\alpha,q_r)}$ is not minimal.

Proof. Assume that $F_{(\alpha,q_r)}$ is developable. Then, $x=0$ and from Equation (15), we have

$$H=\frac{-(a_1^2+a_2^2)\left(\kappa(1+ua_1')-ua_2'\tau\right)}{2\left(a_2^2y^2\right)}\tag{17}$$

(i) If $\kappa=0$, then $\tau=0$ and we have $H=0$ and so, $F_{(\alpha,q_r)}$ is minimal.

(ii) If $\kappa\neq0$, then, $x=0$ implies that $a_1/a_2=\tau/\kappa$. Using that in Equation (17) gives

$$H=-\frac{a_2\kappa+a_1\tau}{2a_2y}\tag{18}$$

Now, $F_{(\alpha,q_r)}$ is minimal if and only if $a_2\kappa+a_1\tau=0$ which implies that $\tau/\kappa=-a_2/a_1$. Then, we have $a_1/a_2=-a_2/a_1$ which gives that $a_1^2+a_2^2=0$ and so, $a_i=0,(i=1,2)$. Then, $q_r=0$ which is a contradiction. So, $F_{(\alpha,q_r)}$ cannot be minimal.

Corollary 3.10. *If $\kappa\neq0$, there exists no developable rectifying helicoid.*

Proof. From Theorem 2.2, we have that only minimal ruled surfaces are helicoids and fragments of helicoids. Then, from Theorem 3.9, we have that there is no developable rectifying helicoid.

Let now $F_{(\alpha,q_r)}$ be developable and $v:F_{(\alpha,q_r)}\to S^2$, $v(p):=U(p)$ be the Gauss map of the surface for each point $p\in F_{(\alpha,q_r)}$ Then, from Equation (12), we get that $v(p)=-N_{\alpha(s)}$ where $N_{\alpha(s)}$ is the principal normal at the point $\alpha(s)$ and that tangent plane of the surface coincides with the rectifying plane of base curve α along q_r. By considering the base $\left\{\partial F_{(\alpha,q_r)}/\partial s,\partial F_{(\alpha,q_r)}/\partial u\right\}$ of tangent space $T_pF_{(\alpha,q_r)}$, the Weingarten map $S_p=-D_pv:T_pF_{(\alpha,q_r)}\to T_{v(p)}S^2$ of $F_{(\alpha,q_r)}$ gives

$$\begin{cases}S_p(\partial F_{(\alpha,q_r)}/\partial s)=D_{\partial F_{(\alpha,q_r)}/\partial s}N=\partial N/\partial s\\ \qquad=-\dfrac{a_2\kappa+a_1\tau}{a_2y}\dfrac{\partial F_{(\alpha,q_r)}}{\partial s}\\ \qquad\quad+\dfrac{ua_2'\kappa+\tau(1+ua_1')}{a_2y}\dfrac{\partial F_{(\alpha,q_r)}}{\partial u}\\ S_p(\partial F_{(\alpha,q_r)}/\partial u)=D_{\partial F_{(\alpha,q_r)}/\partial u}N\\ \qquad=\partial N/\partial u=0\end{cases}\tag{19}$$

Then, the Weingarten map has the matrix form

$$S_p=\begin{pmatrix} -\dfrac{a_2\kappa+a_1\tau}{a_2y} & 0 \\ \dfrac{ua_2'\kappa+\tau(1+ua_1')}{a_2y} & 0 \end{pmatrix} \tag{20}$$

If $\kappa\neq 0$, we get $S_p\neq 0$, $S_p\neq\lambda I$, where I is 2X2 unit matrix. If $\kappa=0$, it follows $S_p=0$. Then, we have the following corollary:

Corollary 3.11. *(i) If $\kappa\neq 0$, there are no umbilical points on developable rectifying surface $F_{(\alpha,q_r)}$.*

(ii) If $\kappa\neq 0$, there are no planar points on developable rectifying surface $F_{(\alpha,q_r)}$. If $\kappa=0$, all points on developable rectifying surface $F_{(\alpha,q_r)}$ are planar. Then, $F_{(\alpha,q_r)}$ is a plane.

It can be easily seen that Equations (14) and (18) can also be obtained from S_p as

$$K=\det(S_p)=0,$$
$$H=\frac{1}{2}tr(S_p)=-\frac{a_2\kappa+a_1\tau}{2a_2y}.$$

Moreover, from the equality $\det(\lambda I-S_p)=0$, the principal curvatures of developable surface $F_{(\alpha,q_r)}$ are $\lambda_1=0$, $\lambda_2=-\dfrac{a_2\kappa+a_1\tau}{a_2y}$. Then, if $\kappa\neq 0$, we have $\lambda_1\lambda_2=0,\ \lambda_2\neq 0$ and if $\kappa=0$, we get $\lambda_1=\lambda_2=0$. Then, the following corollary is obtained.

Corollary 3.12. *(i) If $\kappa\neq 0$, then all points on the developable rectifying surface $F_{(\alpha,q_r)}$ are parabolic and so, quadratic approach of the surface is a parabolic cylinder.*

(ii) If $\kappa=0$, then the quadratic approach of the surface is a plane.

The principal lines $e_i,(i=1,2)$ of the surface are defined by $S_p(e_i)=\lambda_ie_i$ (Do Carmo, 1976). Then a curve on a surface is called a line of curvature if and only if its tangent vector at the point $\alpha(s)$ is a principal line on this point, i.e., $S_p(T)=\lambda T$, where λ is a scalar. Since the unit surface normal U of developable rectifying surface $F_{(\alpha,q_r)}$ along base curve $\alpha(s)$ is $U_\alpha=-N(s)$, we have

$$S_p(T)=-D_TU_\alpha=-U_\alpha{}'=\kappa T-\tau B$$

Then, we have that base curve $\alpha(s)$ is a line of curvature if and only if $\alpha(s)$ is a plane curve. But, since $F_{(\alpha,q_r)}$ is developable, we have $\kappa=(a_2/a_1)\tau$. Then, since $a_1\neq 0$, $\tau=0$ implies that $\kappa=0$, which means $\alpha(s)$ is a line. Then, we get following corollary:

Corollary 3.13. *The base curve $\alpha(s)$ of developable rectifying surface $F_{(\alpha,q_r)}$ is a line of curvature if and only if $\alpha(s)$ is a line.*

Moreover, if $\kappa\neq 0$, the principal line e_2 corresponding to λ_2 is obtained from $S_p(e_2)=\lambda_2e_2$ as follows,

$$e_2=-\left(\frac{a_2\kappa+a_1\tau}{a_2y}\right)\frac{\partial F_{(\alpha,q_r)}}{\partial s}+\left(\frac{ua_2'\kappa+\tau(1+ua_1')}{a_2y}\right)\frac{\partial F_{(\alpha,q_r)}}{\partial u} \tag{21}$$

As in the paragraph given after Equation (18), if we take $a_2\kappa+a_1\tau=0$, we have a contradiction. Then, we have following corollary:

Corollary 3.14. *If $\kappa\neq 0$, for the developable rectifying surface $F_{(\alpha,q_r)}$,*

(i) The parameter curve $F_{(\alpha,q_r)}(s,u_0)$ is line of curvature if and only if $u_0a_2'\kappa+\tau(1+u_0a_1')=0$ where u_0 is non-zero constant.

(ii) The parameter curve $F_{(\alpha,q_r)}(s_0,u)$ cannot be line of curvature.

Furthermore writing Equation (8) in Equation (21), it is obtained that

$$e_2=-\kappa T+\tau B \tag{22}$$

which also gives us Corollary 3.13 again.

Let now v_p be a unit tangent vector on developable surface $F_{(\alpha,q_r)}$. Then, we can write

$$v_p=C(s,u)\frac{\partial F_{(\alpha,q_r)}}{\partial s}+D(s,u)\frac{\partial F_{(\alpha,q_r)}}{\partial u} \tag{23}$$

where *C, D* are smooth functions and $C^2+D^2=1$. Using the linearity of *S* we have

$$S_p(v_p)=C\left[-\left(\frac{a_2\kappa+a_1\tau}{a_2y}\right)\frac{\partial F_{(\alpha,q_r)}}{\partial s}+\left(\frac{ua_2'\kappa+\tau(1+ua_1')}{a_2y}\right)\frac{\partial F_{(\alpha,q_r)}}{\partial u}\right] \tag{24}$$

Writing Equation (8) in Equations (23) and (24), respectively, it follows

$$v_p=\left(C(1+ua_1')+Da_1\right)T+(Cua_2'+Da_2)B \tag{25}$$

$$S_p(v_p)=C(-\kappa T+\tau B) \tag{26}$$

Then, the normal curvature $k_n(v_p)$ in the direction v_p is

$$\begin{aligned} k_n(v_p) &= \langle S_p(v_p), v_p \rangle \\ &= -C\left[\kappa\left(C(1+ua_1')+Da_1\right)+\tau(Cua_2'+Da_2)\right] \end{aligned} \tag{27}$$

If $\kappa \neq 0$, since $F_{(\alpha,q_r)}$ is developable, we have $\tau=\kappa(a_1/a_2)$. Then, Equation (27) becomes $k_n(v_p)=-C^2\kappa y$. If $\kappa=0$, then we have $k_n(v_p)=0$ and we have followings.

Theorem 3.15. *(i) If $\kappa \neq 0$, on the developable rectifying surface $F_{(\alpha,q_r)}$, any unit tangent vector v_p is an asymptotic direction if and only if $y=0$ or v_p is a ruling, i.e., $v_p=q_r$.*

(ii) If $\kappa=0$, then all directions on developable surface $F_{(\alpha,q_r)}$ are asymptotic.

Theorem 3.15 gives the followings.

Corollary 3.16. *Let $\gamma(t):J\rightarrow F_{(\alpha,q_r)}(s,u)$ be an arbitrary curve on developable rectifying surface $F_{(\alpha,q_r)}$ with unit tangent v_p. Then,*

(i) If $\kappa \neq 0$, then $\gamma(t)$ is an asymptotic curve if and only if $\gamma(t)$ is a ruling or $y=0$.

(ii) If $\kappa=0$, then all curves on $F_{(\alpha,q_r)}$ are asymptotic curves.

Corollary 3.17. *On developable rectifying surface $F_{(\alpha,q_r)}$, a non-ruling curve $\gamma(t)$ is an asymptotic curve if and only if the surface $F_{(\alpha,q_r)}$ has singular points.*

Let now consider the frame of rectifying ruled surface $F_{(\alpha,q_r)}$. Assume that $\|q_r\|=1$. Then, we may write

$$q_r(s)=\cos\theta(s)T(s)+\sin\theta(s)B(s),$$

where $\theta=\theta(s)$ is the angle function between unit vectors q_r and T. Differentiating q_r with respect to s gives

$$q_r'(s)=-(\theta'\sin\theta)T+xN+(\theta'\cos\theta)B \tag{28}$$

where $x=\kappa\cos\theta-\tau\sin\theta$. Then, the central normal $h(s)$ and central tangent $a(s)$ of $F_{(\alpha,q_r)}$ are obtained as

$$\begin{cases} h(s)=\dfrac{q_r'(s)}{\|q_r'(s)\|}=\dfrac{1}{\sqrt{x^2+(\theta')^2}}\left[-(\theta'\sin\theta)T\right. \\ \qquad\qquad \left.+xN+(\theta'\cos\theta)B\right], \\ a(s)=q_r(s)\times h(s) \\ \quad =\dfrac{1}{\sqrt{x^2+(\theta')^2}}\left[-(x\sin\theta)T\right. \\ \qquad\qquad \left.-\theta'N+(x\cos\theta)B\right]. \end{cases} \tag{29}$$

Then, we have followings.

Theorem 3.18. *For the rectifying ruled surface $F_{(\alpha,q_r)}$, the followings are equivalent.*

(i) The angle θ between the ruling q_r and tangent T is constant.

(ii) The central normal h(s) of $F_{(\alpha,q_r)}$ coincides with principal normal N(s) of $\alpha(s)$.

(iii) The central tangent a(s) of $F_{(\alpha,q_r)}$ lies on the rectifying plane of $\alpha(s)$.

Proof. The proof is clear from Equation (29).

Theorem 3.19. *For the rectifying ruled surface $F_{(\alpha,q_r)}$, the followings are equivalent.*

(i) $F_{(\alpha,q_r)}$ is developable.

(ii) The central tangent $a(s)$ of $F_{(\alpha,q_r)}$ coincides with principal normal $N(s)$ of $\alpha(s)$.

(iii) The central normal $h(s)$ of $F_{(\alpha,q_r)}$ lies on the rectifying plane of $\alpha(s)$.

Proof. Considering that $F_{(\alpha,q_r)}$ is developable if and only if $x=0$, the proof is clear from Equation (29).

From the last theorems, we have the following corollaries.

Corollary 3.20. *Let the angle θ between the ruling q_r and tangent T be constant. Then, $F_{(\alpha,q_r)}$ is an h-slant ruled surface if and only if $\alpha(s)$ is a slant helix.*

Corollary 3.21. *Let the rectifying ruled surface $F_{(\alpha,q_r)}$ be developable. Then, $F_{(\alpha,q_r)}$ is an q-slant (or a-slant) ruled surface if and only if $\alpha(s)$ is a slant helix.*

4. Examples

Example 4.1. Let consider the general helix $\alpha(s)=\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ in E^3 (Fig 1). Required Frenet apparatus of α are obtained as follows,

$$T_\alpha(s)=\left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$B_\alpha(s)=\left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\kappa_\alpha=\tau_\alpha=\frac{1}{2}.$$

By taking $a_1(s)=a_2(s)=s$, a developable rectifying ruled surface $F_{1(\alpha,q_r)}$ associated to $\alpha(s)$ is obtained as.

$$F_{1(\alpha,q_r)} = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right). \\ +\sqrt{2}u(0,0,s)$$

(Figure 3). Choosing $a_1(s)=\cos s$, $a_2(s)=\sin s$, a non-developable rectifying ruled surface $F_{2(\alpha,q_r)}$ associated to $\alpha(s)$ is obtained as

$$F_{2(\alpha,q_r)} = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) \\ +\frac{1}{\sqrt{2}}u\left(\sin\frac{s}{\sqrt{2}}(\sin s-\cos s), \right. \\ \left. \cos\frac{s}{\sqrt{2}}(\cos s-\sin s), \cos s+\sin s\right)$$

(Figure 4).

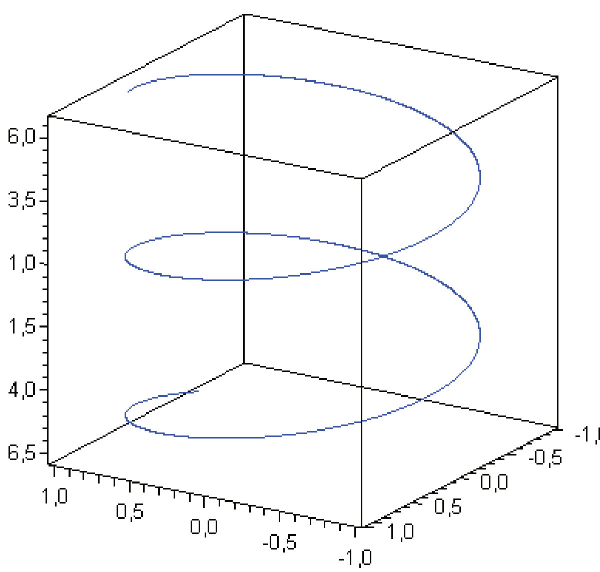

Fig. 1. Helix curve $\alpha(s)$.

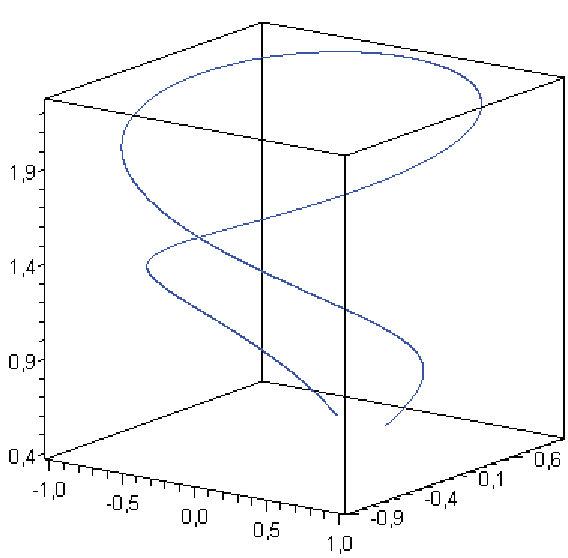

Fig. 2. Slant helix curve $\beta(s)$.

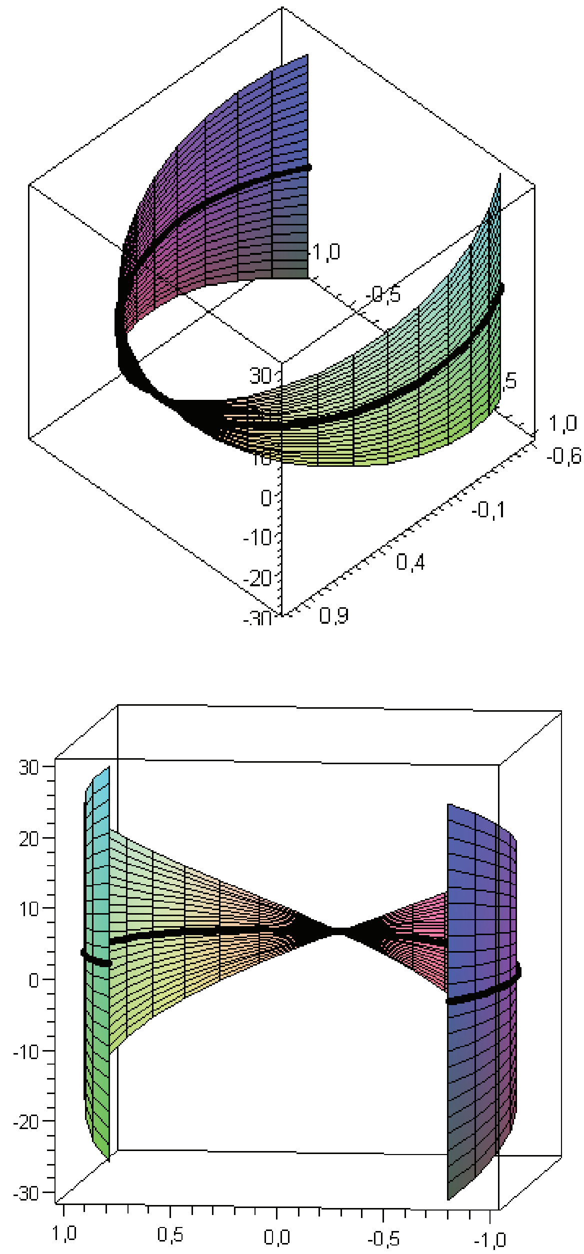

Fig. 3. Two different views of developable rectifying ruled surface $F_{1(\alpha,q_r)}$.

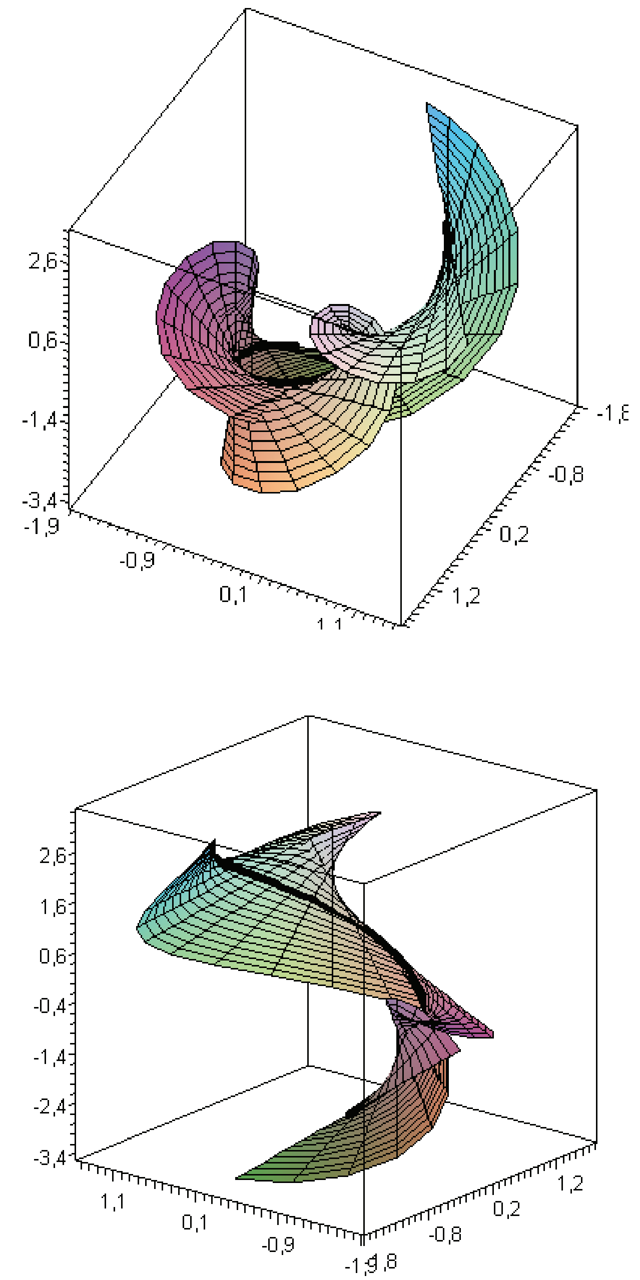

Fig. 4. Two different views of non-developable rectifying ruled surface $F_{2(\alpha,q_r)}$.

Example 4.2. Consider the slant helix $\beta(s)$ given by

$$\beta(s)=\frac{9}{\sqrt{82}}\left(\frac{\sqrt{82}-82}{8(41+\sqrt{82})}\sin\left(\left(1+\frac{\sqrt{82}}{41}\right)s\right)+\frac{\sqrt{82}+82}{8(\sqrt{82}-41)}\sin\left(\left(1-\frac{\sqrt{82}}{41}\right)s\right)-\frac{1}{2}\sin s,\right.$$
$$\frac{82-\sqrt{82}}{8(41+\sqrt{82})}\cos\left(\left(1+\frac{\sqrt{82}}{41}\right)s\right)-\frac{\sqrt{82}+82}{8(\sqrt{82}-41)}\cos\left(\left(1-\frac{\sqrt{82}}{41}\right)s\right)+\frac{1}{2}\cos s,$$
$$\left.\frac{9}{4}\cos\left(\frac{\sqrt{82}}{41}s\right)\right).$$

(Figure 2). The required Frenet apparatus of $\beta(s)$ are obtained as

$$T_\beta(s)=\left(-\cos s\cos\left(\frac{\sqrt{82}}{82}s\right)-\frac{\sqrt{82}}{82}\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$-\sin s\cos\left(\frac{\sqrt{82}}{82}s\right)+\frac{\sqrt{82}}{82}\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),$$
$$\left.-\frac{9\sqrt{82}}{82}\sin\left(\frac{\sqrt{82}}{82}s\right)\right),$$

$$B_\beta(s)=\left(\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\sin s-\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$-\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\cos s-\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),$$
$$\left.\frac{9\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\right)$$

$$\kappa_\beta=1,\ \tau_\beta(s)=\tan\left(\frac{s}{\sqrt{362}}\right).$$

Then, by taking

$$a_1(s)=2\tan\left(\frac{s}{\sqrt{362}}\right),\ a_2=2,$$

a developable rectifying ruled surface $F_{1(\beta,q_r)}$ associated to $\beta(s)$ is obtained as $F_{1(\beta,q_r)}=\beta(s)+uq_r(s)$ where

$$q_r(s)=2\tan\frac{s}{\sqrt{362}}\left(-\cos s\cos\left(\frac{\sqrt{82}}{82}s\right)-\frac{\sqrt{82}}{82}\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$\left.-\sin s\cos\left(\frac{\sqrt{82}}{82}s\right)+\frac{\sqrt{82}}{82}\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),-\frac{9\sqrt{82}}{82}\sin\left(\frac{\sqrt{82}}{82}s\right)\right)$$
$$+2\left(\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\sin s-\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$\left.-\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\cos s-\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),\frac{9\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\right).$$

(Figure 5).

Choosing $a_1=a_2=\sqrt{2}/2$, we obtained a non-developable rectifying ruled surface $F_{2(\beta,q_r)}(s,u)=\beta(s)+uq_r(s)$, where

$$q_r(s)=\frac{\sqrt{2}}{2}\left(-\cos s\cos\left(\frac{\sqrt{82}}{82}s\right)-\frac{\sqrt{82}}{82}\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$\left.-\sin s\cos\left(\frac{\sqrt{82}}{82}s\right)+\frac{\sqrt{82}}{82}\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),-\frac{9\sqrt{82}}{82}\sin\left(\frac{\sqrt{82}}{82}s\right)\right)$$
$$+\frac{\sqrt{2}}{2}\left(\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\sin s-\cos s\sin\left(\frac{\sqrt{82}}{82}s\right),\right.$$
$$\left.-\frac{\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\cos s-\sin s\sin\left(\frac{\sqrt{82}}{82}s\right),\frac{9\sqrt{82}}{82}\cos\left(\frac{\sqrt{82}}{82}s\right)\right).$$

(Figure 6). It is clear that the angle θ between $q_r(s)$ and T_β is constant. Then, from Corollary 3.20, we have that $F_{2(\beta,q_r)}(s,u)$ is also an *h*-slant ruled surface.

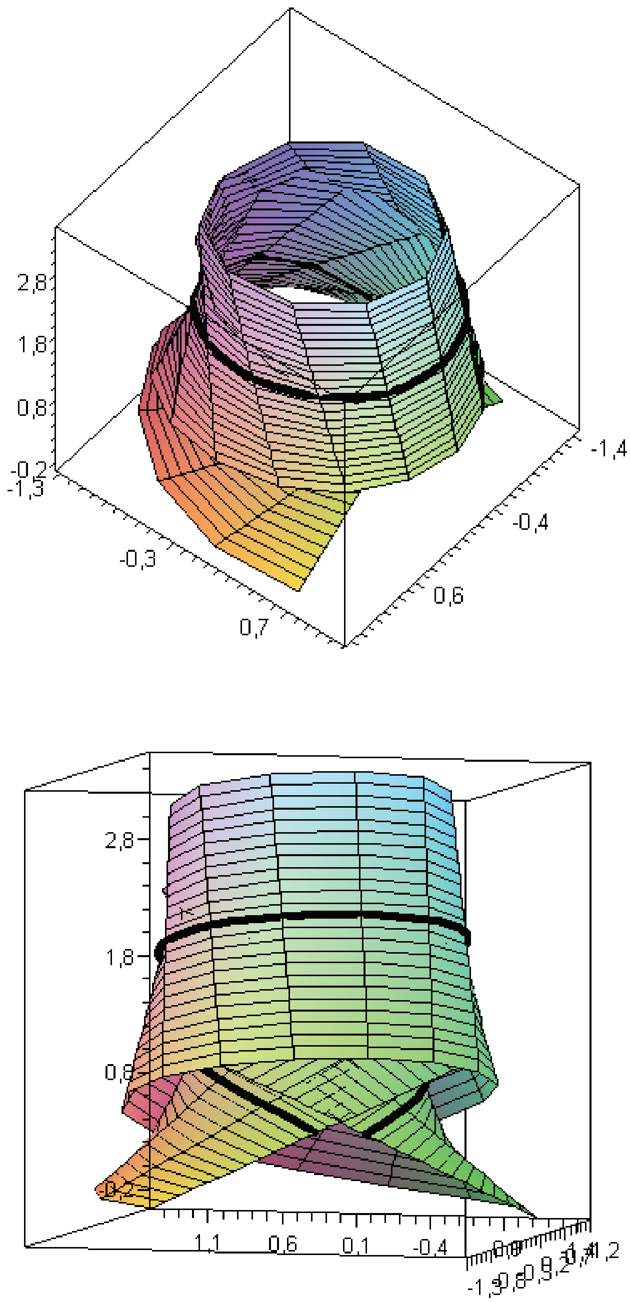

Fig. 5. Two different views of developable rectifying ruled surface $F_{1(\beta,q_r)}$.

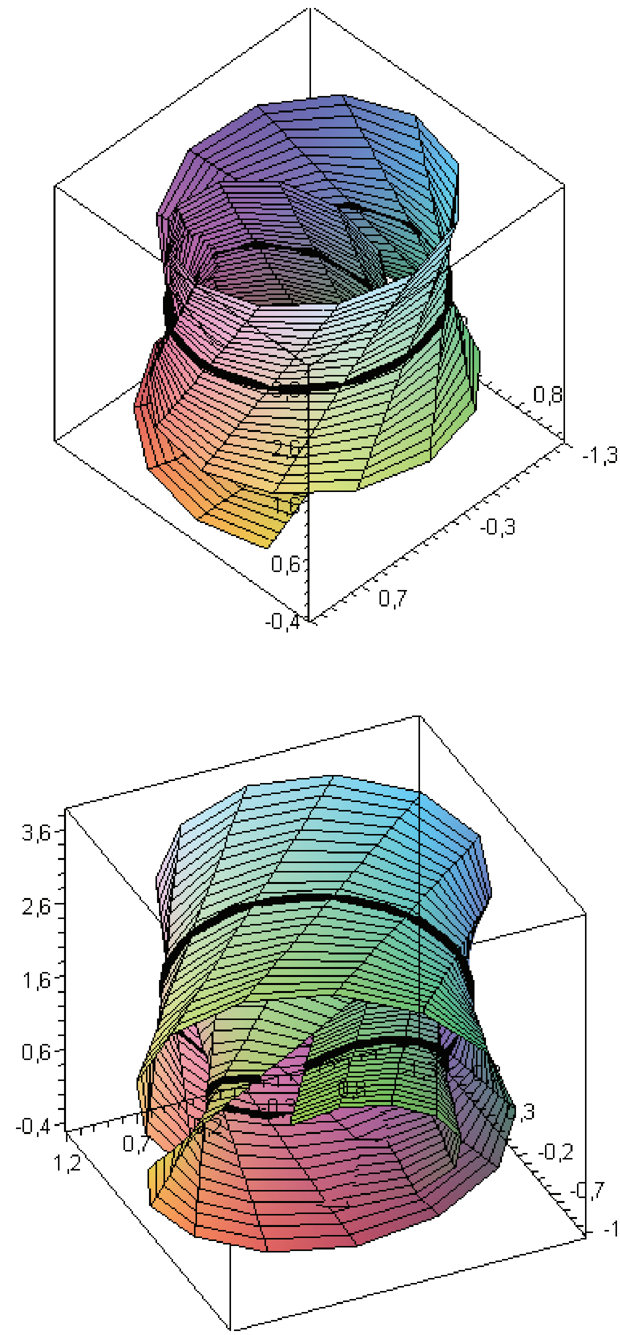

Fig. 6. Two different views of non-developable rectifying ruled surface $F_{2(\beta,q_r)}$.

5. Conclusions

A new type of ruled surface whose ruling always lies on the rectifying plane of the base curve has been introduced and called a "rectifying ruled surface". The geometric properties of this surface have been given by means of the curvatures of the base curve. Of course, this subject can also be studied in different spaces, such as Lorentzian space. Moreover, new kinds of such surfaces can be defined by considering other Frenet planes of the base curve.

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