A generalization of interval-valued optimization problems and optimality conditions by using scalarization and subdifferentials

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Abstract

In this work, interval-valued optimization problems are considered. The ordering cone is used to generalize the interval-valued optimization problems on real topological vector spaces. Some definitions and their properties are obtained for intervals, defined via an ordering cone. Gerstewitz's function is used to derive scalarization for the interval-valued optimization problems. Also, two subdifferentials for interval-valued functions are introduced by using subgradients. Some necessary optimality conditions are obtained via subdifferentials and scalarization. An example is given to demonstrate the results.

Keywords: Interval-valued optimization problem; optimality condition; ordering cone; scalarization; subdifferential

Mathematical Subject Classifications 2010: 90C26, 90C29, 90C30

1. Introduction

We encounter optimization problems (mathematical programming) in our life. When these problems are expressed mathematically, an objective function emerges. Naturally, these problems' solution methods have been attracted more attention by researchers working in the field such as mathematics, engineering, computer science, economics, business, etc. for years. Sometimes, we may encounter uncertainty in optimization problems. In this case, the objective function is an interval-valued.

Optimization problems are classified according to the objective function. For example, the problems are called interval-valued optimization problems (shortly, interval optimization problem) when the objective functions are interval-valued. Interval optimization problems generalize the scalar optimization problems because the interval-valued functions are generalizations of the real-valued functions. Adding to this, the interval optimization problems are a particular form of the set-valued optimization problems.

The coefficients of the objective function for optimization problems are considered as fixed. However, the problem's coefficients may also change due to changing environmental conditions and various kinds of uncertainties. So, we can generate an interval optimization problem. For instance, an investor wants to convert her/his money into foreign currency for investment. If the investor knows the foreign currencies' intervals (it can be obtained via excel forecast application) after a while, then the investor can encounter an interval optimization problem. Therefore, interval optimization problems have an essential place in our life.

Order relations defined on intervals are used to solve the interval optimization problems. These order relations are obtained using natural order relation on the real numbers (Bhurjee & Panda, 2012; Bhurjee & Pandahan, 2016; Chalco-Cano *et al.,* 2013; Costa *et al.*, 2015; Ishibuchi & Tanaka, 1990; Moore, 1966; Karmakar *et al.,* 2009). Using order relations on real numbers, significant results related to the interval optimization problems are obtained by some researcher as Ishibuchi & Tanaka, 1990; Costa *et al.,* 2015, and Moore, 1966, and the references therein. Most of these order relations are partial order relations.

In any vector space, there is no natural order relation, such as on real numbers. So, an order relation is defined with the help of a subset of the space to compare the vector optimization vectors. This set is called ordering cone. According to the ordering cone, this order relation is either preorder relation, partial order relation, or total order relation. So, all definitions and results used in vector optimization problems depend on the ordering cone. Also, all order relations, which are defined on vector space, induce an ordering cone. Thus, each order relation in the vector space corresponds to an ordering cone and vice versa. Naturally, definitions of the minimal and maximal elements can also be changed when the ordering cone or order relation changes.

We use many tools as scalarization (Karaman *et al.,* 2018b), vectorization (Karaman *et al.,* 2018a), derivative (Karaman *et al.,* 2020b), and subdifferential (Chen & Jahn, 1998; Karaman *et al.,* 2020a; Hernández & Rodríguez-Marín, 2011), tangent cones (Zivari-Rezapou,r 2016), embedding space (Karaman, 2019), etc. to obtain the solutions or optimality conditions for optimization problems. Scalarization has significant importance. One of the essential goals in this method is to follow the problem to a scalar optimization problem. We know that there is a natural order relation in the scalar optimization problems. So, the solution of the scalar optimization problem, obtained via the original optimization problem, can be found without using an ordering cone. Gerstewitz's function is recently used to achieve this idea (Gerth & Weidner, 1990; Luc, 1989; Ansari *et al.,* 2018; Khan *et al.,* 2015). This function is also used to obtain vectorization of the vector-valued and set-valued optimization problems (Ansari *et al.,* 2018; Karaman *et al.,* 2018a; Khan *et al.,* 2015; Gerth & Weidner, 1990). Moreover, scalarization is derived using the unified scalarization technique (see Karaman *et al.*, 2018a).

Another method is subdifferential in order to obtain not only the solutions but also the optimality conditions of the optimization problems. Chen & Jahn, 1998, introduced a weak subgradient. They obtained some optimality conditions for set-valued optimization problems concerning the vector approach. Hernández & Rodríguez-Marín, 2011, defined strong subgradient. Some optimality conditions are derived via weak and strong subgradients for set-valued optimization problems concerning the set approach. Additionally, Karaman *et al.*, 2020a, presented necessary and sufficient optimality conditions for set-valued optimization problems concerning an order relation, which is a partial order relation on the family of nonempty and bounded sets.

There are two purposes in this investigation. The first one is to get a general version of the interval optimization problems on real topological vector spaces since there is no work about the intervals on any real topological vector space up to now, to our knowledge. Thus, intervals are defined by using the ordering cone, and two new preorder relations are defined to obtain the solutions to the interval optimization problems. The second purpose is to find the solutions and optimality conditions of the interval optimization problems using scalarization and subdifferentials.

2. Preliminaries

Throughout this paper, X and Y are real topological vector spaces, $C \subset Y$ is convex ($\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$), pointed ($C \cap (-C) = \{0\}$), nonempty interior cone ($\lambda x \in C$ for all $x \in C$ and $\lambda > 0$) and Y is ordered by cone C. Let $A \subset Y$. The topological interior of A is denoted by $int(A)$. We denote the non-negative orthant of $\mathbb{R}^n$ ($n \geq 1$) by $\mathbb{R}^n_+$.

It is known that a convex, pointed ordering cone C induces a partial order relation on Y as follows: For $x, y \in Y$

$$x \leq_C y \Leftrightarrow y - x \in C.$$

Similarly, a partial order relation $\leq_C$ on Y induces a convex, pointed ordering cone $C = \{x \in Y : 0 \leq_C x\}$. Thus, there is a directly relationship between partial order relation $\leq_C$ and convex, pointed ordering cone C. Also, the order relation $\leq_C$ is compatible with the addition ($x_1 \leq_C y_1$ and $x_2 \leq_C y_2$ imply $x_1 + x_2 \leq_C y_1 + y_2$) and the non-negative scalar multiplication ($tx \leq_C ty$ for all $x \leq_C y$ and $t \in \mathbb{R}_+$).

Strict order relation $<_C$ is defined as follows: For $x, y \in Y$

$$x <_C y \Leftrightarrow y - x \in int(C).$$

Let $A \subset Y$ and $a \in A$. If $A \cap (a - C) = \{a\}$ ($A \cap (a + C) = \{a\}$), then a is called a minimal (maximal) element of A concerning the ordering cone C and all minimal (maximal) elements of A are denoted by $\min(A)$ ($max(A)$). We say that a is a weak minimal (weak maximal) element of A concerning the ordering cone C if $A \cap (a - int(C)) = \emptyset$ ($A \cap (a + int(C)) = \emptyset$). All weak minimal (weak maximal) elements of A are denoted by $\text{Wmin}(A)$ ($Wmax(A)$). It is clear that $\min(A) \subset Wmin(A)$ and $max(A) \subset Wmax(A)$.

3. Intervals and arithmetic on real topological vector spaces

A closed interval on Y is defined by using the ordering cone C as: For $a_1, a_2 \in Y$

$$A = [a_1, a_2] = \{x \in Y : a_1 \leq_C x \leq_C a_2\} = (a_1 + C) \cap (a_2 - C),$$

where a_1 and a_2 are called the lower and upper bounds of A, respectively. If $a_1 = a_2$, then the interval A is called degenerate, and this interval equals to a_1 or a_2. The shape of an interval on Y associates with the ordering cone C. That is, an interval on Y may be a line segment, unlimited area, etc., depending on the shape of the ordering cone. In the rest of the study, $I_C(Y)$ denotes a set of all intervals on Y. That is,

$$I_C(Y) = \{[a_1, a_2]: a_1, a_2 \in Y \; and \; a_1 \leq_C a_2\}.$$

Each vector in Y can be considered as a closed and bounded interval. In this case, we can write $a = [a, a]$ for all $a \in Y$.

Let $A = [a_1, a_2], B = [b_1, b_2] \in I_C(Y)$, and $\lambda \in \mathbb{R}$. Addition and difference of intervals A and B and scalar multiplication with λ are defined by

$$A + B = \{x + y \in Y : x \in A \; and \; y \in B\} = [a_1 + b_1, a_2 + b_2],$$

$$A - B = \{x - y \in Y : x \in A \; and \; y \in B\} = [a_1 - b_2, a_2 - b_1]$$

and

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2] & ; \lambda \geq 0 \\ [\lambda a_2, \lambda a_1] & ; \lambda < 0 \end{cases}$$

respectively.

Now, some notations are given used in the interval analysis. Let $A = [a_1, a_2] \in I_C(Y)$. Then, the width, midpoint and radius of A are defined as $w(A) = a_2 - a_1$, $m(A) = \frac{1}{2}(a_1 + a_2)$ and $r(A) = \frac{1}{2}(a_2 - a_1)$, respectively. By using these notations, an interval is also defined as:

$$\begin{aligned} A &= [a_1, a_2] \\ &= [m(A) - r(A), m(A) + r(A)] \\ &= m(A) + \left[-\frac{1}{2}w(A), \frac{1}{2}w(A)\right] \\ &= m(A) + \frac{1}{2}w(A)[-1,1]. \end{aligned}$$

Let $A = [a_1, a_2], B = [b_1, b_2] \in I_C(Y)$. Then, $A + (-A) = 0 = [0_Y, 0_Y]$ if and only if $a_1 = a_2$, where 0_Y is zero element of Y. This says that the difference of an interval and itself in $I_C(Y)$ cannot be the zero element. That is, each element in $I_C(Y)$ cannot have the additive inverse element. Thus, $I_C(Y)$ is not a vector space.

A equals to B if and only if $a_1 = b_1$ and $a_2 = b_2$. When $a_1 = -a_2$, A is called a symmetric interval. The midpoint of a symmetric interval is zero. Let assume that $a_1 \leq_C 0$ and $0 \leq_C a_2$. Non-negative and non-positive part of the interval A are defined as $A^+ = [0, a_2]$ and $A^- = [a_1, 0]$, respectively. Therefore, we have $A = A^+ + A^-$.

4. Interval optimization problems

Interval optimization problems are constructed in this section.

Let $M \subset X$ and $F: M \to I_C(Y)$ be an interval-valued function. That is, $F(x) = [f_L(x), f_U(x)]$ for all $x \in M$, where vector-valued functions $f_L, f_U: M \to Y$ satisfy $f_L(x) \leq_C f_U(x)$ for all $x \in M$. Then, primal interval optimization problem is defined as:

$$(IVOP)\begin{cases}\min(max)F(x)\\ \quad x \in M.\end{cases}$$

To find a solution of $(IVOP)$, an order relation, which is defined on intervals, is needed. To achieve this situation, we should use the order relation $\leq_C$. So, we will use the following order relations defined on the intervals by using the order relation $\leq_C$.

Definition 4.1 Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals defined on Y. *Lower* and *Upper* order relations are defined as:

(i) $A \preccurlyeq^L B \iff a_1 \leq_C b_1$,

(ii) $A \preccurlyeq^U B \iff a_2 \leq_C b_2$,

respectively.

These order relations are preorder relations. Also, they are generalizations given by Karmakar *et al.*, 2009. That is, when we take $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, we obtain the known order relations in the literature.

Now, we introduce the strict version of $\preccurlyeq^L$ and $\preccurlyeq^U$ by using strict order relation $<_C$. The relations $\prec^L$ and $\prec^U$ are used to detect the weak solutions of $(IVOP)$.

Definition 4.2 Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals defined on Y. Strict *Lower* and strict *Upper* order relations are defined by:

(i) $A \prec^L B \iff a_1 <_C b_1$,

(ii) $A \prec^U B \iff a_2 <_C b_2$,

respectively.

We will assume $\star \in \{L, U\}$ in the rest of the work. Therefore, all properties given for order relation $\preccurlyeq^\star$ will be valid the order relations $\preccurlyeq^L$ and $\preccurlyeq^U$.

Proposition 4.3 Order relation $\preccurlyeq^\star$ have the following properties:

(i) $\preccurlyeq^\star$ is compatible with the addition,

(ii) $\preccurlyeq^\star$ is compatible with the non-negative scalar multiplication.

Proof. Let us consider order relation $\preccurlyeq^L$. Similarly, one can obtain for the other order relation.

(i) Let $A = [a_1, a_2], B = [b_1, b_2]$,

(ii) $D = [d_1, d_2], E = [e_1, e_2] \in I_C(Y)$, $A \preccurlyeq^L B$ and $D \preccurlyeq^L E$. Then, we have $a_1 \leq_C b_1$ and $d_1 \leq_C e_1$. Since $\leq_C$ is compatible with the addition, we yield $a_1 + d_1 \leq_C b_1 + e_1$. This gives $A + D \preccurlyeq^L B + E$.

(iii) Proof can be obtained similar to (i) by using the compatibility with the non-negative scalar multiplication of $\leq_C$.

Now, definitions of minimal, maximal, weak minimal, weak maximal, strongly minimal, and strongly maximal intervals for preorder relations are given.

Definition 4.4 Let $\wp \subset I_C(Y)$ and $A \in \wp$. Then, we say that

(i) A is $\star-$minimal interval of $\wp$ if for any $B \in \wp$ such that $B \preccurlyeq^\star A$ implies $A \preccurlyeq^\star B$,

(ii) A is $\star-$maximal interval of $\wp$ if for any $B \in \wp$ such that $A \preccurlyeq^\star B$ implies $B \preccurlyeq^\star A$,

(iii) A is weak $\star-$minimal interval of $\wp$ if for any $B \in \wp$ such that $B \prec^\star A$ implies $A \prec^\star B$, or equivalently $\nexists B \in \wp$ such that $B \prec^\star A$,

(iv) A is weak $\star-$maximal interval of $\wp$ if for any $B \in \wp$ such that $A \prec^\star B$ implies $B \prec^\star A$, or equivalently $\nexists B \in \wp$ such that $A \prec^\star B$,

(v) A is strongly $\star-$minimal (strongly $\star-$maximal) interval of $\wp$ if $A \preccurlyeq^\star B$ ($B \preccurlyeq^\star A$) for all $B \in \wp$.

If we consider the interval optimization problem concerning the order relation $\preccurlyeq^\star$, then we denote it by $(\star-IVOP)$. We say that $x_0 \in M$ is a solution of $(\star-IVOP)$ iff $F(x_0)$ is a $\star$-minimal ($\star$-maximal) interval of the image set $\mathcal{F}(M) := \{F(x) : x \in M\}$. Similarly, x_0 is called a weak solution of $(\star - IVOP)$ iff $F(x_0)$ is a weak $\star$-minimal (weak $\star$-maximal) interval of $\mathcal{F}(M)$. Furthermore, x_0 is called a strongly solution of $(\star - IVOP)$ iff $F(x_0)$ is a strongly $\star$-minimal (strongly $\star$-maximal) interval of $\mathcal{F}(M)$.

Note that every strong solution of $(\star - IVOP)$ is also a solution to the problem. Moreover, every solution of $(\star - IVOP)$ is also a weak solution to the problem.

The solution of an interval optimization problem concerning $\preccurlyeq^L$ and $\preccurlyeq^U$ are obtained in the following example.

Example 4.5 Let ordering cone $C = \mathbb{R}_+$ and interval-valued function $F: \mathbb{R} \to I_C(\mathbb{R})$ be defined as

$$F(x) = \begin{cases} [x^2, |x|] & ; x \in [-1,1] \\ [|x|, x^2] & ; otherwise \end{cases}.$$

Consider the following optimization problem:

$$(IVOP) \begin{cases} \min F(x) \\ x \in \mathbb{R} \end{cases}.$$

Some images of F are given in Figure 1. Because $[0,0] \preccurlyeq^L F(x)$ and $[0,0] \preccurlyeq^U F(x)$ for all $x \in \mathbb{R}$, 0 is a strong solution of $(L - IVOP)$ and $(U - IVOP)$. Also, 0 is also a solution and weak solution of the corresponding problem. There is no solution to the problems other than 0. Really, assume that $x_0 \in \mathbb{R} \setminus \{0\}$ is a solution to the problem. Then, $[0,0] \preccurlyeq^L F(x)$ is not imply $F(x) \preccurlyeq^L [0,0]$ and $[0,0] \preccurlyeq^U F(x)$ is not imply $F(x) \preccurlyeq^U [0,0]$. Also, it is clear that 0 is the unique weak solution of the problems concerning $\preccurlyeq^L$ and $\preccurlyeq^U$.

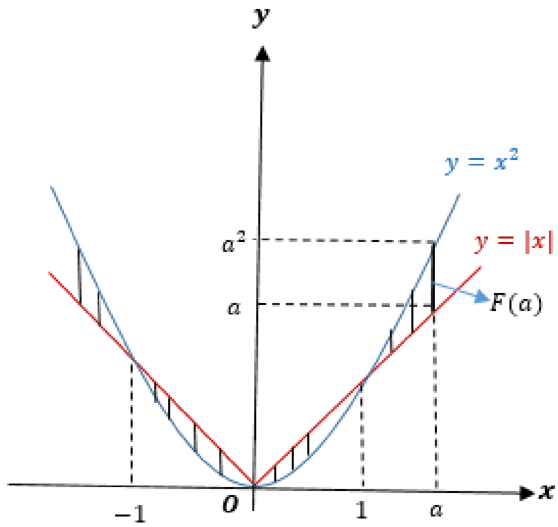

Fig. 1. Graph of the interval-valued function in Example 4.4.

5. Nonlinear Scalarization

In this section, scalarization is obtained for interval optimization problems using nonlinear Gerstewitz's function, used to separate nonconvex sets. This method allows us to characterize the interval optimization problems with scalar optimizations.

Definition 5.1 A given function $t: Y \to \mathbb{R}$ is said to be

(i) increasing if for any $x, y \in Y$ such that $x \leq_C y$ implies $t(x) \leq t(y)$,

(ii) strictly increasing if for any $x, y \in Y$ such that $x <_C y$ implies $t(x) < t(y)$.

Definition 5.2 (Gerth & Weidner, 1990; Luc, 1989) Let $e \in int(C)$ and y be a point in Y. Then, Gerstewitz's function $h_e: Y \to \mathbb{R}$ is defined as

Proposition 5.3 Let $x, y \in Y$ and $r \in \mathbb{R}$. z_e has the following properties:

(ii) $x <_C y \Leftrightarrow z_e(x) < z_e(y)$,

(iii) $x \in C \Rightarrow z_e(x) \geq 0$,

(iv) $z_e(x) \leq r \Leftrightarrow x \in re - C$,

(v) $z_e(x) < r \Leftrightarrow x \in re - int(C)$,

(vi) $x \leq_C y \Leftrightarrow z_e(y - x) \geq 0$,

(vii) $x <_C y \Leftrightarrow z_e(y - x) > 0$.

Proof. *(i)* Assume that $x \leq_C y$ and $z_e(y) = k \in \mathbb{R}$ for a fixed $e \in int(C)$. Then, we have $y \in (k + \varepsilon)e - C$ for all $\varepsilon > 0$. $x \in y - C \subset (k + \varepsilon)e - C - C = (k + \varepsilon)e - C$ because C is convex and $y - x \in C$. This gives $z_e(x) \leq k + \varepsilon$ for all $\varepsilon > 0$. Because $z_e(x) \leq k + \varepsilon$ for all $\varepsilon > 0$, we get $z_e(x) \leq k$ for $\varepsilon \to 0$. Hence $z_e(x) \leq z_e(y)$ yields.

Converse, assume that $t = z_e(x) \leq z_e(y) = k$ and $\varepsilon = t - k < 0$ for $t, k \in \mathbb{R}$. Because $x \in te - C$, $y \in ke - C$ and C is convex ordering cone,

Theorem 5.4 $x_0 \in M$ is a solution of

$$h_{e,y}(x) = \min\{t \in \mathbb{R} : x \in y + te - C\}$$

for any $x \in Y$.

Gerstewitz's function is nonlinear, continuous, real-valued, increasing, strictly increasing, and convex (see Ansari, Köbis & Yao, 2018; Gerth & Weidner, 1990; Luc, 1989; Hernández & Rodríguez-Marín, 2011).

In this work, we will use a particular form of Gerstewitz's function. When we use 0 instead of y in the Gerstewitz's function, we obtain the following well-defined function

$$z_e(x) = \min\{t \in \mathbb{R} : x \in te - C\}.$$

We assume $e \in int(C)$ in the rest of the study. Some properties of z_e are stated in the following proposition.

(i) $x \leq_C y \Leftrightarrow z_e(x) \leq z_e(y)$,

we have $x - y = (t - k)e - C - C = \varepsilon e - C \subset -C$. Therefore, we obtain $x \leq_C y$.

(ii-vii) can be proved similar to (i).

We obtain the relationships between the solutions of interval optimization problems and scalar optimization problems

$$(SP_L)\begin{cases}\min(max)\, z_e(f_L(x)) \\ \quad x \in M\end{cases}$$

and

$$(SP_U)\begin{cases}\min(max)\, z_e(f_U(x)) \\ \quad x \in M\end{cases}.$$

This method has also eliminated the use of the ordering cone to identify minimal (maximal), weak minimal (weak maximal), and strongly minimal (strongly maximal) solutions. Additionally, the solutions of the interval optimization problems are characterized by the solutions of the scalar optimization problems.

$(L - IVOP)$ if and only if x_0 is a solution of (SP_L).

Proof. We will prove for the minimal solutions. Similarly, it can be obtained for maximal solutions.

We assume that x_0 is a solution of $(L-IVOP)$. Then, we have two the following conditions:

(i) $F(x) \preccurlyeq^L F(x_0)$

(ii) $F(x) \not\preccurlyeq^L F(x_0)$

for all $x \in M$.

Let us consider the condition *(i)*. Since x_0 is a solution of $(L-IVOP)$, we get $F(x_0) \preccurlyeq^L F(x)$. Then, $f_L(x) \leq_C f_L(x_0)$ and $f_L(x_0) \leq_C f_L(x)$.

We obtain $z_e(f_L(x)) \leq z_e(f_L(x_0))$ and $z_e(f_L(x_0)) \leq z_e(f_L(x))$ from Proposition 5.3 (i). This gives

$$z_e\big(f_L(x)\big) = z_e(f_L(x_0)) \qquad (1)$$

from anti-symmetrically of $\leq_C$.

Let us consider the condition *(ii)*. Since $F(x) \not\preccurlyeq^L F(x_0)$, we have $f_L(x) \nleq_C f_L(x_0)$. Then, from Proposition 5.3 (i), we get

$$z_e\big(f_L(x)\big) \nleq z_e\big(f_L(x_0)\big). \qquad (2)$$

Finally, $z_e\big(f_L(x_0)\big) \leq z_e(f_L(x))$ for all $x \in M$ from Equation (1) and Equation (2). Then, x_0 is a solution of (SP_L).

Conversely, let x_0 be a solution of (SP_L). We assume that x_0 is not a solution of $(L-IVOP)$. Then, there exists an $\bar{x} \in M \setminus \{x_0\}$ such that $F(\bar{x}) \preccurlyeq^L F(x_0)$ and $F(x_0) \not\preccurlyeq^L F(\bar{x})$. That is $f_L(\bar{x}) \leq_C f_L(x_0)$ and $f_L(x_0) \nleq_C f_L(\bar{x})$.

We have $z_e\big(f_L(\bar{x})\big) \leq z_e(f_L(x_0))$ and $z_e(f_L(x_0)) \nleq z_e(f_L(\bar{x}))$ from Proposition 5.3 (i). Because z_e is real-valued, we obtain $z_e\big(f_L(\bar{x})\big) < z_e(f_L(x_0))$ for an $\bar{x} \in M$. This contradicts with the solution of (SP_L). Therefore, x_0 is a solution of $(L-IVOP)$.

Theorem 5.5 If $x_0 \in M$ is unique the solution of (SP_L), then x_0 is a strong solution of $(L-IVOP)$.

Proof. We will prove the minimum solution of the corresponding problems. Similarly, it can be proved for the maximum solutions.

We assume that x_0 is unique solution of (SP_L). Then, we have $z_e(f_L(x_0)) < z_e\big(f_L(x)\big)$ for all $x \in M \setminus \{x_0\}$. This gives $f_L(x_0) <_C f_L(x)$ for all $x \in M \setminus \{x_0\}$ from Proposition 5.3 (ii). Then, $F(x_0) \prec^L F(x)$ for all $x \in M \setminus \{x_0\}$. That is, $F(x_0) \preccurlyeq^L F(x)$ for all $x \in M$. Hence, x_0 is a strongly solution of $(L-IVOP)$.

Theorem 5.6 Let $x_0 \in M$ be unique solution of (SP_L), then

(i) x_0 is a solution of $(L-IVOP)$,

(ii) x_0 is a weak solution of $(L-IVOP)$.

Theorem 5.7 $x_0 \in M$ is a solution of $(U-IVOP)$ if and only if x_0 is a solution of (SP_U).

Proof. The proof follows immediately from Proposition 5.3 (i), such as the proof of Theorem 5.4.

Theorem 5.8 If $x_0 \in M$ is unique solution of (SP_U), then

(i) x_0 is a strongly solution of $(U-IVOP)$,

(ii) x_0 is a solution of $(U-IVOP)$,

(iii) x_0 is a weak solution of $(U-IVOP)$.

Proof. The proof follows immediately from Proposition 5.3 (i-ii), such as the proof of Theorem 5.4 and Theorem 5.5.

Example 5.9 Let us consider Example 4.4. In this example

$$z_e\big(f_L(x)\big) = \begin{cases} x^2 & ; x \in [-1,1] \\ |x| & ; otherwise \end{cases}$$

and

$$z_e\big(f_U(x)\big) = \begin{cases} |x| & ; x \in [-1,1] \\ x^2 & ; otherwise \end{cases}.$$

Graph of $z_e(f_L(x))$ and $z_e(f_U(x))$ are given in Figure 2 and Figure 3, respectively.

We consider the following scalar optimization problems:

$$(SP_L) \begin{cases} \min z_e(f_L(x)) \\ \quad x \in \mathbb{R} \end{cases}$$

and

$$(SP_U)\begin{cases}\min z_e(f_U(x)) \\ x\in\mathbb{R}\end{cases}.$$

As seen in Figure 2, 0 is the unique solution of (SP_L). Then, we can say that 0 is also a strong solution, solution, and weak solution of $(L-IVOP)$ from Theorem 5.5 and Theorem 5.6.

Similarly, 0 is unique solution of (SP_U). Then, we can say that 0 is also a strong solution, solution, and weak solution of $(U-IVOP)$ from Theorem 5.7 and Theorem 5.8.

We can say that there is no solution to the problems other than 0 from Theorem 5.4 and Theorem 5.7.

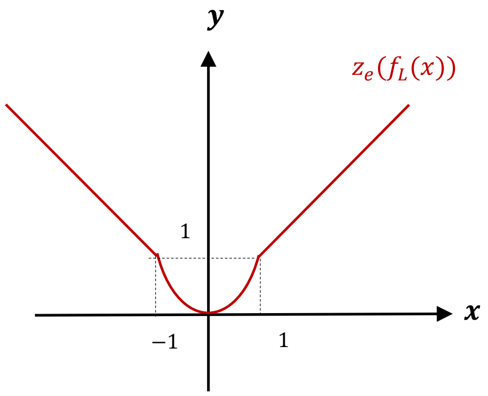

Fig. 2. Graph of the $z_e(f_L(x))$ in Example 5.9.

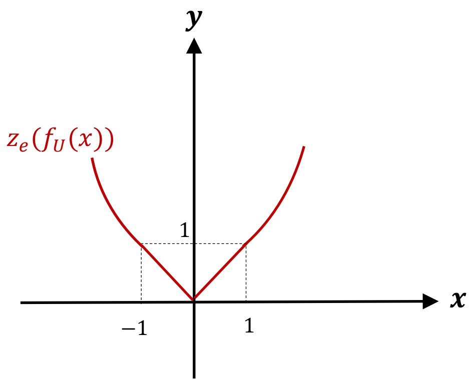

Fig. 3. Graph of the $z_e(f_U(x))$ in Example 5.9.

6. Subdifferentials of interval-valued functions and optimality conditions of interval optimization problems.

Inspired by works in Hernández & Rodríguez-Marín, 2011; Chen & Jahn, 1998, and Karaman *et al.*, 2020a, we present two subgradients for the interval-valued functions in this section. We define two subdifferentials, and some optimality conditions are obtained by using these subdifferentials.

Definition 6.1 A continuous linear operator $T: X \to Y$ is called a weak subgradient of F at $x_0 \in M$ if
$F(x) - F(x_0) - T(x - x_0) \subset Y \setminus \{-int(C)\}$
for all $x \in M$. The class of all weak subgradients of F at x_0 is called weak subdifferential of F at x_0 and denoted by $\partial^w F(x_0)$.

Definition 6.2 A continuous linear operator $T: X \to Y$ is called a subgradient of F at $x_0 \in M$ if

$$F(x) - F(x_0) - T(x - x_0) \subset C$$

for all $x \in M \setminus \{x_0\}$. The class of all subgradients of F at x_0 is called subdifferential of F at x_0 and denoted by $\partial F(x_0)$.

It is clear that $\partial^w F(x_0) \subset \partial F(x_0)$ for $x_0 \in M$.

Proposition 6.3 Let $x_0 \in M$. Then, $\partial F(x_0)$ and $\partial^w F(x_0)$ are convex sets.

Proof. Proof is obtained for subdifferential $\partial F(x_0)$. Similarly, it can be obtained for weak subdifferential.
Let $T_1, T_2 \in \partial F(x_0)$ and $\lambda \in [0,1]$. We shown that $\lambda T_1 + (1-\lambda)T_2 \in \partial F(x_0)$. Since $T_1 \in \partial F(x_0)$, we have

$$F(x) - F(x_0) - T_1(x - x_0) \subset C \qquad (3)$$

for all $x \in M$.
Also, because $T_2 \in \partial F(x_0)$, we get

$$F(x) - F(x_0) - T_2(x - x_0) \subset C \qquad (4)$$

for all $x \in M$. Because C is convex ordering cone, we yield $\lambda F(x) - \lambda F(x_0) - \lambda T_1(x - x_0) + (1-\lambda)F(x) - (1-\lambda)F(x_0) - (1-\lambda)T_2(x - x_0)$ $= F(x) - F(x_0) - (\lambda T_1 + (1-\lambda)T_2)(x - x_0) \subset C + C = C$.
Therefore, $(\lambda T_1 + (1-\lambda)T_2(x - x_0) \in \partial F(x_0)$. Then, $\partial F(x_0)$ is a convex set.

Proposition 6.4
Let $x_0 \in M$ and $F, G: M \to I_C(Y)$ be interval-valued functions.
Then,

(i) $\partial F(x_0) + \partial G(x_0) \subset \partial(F+G)(x_0)$,
(ii) $\partial^w F(x_0) + \partial^w G(x_0) \subset \partial^w(F+G)(x_0)$.

Proof. It follows immediately from Definition 6.1 and Definition 6.2.

Theorem 6.5 Let $x_0 \in M$. If $0 \in \partial F(x_0)$, then x_0 is a strongly minimal solution of $(L-IVOP)$.

Proof. Let $0 \in \partial F(x_0)$. Then, we have $F(x) - F(x_0) \subset C$ for all $x \in M \setminus \{x_0\}$. This gives $[f_L(x), f_U(x)] - [f_L(x_0), f_U(x_0)] \subset C$ for all $x \in M \setminus \{x_0\}$. That is, $f_L(x) - f_U(x_0) \in C$ and $f_U(x_0) \leq_C f_L(x)$ for all $x \in M \setminus \{x_0\}$. Since $f_L(x_0) \leq_C f_U(x_0)$, we obtain $f_L(x_0) \leq_C f_L(x)$ for all $x \in M \setminus \{x_0\}$. So, $F(x_0) \preccurlyeq^L F(x)$ for all $x \in M$. Therefore, x_0 is a strongly minimal solution of $(L-IVOP)$.

Remark 6.6 Let $x_0 \in M$. If $0 \in \partial F(x_0)$, then

(i) x_0 is a minimal solution of $(L-IVOP)$,
(ii) x_0 is a weak minimal solution of $(L-IVOP)$.

Theorem 6.7 Let $x_0 \in M$. If $0 \in \partial F(x_0)$, then x_0 is a strongly minimal solution of $(U-IVOP)$.

Proof. It follows similarly to the proof of Theorem 6.5.

Theorem 6.8 Let $x_0 \in M$. If $0 \notin \partial F(x_0)$, then x_0 is a maximal solution of $(L-IVOP)$.

Proof. Let $0 \notin \partial F(x_0)$. Then, $[f_L(x), f_U(x)] - [f_L(x_0), f_U(x_0)] \not\subset C$ for all $x \in M \setminus \{x_0\}$. That is, $[f_L(x) - f_U(x_0), f_U(x) - f_L(x_0)] \not\subset C$ for all $x \in M \setminus \{x_0\}$. This means $f_L(x) - f_U(x_0) \notin C$ or $f_U(x) - f_L(x_0) \notin C$ for all $x \in M \setminus \{x_0\}$. Then, $f_U(x_0) \nleq_C f_L(x)$ or $f_L(x_0) \nleq_C f_U(x)$ for all $x \in M \setminus \{x_0\}$. Since $f_L(x) \leq_C f_U(x)$ for all $x \in M$, we obtain $f_L(x_0) \nleq_C f_L(x)$ or $f_L(x_0) \nleq_C f_L(x)$ for all $x \in M \setminus \{x_0\}$. Hence, $F(x_0) \not\preccurlyeq^L F(x)$ for all $x \in M \setminus \{x_0\}$. Therefore, x_0 is a maximal solution of $(L-IVOP)$.

Theorem 6.9 Let $x_0 \in M$. If $0 \in \partial^w F(x_0)$, then x_0 is a weak minimal solution of $(U-IVOP)$.

Proof. Assume that $0 \in \partial^w F(x_0)$. Then we have $F(x) - F(x_0) \subset Y \setminus \{-int(C)\}$ for all $x \in M$. This gives $[f_L(x) - f_U(x_0), f_U(x) - f_L(x_0)] \subset Y \setminus \{-int(C)\}$ for all $x \in M$. Then, $f_L(x) - f_U(x_0) \notin -int(C)$ that is, $f_U(x_0) - f_L(x) \notin int(C)$ for all $x \in M$. Since $f_L(x) \leq_C f_U(x)$ for all $x \in M$, we get $f_U(x) \nless_C f_U(x_0)$ for all $x \in M$. So, $F(x) \nprec^U F(x_0)$ for all $x \in M$. This means x_0 is a weak minimal solution of $(U-IVOP)$.

Theorem 6.10 Let $x_0 \in M$. If $0 \in \partial^w F(x_0)$, then x_0 is a weak minimal solution of $(L-IVOP)$.

Proof. It can be proved similarly to the proof of Theorem 6.9.

7. Conclusion

A new generalization of (IVOP) on real topological vector spaces is given in this work. Intervals on the real topological vector space are defined by using the ordering cone. To obtain the solutions and optimality conditions for (IVOP), scalarization and subdifferentials are used. New optimality conditions may be found by using the proposed method, vectorization and directional derivative, etc.

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