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التوصيف الطيفي الوردي للأعداد غير المحددة وفق نظرية لابلاس

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الملخص

يوضح هذا البحث أن الرسم البياني$\mathcal{p}$- الوردي $\mathcal{T}$=RG $(a_3, a_4,\ldots,a_s)$ هو شكل يتكون من دائرتين $\mathcal{p} = a_3 + a_4 + \ldots + a_s \geq 2$ تتقابل جميعها في قمة رأس واحدة، a_i $(3 \leq i \leq s)$ هي عدد الدوائر في $\mathcal{T}$ بطول i. ويُقال إن الرسم البياني G هو DLS (resp. DQS) إذا تم تحديده بطيف مصفوفة لابلاس (الأعداد غير المحددة اللابلاسية .resp) بمعنى إذا كان كل رسم بياني له الطيف نفسه، يكون متماثلاً مع G. أثبت لابلاس من قبل وفان دام مؤخراً أن كل وورد $\mathcal{p}$-، فيما عدا استثنائيين غير متماثلين، يكونوا DLS. نوضح في هذا البحث أنه بما أن P≥3، تكون كل $\mathcal{p}$- وورد تكون DQS.

Signless Laplacian spectral characterization of roses

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Abstract

A p-rose graph $\Gamma = RG(a_3, a_4, \ldots, a_s)$ is a graph consisting of $p = a_3 + a_4 + \cdots + a_s \geqslant 2$ cycles that all meet in one vertex, and a_i $(3 \leqslant i \leqslant s)$ is the number of cycles in Γ of length i. A graph G is said to be DLS (resp. DQS) if it is determined by the spectrum of its Laplacian (resp. signless Laplacian) matrix, i.e., if every graph with the same spectrum is isomorphic to G. He and van Dam recently proved that all p-roses, except for two non-isomorphic exceptions, are DLS. In this paper, we show that, for $p \geqslant 3$ all p-roses are DQS.

Keywords: Rose graph; Signless Laplacian matrix; Signless Laplacian spectrum; Q-cospectral; DQS.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$, and let $A(G)$ denote the adjacency matrix of G. The Laplacian and the signless Laplacian matrix are respectively defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees.

For $M \in \{A, L, Q\}$, we consider the *M-polynomial of G* defined as

$$P_M(G)(x) = det(xI - M(G)),$$

where I is the identity matrix of order n. The multiset $\mathrm{Spec}_M(G)$ consisting of the M-eigenvalues, i.e., the roots of the M-polynomial of G, is called *the M-spectrum of G*.

Two graphs are said to be M-cospectral if they have the same M-spectrum. When the only graphs which are M-cospectral to G are those isomorphic to G, we say that G is determined by its M-spectrum or, equivalently, that G is a DMS-graph.

Establishing whether a graph G in a fixed class $\mathcal{G}$ is DMS or not and, in the latter case, detecting all graphs which are M-cospectral to G is now a classical topic in spectral graph theory. In the last few years, many results have been obtained when $\mathcal{G}$ is the class of p-roses. A p-rose $\Gamma = RG(a_3, a_4, \ldots, a_s)$ (here $p \geqslant 2$) is a graph consisting of $p = a_3 + a_4 + \cdots + a_s$ cycles all sharing a single vertex, a_i $(3 \leqslant i \leqslant s)$ is the number of cycles in Γ of length i, and $a_s > 0$.

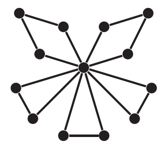

Fig. 1. The 5-rose $RG(3,2)$

Indeed, Wang *et al.* (2010) showed that almost all 2-rose graphs are DQS, the exception being the one with $a_{2k} = a_{2k+1} = 1$ if the order is $4k$, and the one with $a_{2k-1} = a_{2k} = 1$ if the order is $4k-2$. The same result can be now retrieved from (Belardo & Brunetti 2019, Table 1).

3-roses have been investigated in (Liu & Huang 2013; Wang *et al.* 2013); it turned out that they are both DLS and DQS. Ma & Huang (2016) proved that any 4-rose graph is DQS. Finally, He & van Dam (2018) have shown that, apart from two exceptional cases of order 6 and 7, all roses are DLS.

The following theorem is our main result.

Theorem 1.1. *Every p-rose with $p \geqslant 3$ is DQS.*

The remainder of the paper is structured as follows: Section 2 contains preliminary results on the signless Laplacian spectrum of a graph, while Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Let G be a graph of order n. We assume that its vertices $v_1, \ldots, v_n$ are arranged in such a way that $d_1(G) \geqslant d_2(G) \geqslant \cdots \geqslant d_n(G)$, where $d_i(G) = d_G(v_i)$ is the vertex degree of v_i in G (the subscript will be omitted if the graph context is clear). We respectively denote by $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and by $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G)$ the A- and the Q-eigenvalues of G. The latter are nonnegative since $Q(G)$ is positive semidefinite.

We start by recalling the simplest version of the celebrated *Interlacing Theorem*.

Theorem 2.1. (Cvetković *et al.* 2010, Section 1.3) *Let G be a graph of order $n > 1$, and let $G - v$ be the graph obtained from G by deleting the vertex v and all the edges having v as an endpoint. Then,*

$$\lambda_1(G) \geqslant \lambda_1(G-v) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_{n-1}(G-v) \geqslant \lambda_n(G).$$

We now collect some known results on the signless Laplacian of simple graphs. They are all necessary to prove Theorem 1.1. The first one is a consequence of the following well-known fact: the Q-eigenvalues of a graph interlace with the Q-eigenvalues of an edge-deleted subgraph.

Proposition 2.2. (Wang & Belardo 2011, Corollary 2.7) *Let G be a graph of order n and let G' be a subgraph of G of order $n' \leqslant n$. Then, for $1 \leqslant i \leqslant n'$ we have $q_i(G') \leqslant q_i(G)$.*

Proposition 2.3. (Cvetković *et al.* 2010, Section 5) *Let G be a non-empty graph of order $n \geqslant 2$. Then, $q_1(G) \geqslant d_1(G) + 1$, and the equality holds if and only if G is the star $K_{1,n-1}$.*

Proposition 2.4. (Das 2010, Corollary 2.6) *Let G be a graph of order n. Then, $q_1(G) \leqslant d_1(G) + d_2(G)$, where the equality holds if and only if G is either $K_{1,n-1}$ or any regular graph.*

Proposition 2.5. (Das 2010, Corollary 3.2) *For any graph G of order $n \geqslant 2$, we have $q_2(G) \geqslant d_2(G) - 1$. If the equality holds, the maximum and second maximum degree vertices are adjacent, and $d_1(G) = d_2(G)$.*

In Lemma 2.6 below, and in the rest of the paper as well, $n_G(C_3)$ denotes the number of triangles contained in a graph G.

Lemma 2.6. (Cvetković *et al.* 2007b, Corollary 4.3) *Let G be a graph with n vertices, m edges and $D(G) = \mathrm{diag}(d_1, \ldots, d_n)$. Let $T_k(G) = \sum_{i=1}^{n} q_i^k(G)$, $(k \geqslant 0)$ be the k-th spectral moment for the Q-spectrum. Then,*

$$T_0(G) = n, \qquad T_1(G) = \sum_{i=1}^{n} d_i = 2m,$$

$$T_2(G) = 2m + \sum_{i=1}^{n} d_i^2,$$

$$T_3(G) = 6n_G(C_3) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

From Lemma 2.6 it is not hard to obtain the following result.

Proposition 2.7. *Let H be a graph Q-cospectral to G. Then:*

(i) G and H have the same number of vertices and the same number of edges.
(ii) $\sum_{i=1}^{n} d_i^2(G) = \sum_{i=1}^{n} d_i^2(H)$.
(iii) $6\,(n_G(C_3) - n_H(C_3)) = \sum_{i=1}^{n} (d_i^3(H) - d_i^3(G))$.

We now consider the numbers

$$\widetilde{T}_2(G) = \sum_{i=1}^{n} (d_i(G) - 2)^2$$

and

$$\widetilde{T}_3(G) = tr(A(G)^3) + \sum_{i=1}^{n} (d_i(G) - 2)^3.$$

After expanding the powers in the sums and using the well-known equality

$$tr(A(G)^3) = 6n_G(C_3),$$

Proposition 2.8 below follows from Lemma 2.6 and Proposition 2.7.

Proposition 2.8. *If G and H are Q-cospectral, then*

$$\widetilde{T}_2(G) = \widetilde{T}_2(H) \quad \text{and} \quad \widetilde{T}_3(G) = \widetilde{T}_3(H).$$

For any graph G, we recall that the subdivision graph $S(G)$ is obtained from G by replacing each of its edges by a path of length 2, or, equivalently, by inserting an additional vertex into each edge e of G.

Proposition 2.9. (Cvetković *et al.* 2010, Theorem 2.4.4) *Let G be a graph of order n and size m. Then,*

$$P_A(S(G))(x) = x^{m-n} P_Q(G)(x^2).$$

The following corollary is a consequence of Propositions 2.7 and 2.9.

Corollary 2.10. *G and H are Q-cospectral if and only if $S(G)$ and $S(H)$ are A-cospectral.*

Proposition 2.11. (Cvetković *et al.* 1995, p. 73) *The adjacency spectrum* $\mathrm{Spec}_A(P_n)$ *of the path P_n with n vertices is*

$$\left\{ \lambda_i = 2cos\frac{\pi i}{n+1} \,\middle|\, i = 1, 2, \ldots, n \right\}.$$

Given two graphs G and G' we shall denote by $G \cup G'$ the disjoint union of G and G'.

Proposition 2.12. (Liu *et al.* 2014, Proposition 3.2) *Let G be a connected graph with $q_2(G) < 4$, and let the graph H be Q-cospectral to G.*

(1) If G is non-bipartite, then H is connected.
(2) If G is bipartite, then either H is connected or $H = P_r \cup H_1$, where P_r is a path with r vertices and H_1 is connected and non-bipartite.

3. Proof of Theorem 1.1

Our aim is to prove that, for $p \geqslant 3$, every p-rose is DQS.

Lemma 3.1. *Let $\Gamma = RG(a_3, a_4, \ldots, a_s)$ be a p-rose graph. Then,*

(i) $2p + 1 < q_1(\Gamma) < 2p + 2$;
(ii) $q_2(\Gamma) < 4$.

Proof. (*i*) Note that p-roses are neither regular nor they have pendant vertices. Now, the strict inequalities come from Propositions 2.3 and 2.4 since $d_1(\Gamma) = 2p$ and $d_2(\Gamma) = 2$.

(*ii*) Let v be the vertex with maximum degree in $V(\Gamma)$. Then,

$$S(\Gamma) - v = a_3 P_5 \cup a_4 P_7 \cup \cdots \cup a_s P_{2s-1}.$$

By Proposition 2.11, $\lambda_1(S(\Gamma) - v) < 2$. Therefore, $\lambda_2(S(\Gamma)) < 2$ by Theorem 2.1. Finally, by Proposition 2.9 we have $q_2(\Gamma) < 4$. □

Lemma 3.2. *Let the graph H be Q-cospectral to a fixed p-rose Γ. Then,*

$$2p - 2 \leqslant d_1(H) \leqslant 2p \quad \text{and} \quad d_2(H) \leqslant 4.$$

Proof. Proposition 2.3 and Lemma 3.1 imply that

$$d_1(H) < q_1(H) - 1 = q_1(\Gamma) - 1 < 2p + 1.$$

Therefore, $d_1(H) \leqslant 2p$. Moreover, by Proposition 2.5 and Lemma 3.1, we get

$$d_2(H) - 1 \leqslant q_2(H) = q_2(\Gamma) < 4,$$

which is possible only if $d_2(H) \leqslant 4$.

We use such inequality, Proposition 2.4 and Lemma 3.1 to infer that

$$2p + 1 < q_1(\Gamma) = q_1(H) < d_1(H) + d_2(H) \leqslant d_1(H) + 4,$$

from which we deduce $2p - 2 \leqslant d_1(H)$ as claimed. □

Lemma 3.3. *Let the graph H be Q-cospectral to a fixed p-rose Γ with $p \geqslant 4$. Then $d_2(H) \leqslant 3$.*

Proof. Since in our hypotheses $p \geqslant 4$, Lemma 3.2 ensures that $d_1(H) \geqslant 2p - 2 \geqslant 6$ and $d_2(H) \leqslant 4$. We now argue by contradiction assuming that $d_2(H) = 4$. Let v_2 be a vertex of H with degree $d_2(H) = 4$. Clearly, the graph H contains as subgraphs two distinct stars $S' = K_{1,6}$ and $S'' = K_{1,4}$ with centers v_1 and v_2 respectively. Let H' be the (not necessarily disjoint) union of S' and S'', i.e., $V(H') = V(S') \cup V(S'')$ and $E(H') = E(S') \cup E(S'')$.

There are several possibilities: i) S' and S'' are both edge- and vertex-disjoint; ii) S' and S'' are edge-disjoint and have i vertices in common (with $1 \leqslant i \leqslant 4$); iii) S' and S'' have one edge and i vertices in common (with $2 \leqslant i \leqslant 5$). In all the nine cases, a direct computation shows that $q_2(H') \geqslant 4$. This contradicts Proposition 2.2. In fact, $4 > q_2(H)$ by Lemma 3.1. □

Lemma 3.4. *Let the graph H be Q-cospectral to a fixed p-rose Γ with $p \geqslant 4$, and let v_1 be the vertex of H with degree $d_1(H)$. Then, each triangle contained in H has v_1 among its vertices.*

Proof. As in the proof of Lemma 3.3, we argue that $d_1(H) \geqslant 6$. The existence of a triangle in H not having v_1 among its vertices would imply the presence of one of the graphs H_i''s depicted in Figure 2 among the subgraphs of H. But this is not possible, since $q_2(H_i') \geqslant 4$ for $1 \leqslant i \leqslant 4$ (note that the H_i''s all contain $C_3 \cup K_{1,3}$, and $q_2(C_3 \cup K_{1,3}) = 4$). Once again, this contradicts Proposition 2.2, since $4 > q_2(H)$ by Lemma 3.1. □

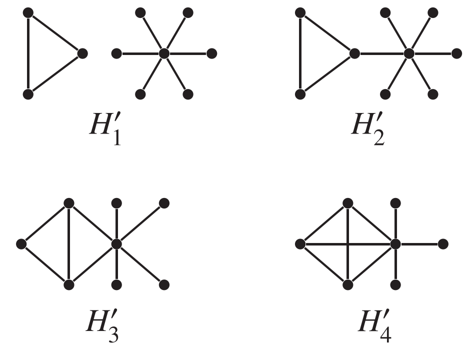

Fig. 2. Four 'forbidden' subgraphs: their second signless Laplacian eigenvalue is at least 4.

Proposition 3.5. *Let the graph H be Q-cospectral to a fixed p-rose Γ with $p \geqslant 4$, and let n_3 be the number of vertices in $V(H)$ of degree 3. Then,*

$$(1) \quad n_H(C_3) \leqslant 2\min\{d_1(H), n_3\} + \max\{d_1(H) - n_3, 0\}.$$

Proof. As before, we denote by v_1 the vertex in H whose vertex degree is $d_1(H)$. Each neighbour of v_1 belongs to at most two triangles. This fact comes from Lemma 3.3 and the absence of the graph H_4' in Figure 2 among the subgraphs of H. On the other hand, we deduce by Lemma 3.4 that each triangle of H has two neighbours of v_1 among its vertices. Therefore, the maximum possible number of triangles is attained if

- all vertices of degree 3 are adjacent to v_1;
- each of them belongs to two triangles;
- each neighbour of v_1 whose degree is not 3, if any, belongs to a triangle (which is necessarily unique by Lemma 3.3).

This means that

$$n_H(C_3) \leqslant \begin{cases} 2n_3 + (d_1(H) - n_3) & \text{if } d_1(H) \geqslant n_3, \\ 2d_1(H) & \text{otherwise.} \end{cases}$$

Such inequality is equivalent to (1). □

The upper bound for the number of triangles in H established in Proposition 3.5 is far from being sharp. In any case, Inequality (1) is restrictive enough to prove Theorem 3.6.

Theorem 3.6. *Suppose that H and the p-rose graph $\Gamma = RG(a_3, a_4, \ldots, a_s)$ are Q-cospectral. Then H and Γ share the same degree sequence.*

Proof. Since it is already known that all 3-roses are DQS (Wang *et al.* 2013, Theorem 5.3) and all 4-roses are DQS as well (Ma & Huang 2016, Theorem 4.8), the statement clearly holds for $p \in \{3, 4\}$.

In any case, we suppose for the rest of the proof that $p \geqslant 4$, and assume that H has n_i vertices of degree i for $1 \leqslant i \leqslant d_1(H)$. According to Proposition 2.7, the graphs H and Γ have the same order n and the same size, which is $n + p - 1$. They also share the other Q-spectral invariants collected in Section 2. The equalities

$$T_0(H)=T_0(\Gamma), \qquad T_1(H)=T_1(\Gamma),$$

$$\sum_{i=1}^{d_1(H)} d_i(H)^2 = \sum_{i=1}^{d_1(\Gamma)} d_i(\Gamma)^2, \quad \widetilde{T}_2(H)=\widetilde{T}_2(\Gamma)$$

lead to the following four equations:

$$\sum_{i=1}^{d_1(H)} n_i = n; \tag{2}$$

$$\sum_{i=1}^{d_1(H)} in_i = 2(n + p - 1); \tag{3}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = 4(n-1) + 4p^2; \tag{4}$$

$$\sum_{i=1}^{n} (d_i(H) - 2)^2 = (2p-2)^2. \tag{5}$$

By adding up (2), (3) and (4) multiplied by 2, −3, and 1 respectively, we obtain

$$\sum_{i=1}^{d_1(H)} (i^2 - 3i + 2)n_i = (2p-1)(2p-2); \tag{6}$$

and if we equate $\widetilde{T}_3(H)$ and

$$\widetilde{T}_3(\Gamma) = 6a_3 + (2p-2)^3,$$

we get

$$n_H(C_3)=a_3+\frac{(2p-2)^3 - \sum_{i=1}^{n} (d_i(H)-2)^3}{6}. \tag{7}$$

A priori, there are just three possibilities for $d_1(H)$. In fact, Lemma 3.2 says in particular that $d_1(H) \in \{2p-2,\ 2p-1,\ 2p\}$.

If $d_1(H) = 2p$, (5) implies that

$$\sum_{i=2}^{n} (d_i(H) - 2)^2 = 0,$$

and this is only possible if $d_i(H) = 2$ for all $i \in \{2, \ldots, n\}$, as claimed. The proof will be over once we show that the cases $d_1(H) = 2p-1$ and $d_2(H) = 2p-2$ cannot occur.

In fact, suppose by contradiction that $d_1(H) = 2p - 1$. Since $d_2(H) \leqslant 3$ by Lemma 3.3, and in our hypotheses $d_1(H) \geqslant 7$, we have $n_{2p-1} = 1$. Therefore, (6) becomes

$$((2p-1)^2 - 3(2p-1) + 2) + 2n_3 = (2p-1)(2p-2),$$

which, together with (2) and (3), gives

$$n_1 = 2p-3,\ n_2 = n-4p+4, \text{ and } n_3 = 2p-2.$$

Finally, by (7), we get

$$n_H(C_3) = a_3 + 2p^2 - 5p + 3.$$

In the case at hand, Proposition 3.5 implies that $n_H(C_3) \leqslant 4p - 3$. But

$$n_H(C_3) = a_3 + 2p^2 - 5p + 3 \leqslant 4p - 3$$

is false for $p \geqslant 4$, being $a_3 \geqslant 0$.

The case $d_1(H) = 2p - 2$ is excluded in a similar way. Assuming $d_1(H) = 2p - 2$ and $p \geqslant 4$, we have $n_{2p-2} = 1$, since $d_2(H) \leqslant 3$. Therefore, (6) becomes

$$((2p-2)^2 - 3(2p-2) + 2) + 2n_3 = (2p-1)(2p-2),$$

which, together with (2) and (3), gives

$$n_1 = 4p-7,\ n_2 = n-8p+11 \text{ and } n_3 = 4p-5.$$

From (7) we get

$$n_H(C_3) = a_3 + 4p^2 - 12p + 9.$$

This time, Proposition 3.5 implies that $n_H(C_3) \leqslant 4p - 4$. In this case too, it is easily seen that

$$n_H(C_3) = a_3 + 4p^2 - 12p + 9 \leqslant 4p - 4$$

is false for $p \geqslant 4$. □

Corollary 3.7. *Let $p \geqslant 3$, and let H be a graph Q-cospectral to a p-rose Γ. Then H is a p-rose itself.*

Proof. Let n be the order of the graph H. By Theorem 3.6 we know that

$$d_i(H) = \begin{cases} 2p & \text{if } i = 1; \\ 2 & \text{if } 2 \leqslant i \leqslant n. \end{cases} \tag{8}$$

The absence of pendant vertices in $V(H)$ implies that H cannot have a path among its connected components. By Proposition 2.12 it follows that H is connected, and the only connected graphs having a degree sequence as given in (8) are p-roses. □

We are now ready to complete the proof of Theorem 1.1. He & van Dam (2018) introduced the *universal Laplacian matrix of a graph G*

$$Q(\alpha, \beta)(G) := \alpha D(G) + \beta A(G),$$

defined for every real number α and for every non-zero real number β. They proved that if two p-roses are $Q(\bar{\alpha}, \bar{\beta})$-cospectral for a fixed $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$, then they are isomorphic (He & van Dam 2018, Proposition 3.1). Now, Corollary 3.7 can be rephrased by saying that if H is a graph Q-cospectral to a p-rose Γ for $p \geqslant 3$, then H is a p-rose which is $Q(1, 1)$-cospectral to Γ. By (He & van Dam 2018, Proposition 3.1), the graphs H and Γ are isomorphic. In other words every p-rose for $p \geqslant 3$ is determined by its signless Laplacian spectrum.

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