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Rotation in four dimensions via generalized Hamilton operators

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ABSTRACT

In this paper, after a brief review of some algebraic properties of generalized quaternions, we investigated the properties of generalized Hamilton operators and we considered how the generalized quaternions can be used to described the rotation in 4-dimensional space $E^4_{\alpha\beta}$.

Keywords: Generalized quaternion; Hamilton operator; rotation.

INTRODUCTION

The quaternions are commonly used in physics, chemistry, robotics, mechanics, electronics and etc., since they serve powerful computational tools to deal with the rotations and translations in the real and dual spaces (Ward, 1997). Arthur Cayley was first who discovered that quaternion could be used to represent the rotations in E^4. Any rotation in E^4 is a product of rotation in a pair of orthogonal two-dimensional subspaces (Weiner & Wilkens, 2005). The quaternion algebra is an associative and non-commutative 4-dimensional Clifford algebra. Some algebraic properties of Hamilton operators are considered in (Agrawal, 1987) where real quaternions have been expressed in terms of 4×4 matrices by means of these operators. These matrices have applications in mechanics, quantum physics and computer-aided geometric design (Adler, 1995). In addition, the homothetic motions has been considered with aid of the Hamilton operators in four-dimensional Euclidean space E^4(Yayli, 1992). The eigenvalues, eigenvectors and the others algebraic properties of these matrices are studied by several authors (Farebrother *et al.,* 2003; Groβ *et al.,* 2001; Zhang, 1997). De-Moivre's and Euler's formulae for matrices associated with real quaternions is studied in (Jafari *et al.,* 2011) and every power of these matrices are immediately obtained. A brief introduction to the generalized quaternions is provided in (Pottman & Wallner, 2000), the subject which have investigated in algebra (Savin *et al.,* 2009; Unger & Markin, 2008). Recently, we have studied the generalized quaternion and some of their algebraic properties (Jafari &Yayli, 2010a). It is shown that the set of all unit generalized quaternions with the group operation of quaternions multiplication is a Lie group of 3-dimension. Their Lie algebra and properties of the bracket multiplication are looked for, e.g. a matrix corresponding to Hamilton operators, defined for the generalized quaternions, determines a Homothetic motion in $E^4_{\alpha\beta}$ (Jafari & Yayli, 2010b). In this paper, we briefly recall some fundamental properties of the generalized quaternions, we investigate some properties of the Hamilton operators and we show how the generalized quaternions can be used to described the rotation in 4-dimensional space $E^4_{\alpha\beta}$.

PRELIMINARIES

In this section, we brief describe some properties of real quaternions. Let C and R be the fields of the complex and the real numbers, respectively. Let H be a four-dimensional vector space over R with an ordered basis $B = \{e, \vec{i}, \vec{j}, \vec{k}\}$.

Real quaternion

A real quaternion is a vector

$$q = a_0 e + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad \in H$$

with real coefficients a_0, a_1, a_2, a_3. Besides the addition and the scalar multiplication of the vector space H over R, the product of any two quaternions $e, \vec{i}, \vec{j}, \vec{k}$ is defined such that $e$acts as an identity and the following properties are fulfilled:

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \vec{i} = -\vec{k}\vec{j},$$

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}, \quad \vec{i}\vec{j}\vec{k} = -1.$$

Real and complex numbers can be thought as quaternions in the natural way. For any $q = a_0 e + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, we define $Req = a_0$, the real part of q; $Co\,q = a_0 + a_1\vec{i}$, the complex part of q; $Imq = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, the imaginary part of q; $\bar{q} = a_0 e - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}$, the conjugate of q.

We can represent a quaternion $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ by a 2×2 complex matrix (with i' being the usual complex imaginary);

$$A = \begin{bmatrix} a_0 + i' a_1 & -i' a_1 + a_2 \\ -i' a_1 - a_2 & a_0 - i' a_3 \end{bmatrix}$$

or by a 4×4 real matrix

$$A = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_2 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \quad \text{(Girard, 2007).}$$

The Euler's and De-Moivre's formulae for the matrix A are studied in (Jafari *et al.,* 2011). From De- Moivre's formula, it results that the equation $A^n = I_4$ for $n > 2$ has an infinity of solutions of matrices of unit quaternions.

Let $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4) \in R^4$ and $\alpha, \beta \in R$. The generalized inner product of $\vec{u}$ and $\vec{v}$ is defined by

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + \alpha u_2 v_2 + \beta u_3 v_3 + \alpha\beta u_4 v_4.$$

We put $E^4_{\alpha\beta} = (R^4, \langle , \rangle)$. (Jafari & Yayli, 2010b)

Special cases:

1. If $\alpha = \beta = 1$, then $E^4_{\alpha\beta}$ is an Euclidean 4-space E^4.
2. If $\alpha = 1$, $\beta = -1$, then $E^4_{\alpha\beta}$ is a semi-Euclidean 4-space with 2-index E^4_2. (Kula & Yayli, 2007)

Generalized quaternion

A generalized quaternion q is an expression of the form

$$q = a_0 + a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^2 = -\alpha, \quad \vec{j}^2 = -\beta, \quad \vec{k}^2 = -\alpha\beta,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i}=\alpha\vec{j}=-\vec{i}\vec{k}, \quad \alpha,\beta \in R.$$

The set of all generalized quaternions is denoted by $\mathrm{H}_{\alpha\beta}$. A generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $\vec{V}_q = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$. Therefore $\mathrm{H}_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis R and a 3-dimensional real linear space $E^3_{\alpha\beta}$, so that, $\mathrm{H}_{\alpha\beta} = R \oplus E^3_{\alpha\beta}$.

Special cases:

1. $\alpha = \beta = 1$, is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of real quaternions.
2. $\alpha = 1,\ \beta = -1$, is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of split quaternions.
3. $\alpha = 1,\ \beta = 0$, is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of semi-quaternions.
4. $\alpha = -1,\ \beta = 0$, is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of split semi-quaternions.
5. $\alpha = 0,\ \beta = 0$, is considered, then $\mathrm{H}_{\alpha\beta}$ is the algebra of $\frac{1}{4}$ -quaternions. (Rosenfeld, 1997)

The multiplication rule for generalized quaternions is defined as

$$qp = S_qS_p - \left\langle \vec{V}_q, \vec{V}_p \right\rangle + S_q\vec{V}_p + S_p\vec{V}_q + \vec{V}_p \times \vec{V}_q$$

where

$$S_q = a_0, \quad S_p = b_0, \quad \left\langle \vec{V}_q, \vec{V}_p \right\rangle = \alpha a_1b_1 + \beta a_2b_2 + \alpha\beta a_3b_3,$$
$$\vec{V}_q \times \vec{V}_p = \beta(a_2b_3 - a_3b_2)\vec{i} + \alpha(a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}.$$

It could be written

$$qp = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Obviously, quaternions multiplication is associative and distributive with respect to addition and subtraction, but the commutative law does not hold in general.

$H_{\alpha\beta}$ with addition and multiplication has all the properties of a number field without commutativity of the multiplication. Therefore it is called the skew field of quaternions.

The conjugate of the quaternion $q = S_q + \vec{V}_q$ is denoted by $\bar{q}$ and defined as $\bar{q} = S_q - \vec{V}_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $N_q = q\,\bar{q} = \bar{q}\,q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2$ and we say that $q_0 = q/N_q$ is a unit generalized quaternion where $q \neq 0$. The set of unit generalized quaternions, G, with the group operation of quaternions multiplication is a Lie group of 3-dimension. The scalar product of two generalized quaternions $q = S_q + \vec{V}_q$ and $p = S_p + \vec{V}_p$ is defined as

$$\langle q, p \rangle_s = S_q S_p + \left\langle \vec{V}_q, \vec{V}_p \right\rangle = S(q\bar{p}).$$

Also, using the scalar product we can defined an angle λ between two quaternions q, $p$to be such

$$\cos\lambda = \frac{S(q\bar{p})}{\sqrt{N_q}\sqrt{N_p}},$$

(Jafari & Yayli, 2010a).

A matrix A is called a quasi-orthogonal matrix if $A^T \varepsilon A = \varepsilon$ and $\det A = 1$ where

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix},$$

and $\alpha, \beta \in R$. The set of all quasi-orthogonal matrices with the operation of matrix multiplication is called rotations group in 4-spaces $E^4_{\alpha\beta}$ (Jafari & Yayli, 2010b).

GENERALIZED HAMILTON OPERATORS

In this section, we introduce the $R-$linear transformations representing left and right multiplication in $H_{\alpha\beta}$. Let q be a unit generalized quaternion, then $\overset{+}{\underset{q}{h}} : H_{\alpha\beta} \rightarrow H_{\alpha\beta}$ and $\overset{-}{\underset{q}{h}} : H_{\alpha\beta} \rightarrow H_{\alpha\beta}$ are defined as follows:

$$\overset{+}{\underset{q}{h}}(x) = qx, \qquad \overset{-}{\underset{q}{h}}(x) = xq \qquad x \in H_{\alpha\beta}.$$

In both cases, considering $H_{\alpha\beta}$ to be $E^4_{\alpha\beta}$ spanned by the usual basic elements. We suspect that both these maps correspond to rotation, since it easy to show that they are norm and angle preserving. For example, considering the map $\overset{+}{\underset{q}{h}}$, we have already seen that if $x, y, q \in H_{\alpha\beta}$ and $N_q = 1$, then

$$N_{qx} = N_q N_x = N_x.$$

The generalized Hamilton operators $\overset{+}{H}$ and $\overset{-}{H}$, could be represented as the matrices;

$$\overset{+}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \tag{1}$$

and

$$\overset{-}{H}(q) = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \tag{2}$$

A direct consequence of the above operators is the following identities:

$$\overset{+}{H}(1) = I_4, \ \ \overset{+}{H}(\vec{i}) = E_1, \ \ \overset{+}{H}(\vec{j}) = E_2, \ \ \overset{+}{H}(\vec{k}) = E_3 \tag{3}$$

and

$$\overset{-}{H}(1) = I_4, \ \ \overset{-}{H}(\vec{i}) = F_1, \ \ \overset{-}{H}(\vec{j}) = F_2, \ \ \overset{-}{H}(\vec{k}) = F_3 \quad , \tag{4}$$

where I_4 is a 4×4 identity matrix. Note, that the properties of the E_n and $F_n \quad (n = 1, 2, 3)$ are identical to those of generalized quaternionic unit $\vec{i}, \vec{j}, \vec{k}$. Since $\overset{+}{H}$ and $\overset{-}{H}$ are linear, it follows that;

$$\begin{aligned} \overset{+}{H}(q) &= a_0 \overset{+}{H}(1) + a_1 \overset{+}{H}(\vec{i}) + a_2 \overset{+}{H}(\vec{j}) + a_3 \overset{+}{H}(\vec{k}) \\ &= a_0 I_4 + a_1 E_1 + a_2 E_2 + a_3 E_3, \end{aligned}$$

and

$$\begin{aligned}\overset{-}{H}(q) &= a_0 \overset{-}{H}(1) + a_1 \overset{-}{H}(\vec{i}) + a_2 \overset{-}{H}(\vec{j}) + a_3 \overset{-}{H}(\vec{k}) \\ &= a_0 I_4 + a_1 F_1 + a_2 F_2 + a_3 F_3.\end{aligned}$$

Using the definitions of $\overset{+}{H}$ and $\overset{-}{H}$, the multiplication of the two generalized quaternions q and p is given by

$$qp = \overset{+}{H}(q)p = \overset{-}{H}(p)q.$$

Theorem 1: If q and p are two generalized quaternions, λ is a real number and $\overset{+}{H}$ and $\overset{-}{H}$ are operators as defined in equations (1) and (2), respectively, then the following identities hold:

i. $q = p \Leftrightarrow \overset{+}{H}(q) = \overset{+}{H}(p) \Leftrightarrow \overset{-}{H}(q) = \overset{-}{H}(p).$

ii. $\overset{+}{H}(q+p) = \overset{+}{H}(q) + \overset{+}{H}(p), \quad \overset{-}{H}(q+p) = \overset{-}{H}(q) + \overset{-}{H}(p).$

iii. $\overset{+}{H}(\lambda q) = \lambda \overset{+}{H}(q), \quad \overset{-}{H}(\lambda q) = \lambda \overset{-}{H}(q).$

iv. $\overset{+}{H}(qp) = \overset{+}{H}(q) \overset{+}{H}(p), \quad \overset{-}{H}(qp) = \overset{-}{H}(p) + \overset{-}{H}(q).$

v. $\overset{+}{H}(q^{-1}) = \left[\overset{+}{H}(q)\right]^{-1}, \quad \overset{-}{H}(q^{-1}) = \left[\overset{-}{H}(q)\right]^{-1}, \quad (N_q)^2 \neq 0.$

vi. $\overset{+}{H}(\bar{q}) = \left[\overset{+}{H}(q)\right]^{T}, \quad \overset{-}{H}(\bar{q}) = \left[\overset{-}{H}(q)\right]^{T}.$

vii. $\det\left[\overset{+}{H}(q)\right] = (N_q)^2, \quad \det\left[\overset{-}{H}(q)\right] = (N_q)^2.$

viii. $tr\left[\overset{+}{H}(q)\right] = 4a_0, \quad tr\left[\overset{-}{H}(q)\right] = 4a_0.$

Proof: Identities $(i), (ii)$ and (iii) can be proved easily. Using the associative property of the quaternions multiplication, it is clear that following identities hold:

$$(qp)r \quad = q(pr) = qpr$$

In terms of operator $\overset{+}{H}$, the above identities can be written as

$$\begin{aligned}\overset{+}{H}(qp)r &= \overset{+}{H}(\overset{+}{H}(q)p)r \\ &= \overset{+}{H}(q)(\overset{+}{H}(p)r) = \overset{+}{H}(q) \overset{+}{H}(p)r.\end{aligned}$$

and similarly,

$$\begin{aligned}\overset{-}{H}(qp)r &= \overset{-}{H}(\overset{-}{H}(q)p)r \\ &= \overset{-}{H}(p)(\overset{-}{H}(q)r) = \overset{-}{H}(p)\,\overset{-}{H}(q)r.\end{aligned}$$

Since r is arbitrary, the above relation employs equation (iv). Using the inverse property, we have

$$qq^{-1} = q^{-1}q = I_4$$

and in terms of operator $\overset{+}{H}$, the above identities can be written as

$$\begin{aligned}\overset{+}{H}(qq^{-1}) &= \overset{+}{H}(q)\,\overset{+}{H}(q^{-1}) = \overset{+}{H}(I_4) = I_4, \\ \overset{-}{H}(q^{-1}q) &= \overset{-}{H}(q^{-1})\,\overset{-}{H}(q) = \overset{-}{H}(I_4) = I_4,\end{aligned}$$

therefore, the above relation employs equation(v). Identities (vi), (vii) and $(viii)$ can be proved easily.

Theorem 2: The map φ defined as

$$\varphi : (H_{\alpha\beta}, +, .) \to (M_{(4,R)}, \oplus, \otimes)$$

$$\varphi(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \mapsto \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}$$

is an isomorphism of algebras.

Proof: See Ward (1997) for a similar proof.

Theorem 3: Let q be a unit generalized quaternion. Matrices generated by operators $\overset{+}{H}$ and $\overset{-}{H}$ are quasi-orthogonal matrices, *i.e.*

i) $\left[\overset{+}{H}(q)\right]^T \varepsilon\, \overset{+}{H}(q) = \varepsilon,$

ii) $\left[\overset{-}{H}(q)\right]^T \varepsilon\, \overset{-}{H}(q) = \varepsilon, \varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}.$

Corollary 1: Let $q = \cos\theta + \vec{u}\sin\theta$ be a unit generalized quaternion. Then the generalized Hamilton operators $\overset{+}{h_q}$ and $\overset{-}{h_q}$ represent rotations of x in $E^4_{\alpha\beta}$.

The angle of rotation (using $\overset{+}{h_q}$) is easily determined. This is the angle ω between x and qx:

$$\begin{aligned}\cos\omega &= \frac{S(x(\overline{qx}))}{\sqrt{N_x}\sqrt{N_{qx}}} \\ &= \frac{S(x(\overline{xq}))}{N_x\sqrt{N_q}} = \frac{S(q)}{\sqrt{N_q}} = S(q) = \cos\theta.\end{aligned}$$

Therefore that the angle of rotation ω is the angle of q.

Example 1: Let $q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, 0)$ be a unit generalized quaternion and $\alpha, \beta > 0$. The matrix corresponding to this quaternion is

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{\alpha} & -\sqrt{\beta} & 0 \\ \frac{1}{\sqrt{\alpha}} & \sqrt{2} & 0 & \sqrt{\beta} \\ \frac{1}{\sqrt{\beta}} & 0 & \sqrt{2} & -\sqrt{\alpha} \\ 0 & -\frac{1}{\sqrt{\beta}} & \frac{1}{\sqrt{\alpha}} & \sqrt{2} \end{bmatrix}.$$

A is a quasi-orthogonal matrix and therefore it represents a rotation in 4-space $E^4_{\alpha\beta}$.

CONCLUSION

Rotations in four dimensions can be represented by 4th order orthogonal matrices, as a generalization of the rotation matrix. Quaternions can also be generalized into four dimensions, as even multivectors of the four dimensional geometric algebra. In this paper, we showed how the generalized quaternions can be used to described the rotation in 4-dimensional space $E^4_{\alpha\beta}$. In the next work, we will introduce the quaternion rotation operator in 3-space $E^3_{\alpha\beta}$.

APPENDIX

Matrices E_n and F_n, $(n = 1, 2, 3)$ defined in equations (3) and (4) are as follows;

$$E_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & \beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad F_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & \beta & 0 \\ 0 & -\alpha & 0 & \alpha \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Following the definition of quaternionic unit and operators $\overset{+}{H}$ and $\overset{-}{H}$ it can be proved that

$$\begin{aligned} E_1E_1 &= -\alpha I_4; \quad E_2E_2 = -\beta I_4; \quad E_3E_3 = -\alpha\beta I_4 \\ &E_1E_2 = E_3 = -E_2E_1 \\ &E_2E_3 = \beta E_1 = -E_3E_2 \\ &E_3E_1 = \alpha E_3 = -E_1E_3 \end{aligned}$$

and

$$\begin{aligned} F_1F_1 &= -\alpha I_4; \quad F_2F_2 = -\beta I_4; \quad F_3F_3 = -\alpha\beta I_4 \\ &F_1F_2 = -F_3 = -F_2F_1 \\ &F_2F_3 = -\beta F_1 = -F_3F_2 \\ &F_3F_1 = -\alpha F_3 = -F_1F_3, \end{aligned}$$

In addition, we have

$$E_1^T \varepsilon E_1 = \alpha\varepsilon, \quad E_2^T \varepsilon E_2 = \beta\varepsilon, \quad E_3^T \varepsilon E_3 = \alpha\beta$$

and

$$F_1^T \varepsilon F_1 = \alpha\varepsilon, \quad F_2^T \varepsilon F_2 = \beta\varepsilon, \quad F_3^T \varepsilon F_3 = \alpha\beta\varepsilon.$$

Not that the properties of E_n and $F_n\,(n = 1, 2, 3)$ are identical to those of quaternionic units $\vec{i}, \vec{j}, \vec{k}$. Further, The theorems presented above for operators $\overset{+}{H}$ and $\overset{-}{H}$ equally apply for matrices E_n and $F_n \quad (n = 1, 2, 3)$, since E_n and F_n are special cases of matrices generated by $\overset{+}{H}$ and $\overset{-}{H}$.

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تدوير في أربعة عبر مؤثرات هاملتون المعممة

مهدي جعفري و يوسف يايلي
قسم الرياضيات - كلية العلوم - جامعةأنقرة - تركيا

خلاصة

في هذا البحث، بعد مراجعة مختصرة لبعض الخواص الجبرية للمِربَاعات (Quaternions)، قمنا بالبحث في خواص مؤثرات هاملتون المعممة ونظرنا في الكيفية التي وفقها يمكن للمرباعات المعممة أن تستخدم لوصف التدوير في فضاء ذي أربعة أبعاد $E^4_{\alpha\beta}$.

المجلــة التربــوية

مجلة فصلية، تخصصية، محكّمة

تصدر عن مجلس النشر العلمي - جامعة الكويت

رئيس التحرير: أ. د. **عبدالله محمد الشيخ**

تنشــر:

- **البحــوث التربــوية المحكمة**
- **مراجـعـات الكتب التربوية الحديثة**
- **محاضــر الحوار التربوي**
- **التقارير عــن المؤتمـرات التربــوية وملخصــات الرســائل الجــامعية**

- **تقبل البحوث باللغتين العربية والإنجليزية.**
- **تنشر لأساتذة التربية والمختصين بها من مختلف الأقطار العربية والدول الأجنبية.**

الاشتراكات:

في الكويـــــت: ثلاثة دنانير للأفـراد، وخمسة عشر ديناراً للمؤسسات.

في الدول العربيــة: أربعـة دنـانير للأفراد، وخمسة عشر ديناراً للمؤسسات.

في الدول الأجنبية: خمسة عشـر دولاراً للأفراد، وستون دولاراً للمؤسسات.

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