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Adaptive multi-step differential transformation method to solve ODE systems

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ABSTRACT

In this paper, it is given a fast algorithm to solve chaotic differential systems using the multi-step differential transforms method (MsDTM). The approach is applied to a number of chaotic nonlinear differential equations and numerical results are given. Performance analyses reveal that the proposed approach is an efficiency tool to solve using fewer time step to the considered equation systems.

Keywords: The differential transform method; adaptive; Runge-Kutta method; chaotic systems.

INTRODUCTION

The differential transform method (DTM) is a semi-analytical-numerical methodfor solving integral equations, ordinary, partial differential equations and differential equation systems. The method enables the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou (1986) and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method generates, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in Jang *et al.* (1997); Chen & Liu (1998); Ho & Chen (1998); Jordan & Smith (1999); Chen & Ho (1999a, 1999b); Jang *et al.* (2000a, 2000b, 2001); Köksal & Herdem (2002); Merdan *et al.* (2011a); Abdel-Halim Hassan (2002); Yildirim *et al.* (2012); Ayaz (2004); Arikoğlu & Özkol (2005); Merdan *et al.* (2011b); Ertürk & Momani (2007); Chang & Chang (2008); Ravi & Aruna (2008); Momani & Ertürk (2008);

El-Shahed (2008); Ravi & Aruna (2009a,2009b); Ebaid (2010); Merdan & Gökdoğan (2011).

However, DTM has some drawbacks. By using the DTM, we obtain a series solution, actually a truncated series solution. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. To overcome the shortcoming, MsDTM was presented in Odibat *et. al.* (2010); Gökdoğan *et al.* (2012a, 2012b).On the other hand, MsDTM has also some drawbacks. By using the DTM, the interval $[0,\ T]$ is divided into M sub-interval and the series solutions is obtained in $t \in [t_i, t_{i+1}],\ i = 0, 1, ..., M-1$. In some problems, MsDTM can be required a very small sub-division of intervals. In this case, both the solution time lengthens and series solutions are obtained for a great number of sub-intervals.

In this study, we present the fast and effective algorithm to eliminate the above-mentioned disadvantages of classical MsDTM (Gökdoğan *et al.*, 2012a). The speed and efficiency of the provided algorithm were evaluated on the chaotic differential equations.

DIFFERENTIAL TRANSFORM METHOD

The basic definitions of differential transform are introduced as follows. Let $x(t)$ be analytic in a domain D and let $t = t_0$ represent any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_0. The differential transform of the k th derivative of a function $x(t)$is defined as follows:

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \forall t \in D \tag{1}$$

In (1), $x(t)$is the original function and $X(k)$ is the transformed function. The differential inverse transformation of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} X(k)(t - t_i)^k,\ \forall t \in D \tag{2}$$

From (1) and (2), we obtain

$$x(t) = \sum_{k=0}^{\infty} \frac{(t - t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0},\ \forall t \in D \tag{3}$$

The fundamental theorems of the one-dimensional differential transform are:

Theorem 1. If $z(t)=x(t)\pm y(t)$, then $Z(k)=X(k)\pm Y(k)$.

Theorem 2. If $z(t)=\mathrm{c}y(t)$, then $Z(k)=\mathrm{c}\,Y(k)$.

Theorem 3. If $z(t)=\dfrac{dy(t)}{dt}$, then $Z(k)=(\mathrm{k}+1)\,Y(k+1)$.

Theorem 4. If $z(t)=\dfrac{d^{n}y(t)}{dt}$, then $Z(k)=\dfrac{(\mathrm{k}+\mathrm{n})!}{k!}\,Y(k)$.

Theorem 5. If $z(t)=x(t)y(t)$, then $Z(k)=\sum_{k_1=0}^{k}X(k_1)\,Y(k-k_1)$.

Theorem 6. If $z(t)=t^{n}$, then $Z(k)=\delta(k-n)=\begin{cases}1 & k=n\\ 0 & k\neq n\end{cases}$.

In real applications, the function $x(t)$is expressed by a finite series and (3) can be written as

$$x(t)=\sum_{k=0}^{N}X(k)(t-t_i)^{k},\ \forall t\in D \tag{4}$$

Equation (4) implies that$\sum_{k=N+1}^{\infty}X(k)(t-t_i)^{k}$is negligibly small.To illustrate the differential transform method (DTM) for solving differential equations systems, Consider a general system of first-order ODES

$$\begin{aligned}&\frac{dx_1}{dt}+h_1(t,x_1,x_2,...,x_m)=g_1(t),\\ &\frac{dx_2}{dt}+h_2(t,x_1,x_2,...,x_m)=g_2(t),\\ &\vdots\\ &\frac{dx_m}{dt}+h_m(t,x_1,x_2,...,x_m)=g_m(t),\end{aligned} \tag{5}$$

subject to the initial conditions

$$x_1(t_0)=d_1,\qquad x_2(t_0)=d_2,\cdots,x_m(t_0)=d_m. \tag{6}$$

According to DTM, by taking differential transformed both sides of the systems of equations given Eq. (5) and (6) is transformed as follows:

$$\begin{aligned}&(k+1)X_1(k+1)+H_1(k)=G_1(k),\\ &(k+1)X_2(k+1)+H_2(k)=G_2(k),\\ &\vdots\\ &(k+1)X_m(k+1)+H_m(k)=G_m(k).\end{aligned} \tag{7}$$

$$X_1(0) = d_1, \qquad X_2(0) = d_2, \cdots, X_m(0) = d_m. \tag{8}$$

Therefore, according to DTM the $N-$term approximations for (5) can be expressed as

$$\begin{aligned} &\varphi_{1,n}(t) = x_1(t) = \textstyle\sum_{k=1}^{N} X_1(k)t^k, \\ &\varphi_{2,n}(t) = x_2(t) = \textstyle\sum_{k=1}^{N} X_2(k)t^k, \\ &\vdots \\ &\varphi_{m,n}(t) = x_m(t) = \textstyle\sum_{k=1}^{N} X_m(k)t^k. \end{aligned} \tag{9}$$

MULTI-STEP DIFFERENTIAL TRANSFORM METHOD

The approximate solutions (3) are generally, as will be shown in the numerical experiments of this paper, not valid for large t. A simple way of ensuring validity of the approximations for large t is to treat (9) as an algorithm for approximating the solutions of (4) in a sequence of intervals choosing the initial approximations as

$$\begin{aligned} &x_{1,0}(t) = x_1(t^*) = d_1^*, \\ &x_{2,0}(t) = x_2(t^*) = d_2^*, \\ &\vdots \\ &x_{m,0}(t) = x_m(t^*) = d_m^*. \end{aligned} \tag{10}$$

In order to carry out the iterations in every subinterval $[0, t_1),\ [t_1,\ t_2),\ [t_2, t_3), ..., [t_{j-1}, t)$ of equal length h, we would need to know the values of the following (Odibat *et al.*, 2010),

$$x_{1,0}^*(t) = x_1(t^*), \quad x_{2,0}^*(t) = x_2(t^*), \quad ...., x_{m,0}^*(t) = x_m(t^*). \tag{11}$$

But, in general, we do not have these information at our clearance except at the initial point $t^* = t_0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\varphi_{1,n}, \varphi_{2,n}, ..., \varphi_{m,n}$ of the preceding subinterval, i.e.,

$$x_{1,0}^* \cong \varphi_{1,n}(t^*),\ x_{2,0}^* \cong \varphi_{2,n}(t^*),\ ..., x_{m,0}^* \cong \varphi_{m,n}(t^*). \tag{12}$$

THE ALGORITHM

By sub-dividing interval $[0, T]$ into equilength M numbered sub-intervals, MsDTM method finds DTM solutions of these sub-ranges. But, it is necessary to select the step length of the small to solve some differential equations and this requires finding DTM solutions in more sub-intervals. Hence, the solution time prolongs in large values of T. A new approach is needed to overcome this problem. For this, we recommend a fast algorithm (Gökdoğan *et al.,* 2012a).

We apply the DTM to system (5) over the interval$[0, t_1]$, we will obtain the following approximate solution,

$$\begin{aligned}
&\varphi_{1,1}(t) = x_1(t) = \sum_{k=1}^{n} X_1(k)t^k, \ t \in [0, t_1] \\
&\varphi_{2,1}(t) = x_2(t) = \sum_{k=1}^{n} X_2(k)t^k, \ t \in [0, t_1] \\
&\vdots \\
&\varphi_{m,1}(t) = x_m(t) = \sum_{k=1}^{n} X_m(k)t^k, \ t \in [0, t_1]
\end{aligned} \tag{13}$$

using the initial conditions $x_k(0) = d_k, \ k = 1, 2, ..., m$.For $i \geq 2$, at each subinterval $[t_{i-1}, t_i]$, if $max_{i=1,...,m}\left|\phi_{i,j-1}^{N}(t_{i-1}) - \phi_{i,j-1}^{N-1}(t_{i-1})\right| \geq \varepsilon$, we will use the initial conditions $x_k(t_{i-1}) = x_{k-1}(t_{i-1}), \ k = 1, 2, ..., m$ and apply the DTM to Eq. (5) over the interval $[t_{i-1}, t_i]$, where t_0 in Eq. (1) is replaced by t_{i-1}. If $\underset{i=1,...,m}{max}\left|\phi_{i,j-1}^{N}(t_{i-1}) - \phi_{i,j-1}^{N-1}(t_{i-1})\right| < \varepsilon$, we get $x_k(t) = x_{k-1}(t), \ k = 1, 2, ..., m,$ not apply the DTM to Eq. (5) over the this interval. The process is repeated and generates a sequence of approximate solutions $\phi_{k,i}(t), \ i = 1, 2, ..., K \leq M, \ k = 1, 2, ..., m$ for the solution $x_k(t), \ k = 1, 2, ..., m$.

NUMERICAL RESULTS

In this section, some chaotic systems are solved by using the proposed approach and MsDTM. The performance analyses are given for both methods. For performance analyses, only the operations up to the finding step of $x_m(t)$'s are taken, the elapsed time for graphic drawings are not included in the analysis. Analyses are carried out by MAPLE 7 software in a PC with a Pentium 1.6 GHz and 512 MB of RAM.

Example 1

The Genesio system, proposed by Genesio (Olek 1994) and Tesi (Genesio &

Tesi, 1992), is one of paradigms of chaos since it captures many features of chaotic systems. The dynamic equations of the system is given by

$$\begin{cases} x' = y \\ y' = z \\ z' = az + by + cx + dx^2 \end{cases} \tag{14}$$

when $a = -1.2, b = -2.92, c = -6, d = 1$, by taking with the initial conditions

$$x(0) = 0.2,\ y(0) = -0.3, z(0) = 0.1 \tag{15}$$

We will apply classic DTM and the adaptive MsDTM to nonlinear ordinary differential Eq. (14)-(15). Applying classic DTM for Eq.(14)- (15)

$$\begin{aligned} X(k+1) &= Y(k)k + 1 \\ Y(k+1) &= Z(k)k + 1 \\ Z(k+1) &= \frac{\left(aZ(k) + bY(k) + cX(k) + d\sum_{k_1=0}^{k} X(k_1)X(k-k_1)\right)}{k+1} \\ X(0) &= 0.2,\ Y(0) = -0.3, Z(0) = 0.1 \end{aligned} \tag{16}$$

By applying the multi-step DTM to (14)- (15), $X_i(n), Y_i(n)$ and $Z_i(n)$satisfy the following recurrence relations for $n = 0, 1, 2...N-1$.

$$\begin{aligned} X_i(k+1) &= \frac{Y_i(k)}{k+1} \\ Y_i(k+1) &= \frac{Z_i(k)}{k+1} \\ Z_i(k+1) &= \frac{\left(aZ_i(k) + bY_i(k) + cX_i(k) + d\sum_{k_1=0}^{k} X_i(k_1)X_i(k-k_1)\right)}{k+1} \\ X_0(0) &= 0.2,\ X_i(0) = x_{i-1}(t_i), Y_0(0) = -0.3, Y_i(0) = y_{i-1}(t_i), Z_0(0) \\ &= 0.1, Z_i(0) = z_{i-1}(t_i),\ i = 1, 2, ..., K \le M \end{aligned} \tag{17}$$

The solutions obtained from the proposed approach are given below.The performance analyses are obtained by MsDTM and the approach summarized in Table 1. It is showed that the proposed approach is very fast and effective.

Table 1. Comparison processing time and time-step for $t \in [0, 20]$

5-term MsDTM($h = 0.1$)		**5-term the proposed approach ($h = 0.1$)**		
processing time	**time-step**		**processing time**	**time-step**
0.530s	200	$\varepsilon = 0.01$	0.290s	53
		$\varepsilon = 0.001$	0.370s	98
		$\varepsilon = 0.0001$	0.580s	149
MsDTM($h = 0.01$)		**5-term the proposed approach ($h = 0.01$)**		
Proessing time	**time-step**		**processing time**	**time-step**
5.038s	2000	$\varepsilon = 0.01$	0.761s	47
		$\varepsilon = 0.001$	0.831s	75
		$\varepsilon = 0.0001$	0.951s	121
MsDTM($h = 0.001$)		**5-term the proposed approach ($h = 0.001$)**		
processing time	**time-step**		**processing time**	**time-step**
274.334s	20000	$\varepsilon = 0.01$	4.517s	46
		$\varepsilon = 0.001$	4.667s	73
		$\varepsilon = 0.0001$	6.119s	117

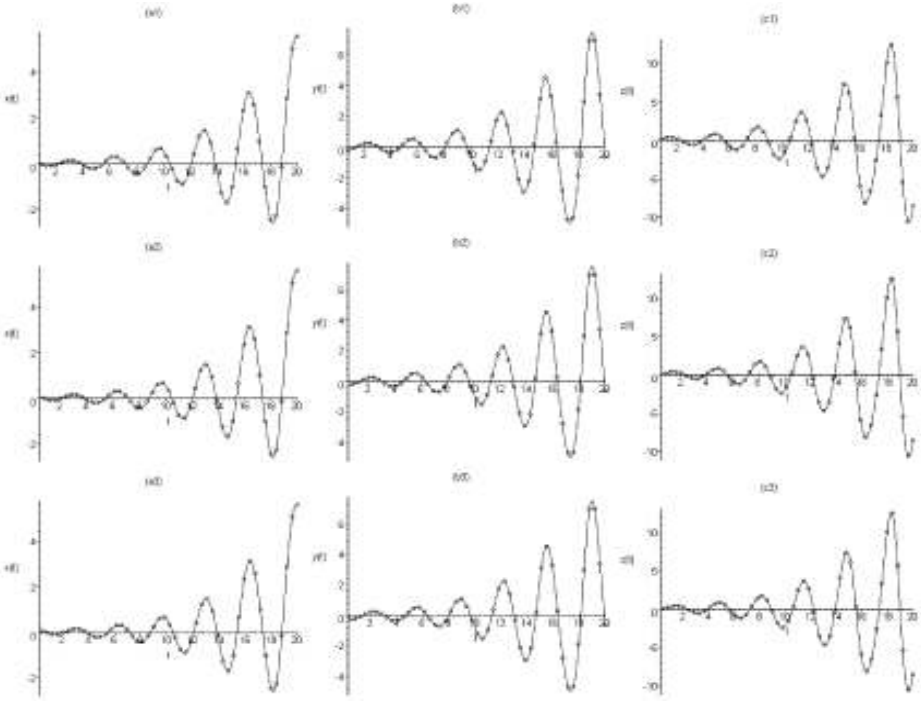

Fig. 1. Comparison 5-term MsDTM with $h = 0.1$ and $RK4$ with $h = 0.001$ (a1) $x(t)$, (b1) $y(t)$, (c1) $z(t)$ when $\varepsilon = 0.01$, (a2) $x(t)$, (b2) $y(t)$, (c2)$z(t)$ when $\varepsilon = 0.001$ and (a3) $x(t)$, (b3) $y(t)$, (c3)$z(t)$ when $\varepsilon = 0.0001$

Figure 1 show the approximate solutions for chaotic Genesio system obtained using the proposed approach with $h = 0.1$ and RK4 method with $h = 0.001$. We can observe that local changes obtained using the approach are in high adaptation with RK4 method for small value of ε. At the same time, we present the absolute errors between the 5-term MsDTM (h = 0.1) solutions and the RK4(h = 0.001) solutionin Table 2. It is showed that results obtained for small values of ε are compatible with RK4 even at large time step.

Table 2. Differences between 5-term MsDTM with $h = 0.1$ and RK4 with $h = 0.001$ for different ε values in $t \in [0, 20]$

	$\Delta = \lvert \boldsymbol{MsDTM_{0.1} - RK4_{0.001}} \rvert$ **and** $\varepsilon = \mathbf{0.01}$			$\Delta = \lvert \boldsymbol{MsDTM_{0.1} - RK4_{0.001}} \rvert$ **and** $\varepsilon = \mathbf{0.001}$			$\Delta = \lvert \boldsymbol{MsDTM_{0.1} - RK4_{0.001}} \rvert$ **and** $\varepsilon = \mathbf{0.0001}$		
t	Δx	Δy	Δz	Δx	Δy	Δz	Δx	Δy	Δz
2	1.783e-03	3.201e-03	4.375e-03	2.528e-04	1.803e-04	8.008e-04	1.687e-05	2.155e-06	5.926e-05
4	6.136e-03	9.659e-04	2.151e-02	5.348e-04	3.965e-04	2.089e-03	2.436e-05	3.681e-05	1.033e-04
6	9.748e-03	1.345e-02	4.124e-02	4.297e-04	1.771e-03	2.304e-03	7.440e-06	1.224e-04	7.986e-05
8	4.148e-03	4.055e-02	3.675e-02	3.834e-04	3.497e-03	2.619e-04	5.435e-05	2.027e-04	1.098e-04
10	1.551e-02	6.128e-02	3.231e-02	2.350e-03	3.861e-03	6.403e-03	1.858e-04	1.635e-04	5.418e-04
12	5.412e-02	6.020e-02	1.768e-01	4.707e-03	1.291e-03	1.791e-02	2.835e-04	8.540e-05	1.183e-03
14	7.782e-02	2.191e-03	2.752e-01	6.366e-03	4.817e-03	2.330e-02	3.764e-04	4.683e-04	1.367e-03
16	9.427e-02	1.520e-01	2.834e-01	3.811e-03	1.823e-02	1.510e-02	6.664e-05	1.291e-03	5.866e-04
18	4.329e-02	2.406e-01	3.484e-01	3.318e-03	2.865e-02	3.047e-02	1.913e-04	2.043e-03	1.878e-03
20	9.535e-03	1.893e-01	2.367e-01	4.073e-03	2.370e-02	2.403e-02	4.558e-04	1.663e-03	1.363e-03

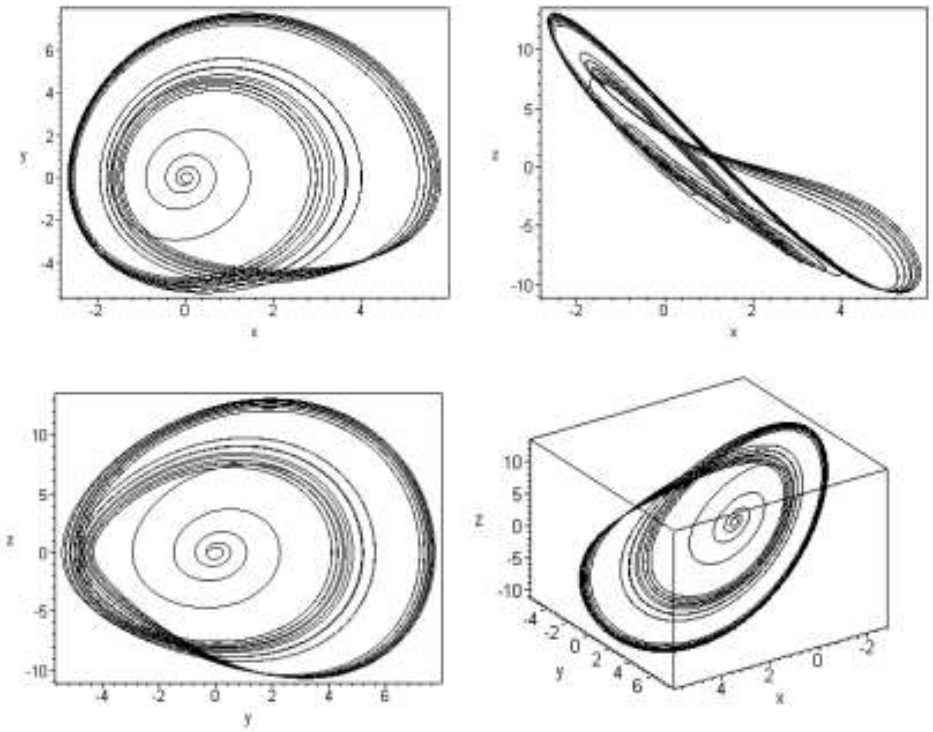

Fig. 2. Chaotic attractors for Genesio system with time span $[0, 100]$ using adaptive MsDTM

In Figure 2, we reproduce the well-known chaotic attractors of the Genesio system using 5-term the proposed approach solutions with $h = 0.01$.

Example 2

Consider chaotic Coullet system (Coullet *et al.* 1979; Arnodo *et al.,* 1981)

$$\begin{cases} x' = y \\ y' = z \\ z' = az + by + cx + dx^3 \end{cases} \tag{18}$$

when $a = 0.8, b = -1.1, c = -0.45, d = -1$, by taking with the initial conditions

$$x(0) = 0.1,\ \ y(0) = 0.41, z(0) = 0.31 \tag{19}$$

We will apply classic DTM and the adaptive MsDTM to nonlinear ordinary differential Eq. (18)-(19). Applying classic DTM for Eq.(18)-(19)

$$X(k+1) = \frac{Y(k)}{k+1}$$
$$Y(k+1) = \frac{Z(k)}{k+1}$$
$$Z(k+1) = \frac{\begin{pmatrix} aZ(k) + bY(k) + cX(k) + \\ d\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X(k_1)X(k_2-k_1)X(k-k_2) \end{pmatrix}}{k+1} \tag{20}$$

$$X(0) = 0.1,\ \ Y(0) = 0.41, Z(0) = 0.31$$

By applying the multi-step DTM toEq. (18)-(19), $X_i(n), Y_i(n)$ and $Z_i(n)$ satisfy the following recurrence relations for $n = 0, 1, 2...N-1$.

$$X_i(k+1) = \frac{Y_i(k)}{k+1}$$
$$Y_i(k+1) = \frac{Z_i(k)}{k+1}$$
$$Z_i(k+1) = \frac{\begin{pmatrix} aZ_i(k) + bY_i(k) + cX_i(k) \\ +d\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X_i(k_1)X_i(k_2-k_1)X_i(k-k_2) \end{pmatrix}}{k+1} \tag{21}$$

$$X_0(0) = 0.1,\ \ X_i(0) = x_{i-1}(t_i), Y_0(0) = 0.41, Y_i(0) = y_{i-1}(t_i), Z_0(0)$$

$$= 0.31, Z_i(0) = z_{i-1}(t_i),\ \ i = 1, 2, ..., K \leq M$$

The performance analyses are obtained by MsDTM and the approach summarized in Table 3. It is showed that the proposed approach is very fast and effective.

Table 3. Comparison processing time and time-step for $t \in [0, 20]$

5-term MsDTM(h = 0.1)		**5-term the proposed approach MsDTM(h = 0.1)**		
processing time	**time-step**		**processing time**	**time-step**
1.262s	200	$\varepsilon = 0.01$	0.491s	48
		$\varepsilon = 0.001$	1.002s	137
		$\varepsilon = 0.0001$	1.352s	199
MsDTM(h = 0.01)		**5-term the proposed approach (h = 0.01)**		
processing time	**time-step**		**processing time**	**time-step**
14.421s	2000	$\varepsilon = 0.01$	0.911s	43
		$\varepsilon = 0.001$	1.021s	102
		$\varepsilon = 0.0001$	1.943s	228
MsDTM(h = 0.001)		**5-term the proposed approach (h = 0.001)**		
processing time	**time-step**		**processing time**	**time-step**
614.233s	20000	$\varepsilon = 0.01$	2.974s	43
		$\varepsilon = 0.001$	3.244s	100
		$\varepsilon = 0.0001$	4.066s	216

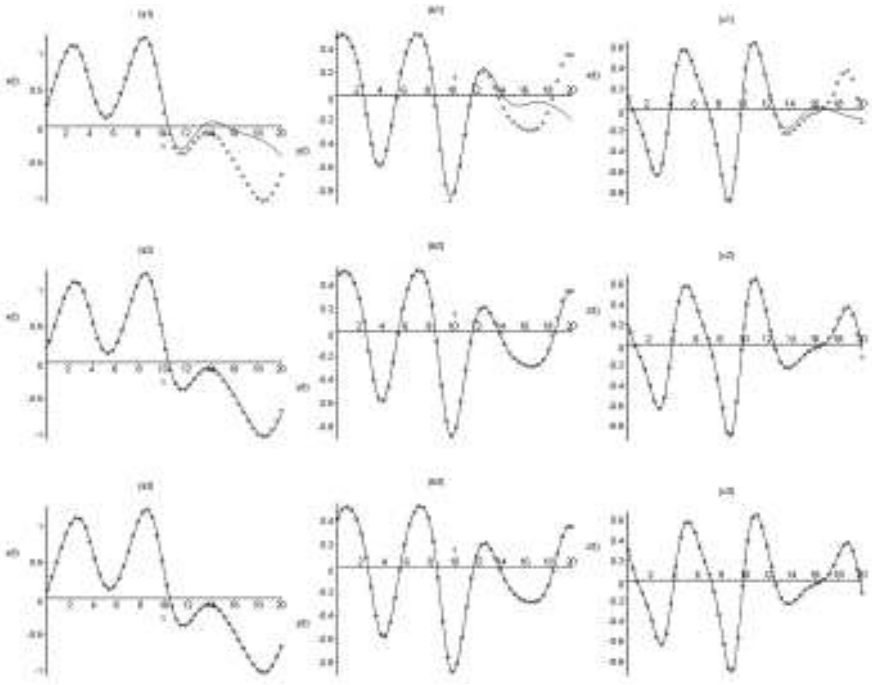

Fig. 3. Comparison MsDTM with $h = 0.1$ and $RK4$ with $h = 0.001$ (a1) $x(t)$, (b1) $y(t)$, (c1) $z(t)$ when $\varepsilon = 0.01$, (a2) $x(t)$, (b2) $y(t)$, (c2)$z(t)$ when $\varepsilon = 0.001$ and (a3) $x(t)$, (b3) $y(t)$, (c3)$z(t)$ when $\varepsilon = 0.0001$

Figure 3 show the approximate solutions for chaotic Coullet system obtained using adaptive MsDTM with $h = 0.1$ and RK4 method with $h = 0.001$. We can observe that local changes obtained using the approach are in adaptation with RK4 method for small value of ε.

Table 4. Differences between 5-term MsDTM with $h = 0.1$ and RK4 with $h = 0.001$ for different ε values in $t \in [0, 20]$

	$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.01$			$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.001$			$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.0001$		
t	Δx	Δy	Δz	Δx	Δy	Δz	Δx	Δy	Δz
2	2.776e-03	4.529e-03	3.159e-03	3.266e-04	2.717e-04	6.930e-04	4.255e-05	2.537e-05	9.442e-05
4	1.099e-02	9.492e-03	8.943e-03	8.381e-04	7.566e-04	6.223e-04	1.469e-04	9.007e-05	2.441e-04
6	1.658e-02	8.172e-03	5.696e-03	1.396e-03	3.834e-04	3.726e-04	1.844e-04	3.928e-05	6.347e-05
8	2.634e-02	1.693e-02	4.631e-02	1.677e-03	1.161e-03	2.747e-03	2.127e-04	1.406e-04	4.287e-04
10	6.142e-02	2.859e-02	5.362e-02	4.058e-03	1.879e-03	3.522e-03	5.604e-04	1.948e-04	6.672e-04
12	7.266e-02	7.440e-03	2.320e-02	4.782e-03	4.977e-04	1.269e-03	6.558e-05	2.993e-05	2.845e-04
14	1.504e-01	8.325e-02	5.004e-02	9.341e-03	5.031e-03	3.078e-03	2.163e-04	1.484e-04	3.850e-05
16	4.331e-01	2.053e-01	5.878e-02	2.581e-02	1.034e-02	5.794e-04	5.004e-04	1.863e-04	3.047e-05
18	8.356e-01	7.264e-02	2.990e-01	2.265e-02	2.801e-02	3.931e-02	4.521e-04	5.983e-04	8.671e-04
20	2.422e-01	5.472e-01	3.564e-02	5.059e-02	1.385e-03	7.466e-02	1.102e-03	2.455e-04	1.599e-03

we submit the absolute errors between the 5-term MsDTM (h = 0.1) solutions and the RK4(h = 0.001) solution in Table 4. It is showed that results obtained for small values of ε are compatible with RK4 even at large time step.

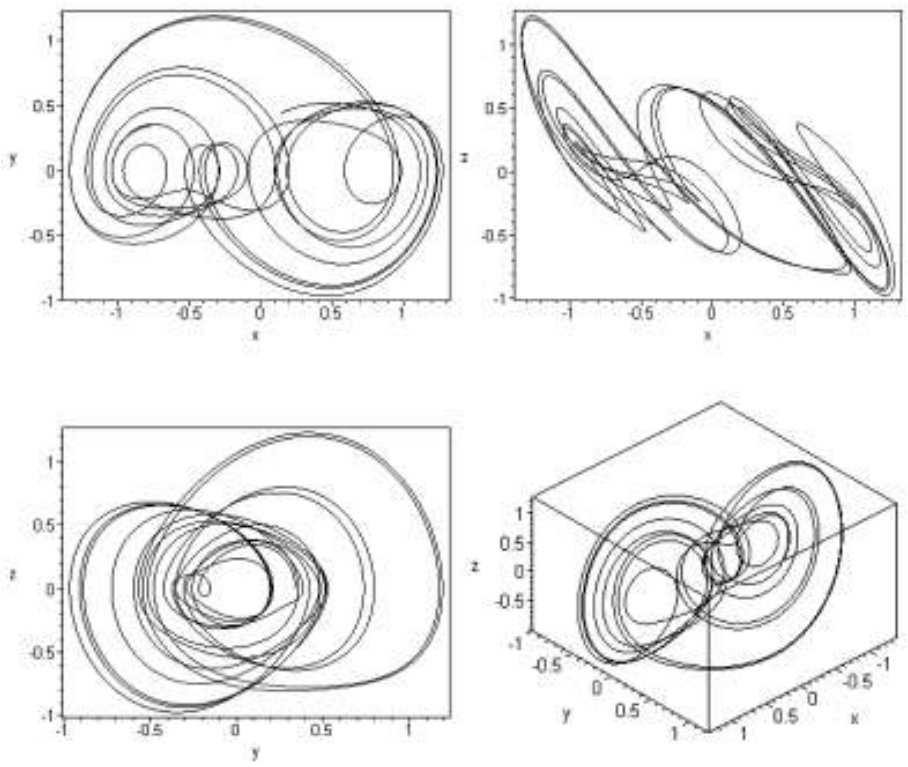

Fig. 4. Chaotic attractors for Coullet system with time span $[0, 100]$ using adaptive MsDTM

In Figure 4, we reproduce the well-known chaotic attractors of the Coullet system using 5-term the approach solutions with $h = 0.01$.

Example 3

Here, we take Lorenz system into consideration. In 1963, the Lorenz system is introduced by Edward Lorenz, (1963). The nonlinear differential equations that describe the Lorenz system are

$$\begin{cases} x' = \rho(y - x) \\ y' = Rx - y - xz \\ z' = xy + bz \end{cases} \tag{22}$$

when $\rho = 10, b = -8/3, R = 28$, by taking with the initial conditions

$$x(0) = -15.8,\ y(0) = -17.48, z(0) = 35.64 \tag{23}$$

We will apply classic DTM and the adaptive MsDTM to nonlinear ordinary

differential Eq. (22)-(23). Applying classic DTM for Eq. (22)-(23)

$$\begin{aligned} X(k+1) &= \frac{\rho(Y(k)-X(k))}{k+1} \\ Y(k+1) &= \frac{\left(RX(k)-Y(k)-\sum_{k_1=0}^{k} X(k_1)Z(k-k_1)\right)}{k+1} \\ Z(k+1) &= \frac{\left(bZ(k)+\sum_{k_1=0}^{k} X(k_1)Y(k-k_1)\right)}{k+1} \end{aligned} \tag{24}$$

$$X(0) = 15.8,\ \ Y(0) = -17.48, Z(0) = 35.64$$

By applying the multi-step DTM to Eq. (22)-(23), $X_i(n), Y_i(n)$ and $Z_i(n)$ satisfy the following recurrence relations for $n = 0, 1, 2...N-1$.

$$\begin{aligned} X_i(k+1) &= \frac{\rho(Y_i(k)-X_i(k))}{k+1} \\ Y_i(k+1) &= \frac{\left(RX_i(k)-Y_i(k)-\sum_{k_1=0}^{k} X_i(k_1)Z_i(k-k_1)\right)}{k+1} \\ Z_i(k+1) &= \frac{\left(bZ_i(k)+\sum_{k_1=0}^{k} X_i(k_1)Y_i(k-k_1)\right)}{k+1} \end{aligned} \tag{25}$$

$$X_0(0) = 15.8,\ \ X_i(0) = x_{i-1}(t_i), Y_0(0) = -17.48, Y_i(0) = y_{i-1}(t_i), Z_0(0)$$

$$= 35.64, Z_i(0) = z_{i-1}(t_i),\ \ i = 1, 2, ..., K \leq M$$

The performance analyses are obtained by MsDTM and the approach summarized in Table 5. It is showed that the proposed approach is very fast and effective.

Table 5. Comparison processing time and time-step for $t \in [0, 10]$

5-term MsDTM(h = 0.1)		**5-term the proposed approach MsDTM(h = 0.1)**		
processing time	**time-step**		**processing time**	**time-step**
0.641s	100	$\varepsilon = 0.01$	0.471s	98
		$\varepsilon = 0.001$	0.480s	100
		$\varepsilon = 0.0001$	0.480s	100
MsDTM($h = 0.01$)		**Adaptive MsDTM($h = 0.01$)**		
processing time	**time-step**		**processing time**	**time-step**
6.529s	1000	$\varepsilon = 0.01$	0.981s	100
		$\varepsilon = 0.001$	1.112s	127
		$\varepsilon = 0.0001$	1.282s	162
MsTM($h = 0.001$)		**Adaptive MsDTM($h = 0.001$)**		
processing time	**time-step**		**processing time**	**time-step**
315.785s	10000	$\varepsilon = 0.01$	5.778s	94
		$\varepsilon = 0.001$	5.888s	120
		$\varepsilon = 0.0001$	6.159s	150

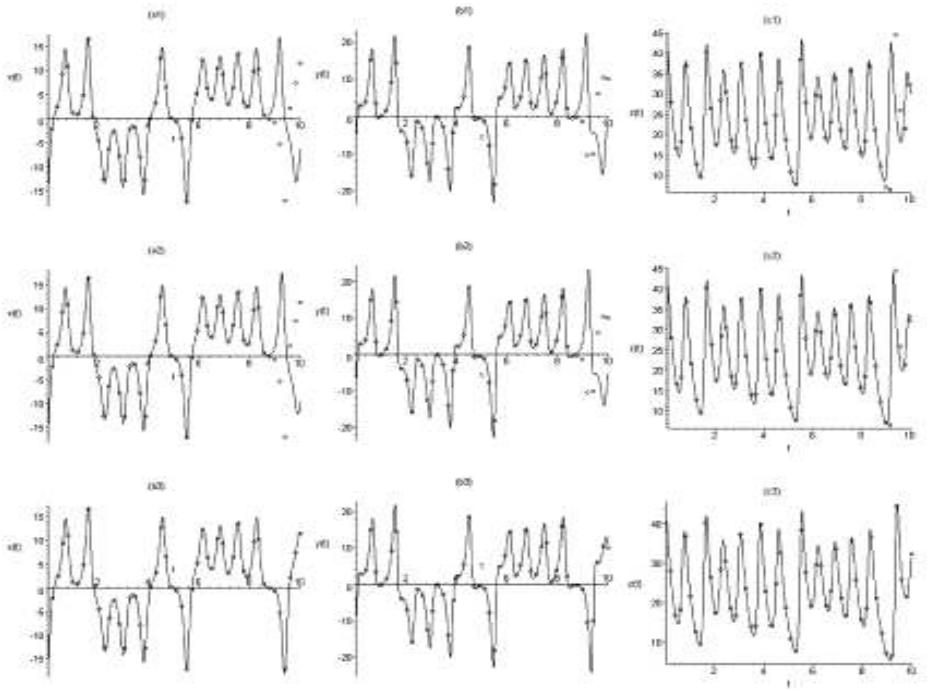

Fig. 5. Comparison 10-term MsDTM with $h = 0.1$ and *RK*4 with $h = 0.001$ (a1) $x(t)$, (b1) $y(t)$, (c1) $z(t)$ when $\varepsilon = 0.01$, (a2) $x(t)$, (b2) $y(t)$, (c2)$z(t)$ when $\varepsilon = 0.001$ and (a3) $x(t)$, (b3) $y(t)$, (c3)$z(t)$ when $\varepsilon = 0.0001$

Figure 5 show the approximate solutions for chaoticLorenz system obtained using the approach with $h = 0.1$ and RK4 method with $\mathrm{h} = 0.001$. We can observe that local changes obtained using the approach are in high adaptation with RK4 method for small value of ε.

Table 6. Differences between 10-term MsDTM with $h = 0.01$ and RK4 with $h = 0.001$ for different ε values in $t \in [0, 10]$

	$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.01$			$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.001$			$\Delta = \lvert \boldsymbol{MsDTM}_{0.1} - \boldsymbol{RK4}_{0.001} \rvert$ **and** $\varepsilon = 0.0001$		
t	Δx	Δy	Δz	Δx	Δy	Δz	Δx	Δy	Δz
2	7.490e-04	1.758e-03	4.903e-03	7.820e-05	3.861e-04	1.072e-04	3.350e-07	9.623e-06	1.529e-05
4	4.153e-04	2.449e-03	1.098e-02	4.234e-04	7.867e-04	2.777e-03	2.031e-05	1.308e-05	5.638e-05
6	2.128e-03	4.930e-03	9.280e-03	9.715e-04	3.165e-03	8.144e-04	2.646e-05	3.327e-04	2.834e-04
8	1.515e-03	5.008e-03	8.055e-03	7.427e-03	3.554e-03	1.501e-02	5.309e-04	2.024e-04	1.023e-03
10	2.150e-02	3.501e-02	9.312e-03	1.113e-02	1.817e-02	2.074e-03	6.323e-04	1.024e-03	3.757e-05
12	1.472e-01	2.477e-01	2.164e-02	8.109e-02	1.327e-01	2.128e-02	4.511e-03	7.378e-03	1.257e-03
14	1.446e-01	1.590e-01	1.055e-01	7.958e-02	7.959e-02	6.547e-02	4.358e-03	4.408e-03	3.521e-03
16	3.952e-01	6.251e-01	2.349e-01	1.882e-01	2.996e-01	1.117e-01	1.071e-02	1.713e-02	6.000e-03
18	6.305e+00	1.204e+01	2.465e+00	2.971e+00	5.794e+00	2.391e-01	1.611e-01	3.140e-01	2.897e-02
20	1.770e+01	1.188e+01	2.340e+00	2.106e+01	1.542e+01	7.081e-01	4.305e-02	8.495e-01	7.908e-01

we present the absolute errors between the 10-term MsDTM ($\mathrm{h} = 0.1$) solutions and the RK4($\mathrm{h} = 0.001$) solution in Table 6. It is showed that results obtained for small values of ε are compatible with RK4.

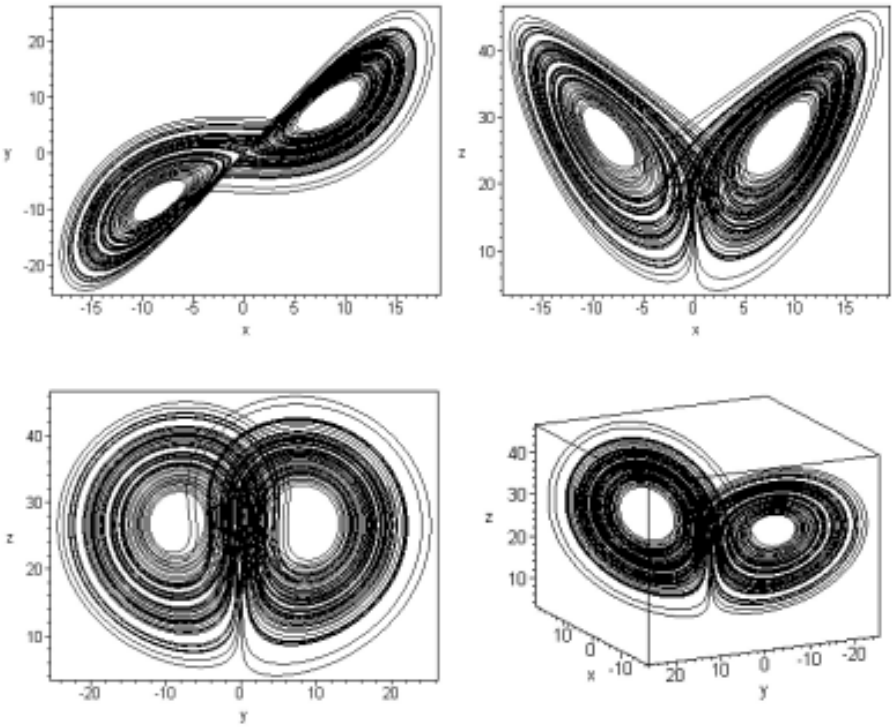

Fig. 6. Chaotic attractors for Lorenz system with time span $[0, 100]$ using adaptive MsDTM

In Figure 6, we reproduce the well-known chaotic attractors of the Genesio system using 5-term the approach solutions with $h = 0.01$.

CONCLUSION

In this work, we suggested a new efficient and fast algorithm for the MsDTM, a reliable modification of the DTM that develops the convergence of the series solution. Comparisons between the proposed method solutions and RK4's numerical solutions were made. The method provides immediate and visible symbolic terms of analytic solutions. The validity of the proposed method has been successfully demonstrated by applying it for the well-known chaotic systems (Genesio, Coullet, Lorenz). The results show that adaptive MsDTM provides the user remarkable performance in terms of processing time and time step.

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طريقة تحويل تفاضلي تكيفي متعدد الخطوات لحل أنظمة المعادلات التفاضلية الاعتيادية

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خلاصة

نقدم في هذا البحث خوارزمية سريعة لحل الأنظمة التفاضلية الشواشية وذلك باستخدام طريقة تحويل تفضالي تكيفي متعدد الخطوات. ونطبق هذه الخوارزمية على عدد من المعادلات التفاضلية الشواشية وغير الخطية فنحصل على نتائج عددية. ويكشف تحليل الأداء أن طريقتنا المقترحة ذات فعالية وتحتاج إلى وقت وخطوات أقل لحل الأنظمة موضوع الدراسة.

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