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On the curvatures of spacelike circular surfaces

Rashad A. Abdel-Baky[1,2], Yasin Ünlütürk[3,*]

[1]*Present address: Dept. of Mathematics, Sciences Faculty for Girls, King Abdulaziz University, P.O. Box 126300, Jeddah 21352, Saudi Arabia*

[2]*Dept. of Mathematics, Faculty of Science, University of Assiut, Assiut 71516 , Egypt*

[3]*Dept. of Mathematics, Faculty of Art and Science, Kırklareli University, 39100, Kırklareli, Turkey*

**Corresponding author: yasinunluturk@klu.edu.tr*

Abstract

In this paper a complete system of invariants is presented to study spacelike circular surfaces with fixed radius. The study of spacelike circular surfaces is simplified to the study of two curves: the Lorentzian spherical indicatrix of the unit normals of circle planes and the spacelike spine curve. Then the geometric meanings of these invariants are used to give corresponding properties of spacelike circular surfaces with classical ruled surfaces. Later, we introduce spacelike roller coaster surfaces as a special class of spacelike circular surfaces.

Keywords: Ruled surface; singularities; spacelike canal surface; spacelike circular surface; spacelike roller coaster surface.

1. Introduction

At the beginning of the twentieth century, Einstein's theory constructed a bridge between mathematical physics and modern differential geometry. It has been observed that Lorentz-Minkowski geometry is an attractive area for geometers who can penetrate surprisingly quickly into cosmology (redshift, expanding universe and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light and black holes) (O'Neill, 1983; McNerty, 1980; Blum, 1980).

Despite its long history, the theory of surface is still one of the most important interesting topics in differential geometry and it is still studied by many mathematicians until now, see for example (Izumiya *et al*., 2007; Izumiya & Takeuchi, 2001; Izumiya, 2003a; Xu *et al*., 2006). Among the surfaces, a circular surface with constant radius has been drawing attention to scientists as well as mathematicians because of its various applications such as networks of blood vessels and neurons in medicine, or tube and hose systems in industrial environments.

In this paper we study geometric properties and singularities of circular surfaces with constant radius in Minkowski 3-space E_1^3. In section 3, we introduce spacelike circular surfaces and give their Gaussian and mean curvatures. Also we show that the parameter curves on spacelike circular surface are lines of curvature. Then we study local singularities of a spacelike circular surface. In section 4, we introduce spacelike roller coaster surfaces as a special class of spacelike circular surfaces. Also, we give some examples of spacelike roller coaster surface by using computer aided graphics.

2. Basic concepts

Let E_1^3 be the three-dimensional Minkowski space, that is, the three-dimensional real vector space E^3 with the metric

$$< dx, dx >= dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in E^3. An arbitrary vector x of E_1^3 is said to be spacelike if $< x, x >> 0$ or x = 0, timelike if $< x, x >< 0$ and lightlike or null if $< x, x >= 0$ and $x \neq 0$. A timelike or light-like vector in E_1^3 is said to be causal. For $x \in {E_1}^3$ the norm is defined by $\|x\| = \sqrt{| < x, x > |}$), then the vector x is called a spacelike unit vector if $< x, x >= 1$ and a timelike unit vector if $< x, x >= -1$. Similarly, a regular curve in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively (O'Neill, 1983; McNerty, 1980; Blum, 1980). For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of E_1^3, the inner product is the real number $< x, y >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by

$$x \times y = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1)).$$

Definition 1. The angle between two vectors in Minkowski space is defined as follows:

i) *Spacelike angle:* Let x and y be spacelike vectors in E_1^3 that span a spacelike vector subspace; then we have

$$| < x, y > | \leq \|x\|\|y\|,$$

and hence, there is a unique real number $\theta \geq 0$ such that

$$< x, y > = \|x\|\|y\| cos\theta.$$

This number is called the spacelike angle between the vectors x and y.

ii) *Central angle:* Let x and y be spacelike vectors in E_1^3 that span a timelike vector subspace; then we have

$$| < x, y > | > \|x\|\|y\|,$$

and hence, there is a unique real number $\theta \geq 0$ such that

$$< x, y >= \|x\|\|y\| cosh\theta.$$

This number is called the central angle between the vectors x and y.

iii) *Lorentzian timelike angle:* Let x be spacelike vector and y be timelike vector in E_1^3. Then there is a unique real number $\theta \geq 0$ such that

$$< x, y >= \|x\|\|y\| sinh\theta.$$

This number is called the Lorentzian timelike angle between the vectors x and y (Ratcliffe, 2006).

On the other hand, the hyperbolic and Lorentzian unit spheres are

$$H_+^2(1) = \{x \in E_1^3 \mid x_1^2 + x_2^2 - x_3^2 = -1, x_3 \geq 1\},$$

is called the hyperbolic unit sphere in $E_1^3$and

$$S_1^2(1) = \{x \in E_1^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\}$$

is called the hyperbolic unit sphere in E_1^3 (O'Neill, 1983).

A surface in the Minkowski 3-space E_1^3 is called a spacelike surface, if the induced metric on the surface is a positive definite Riemannian metric. This is equivalent to saying that the normal vector on the spacelike surface is a timelike vector (O'Neill, 1983).

3. Spacelike circular surfaces

In this section, we introduce the notion of spacelike circular surface in E_1^3.

A spacelike circular surface with a constant radius is swept out by moving a spacelike circle with its center following a curve, which acts as the spine curve. Each circle is called a generating circle, which lies on a plane called circle plane. A non null curve $C = C(u)$ is called the spine curve, whose tangent vector is $t = C'(u)$ such that $\|C'\| \neq 0$ for every $u \in I$, and a positive number $r > 0$.

Let e_1 denote the timelike unit normal vector of a circle plane. The vector e_1 is attached to each point of the spine curve C for a generating spacelike circle with the radius r. Hence, a spacelike circular surface is determined by both C and e_1.

If u is the arc-length of the spherical curve $e_1(u) \in H_+^2$, then the unit tangent vector of $e_1(u)$ is $e_2 = e_1'(u)$. We define a vector $e_3 = e_1 \times e_2$, then we have

$$\begin{gathered} -< e_1, e_1 >=< e_2, e_2 >=< e_3, e_3 >= 1, \\ < e_1, e_2 >=< e_2, e_3 >=< e_3, e_1 >= 0, \qquad (1) \\ e_1 \times e_2 = e_3, e_3 \times e_1 = e_2, e_2 \times e_3 = -e_1. \end{gathered}$$

Excluding $e_1(u)$ is constant or null or $e_1'(u)$ null. This parametrization gives rise to exploring the kinematic geometry and relevant geometric characteristics.

Depending on the causal character of the curve $e_1(u)$, we have the following Frenet formulae

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \qquad (2)$$

where γ is called the geodesic curvature of the curve $e_1(u)$ on H_+^2 (Walrave, 1995). The derivative with respect to u will be denoted by a dash over functional symbol.

In the parametrized representation, the tangent vector C' can be expressed by

$$C' = \alpha e_1 + \sigma e_2 + \eta e_3, \qquad (3)$$

where $\alpha = \alpha(u), \sigma = \sigma(u)$ and $\eta = \eta(u)$ are the coordinate functions of the tangent vector C'. Let e_2 and e_3 form a pseudo-orthonormal basis of the corresponding circle plane at each point of the spine curve $C(u)$. Then, by means of the solutions of the differential system (2), the spacelike circular surface M is described by

$$M: P(u,\theta) = C(u) + r(cos\theta e_2(u) + sin\theta e_3(u)), 0 \leq \theta \leq 2\pi. \tag{4}$$

The four functions $\gamma(u), \alpha(u), \sigma(u)$ and $\eta(u)$ constitute a complete system of curvature functions of the surface M. The standard circles

$$\theta \to C(u) + r(cos\theta e_2(u) + sin\theta e_3(u))$$

are called generating circles.

Differentiating the vector equation of the circular surface (4) by the parameters u, θ, respectively, we get the tangent vectors of the parametric curves:

$$P_\theta = r(-sin\theta e_2 + cos\theta e_3),$$

$$P_u = (rcos\theta + \alpha)e_1 + \gamma P_\theta + \sigma e_2 + \eta e_3. \tag{5}$$

Then, the coefficients E, F, and G of the first fundamental form with respect to (u, θ) are given by:

$$E = -(rcos\theta + \alpha)^2 + r^2\gamma^2 + \sigma^2 + \eta^2 + 2r\gamma(\eta cos\theta - \sigma sin\theta),$$

$$F = r(r\gamma - \sigma sin\theta + \eta cos\theta), G = r^2, \tag{6}$$

and thus the timelike unit normal vector of the surface for regular points is given by:

$$N(u,\theta) = \frac{P_u \times P_\theta}{\|P_u \times P_\theta\|} = \frac{-(\sigma cos\theta + \eta sin\theta)e_1 - (rcos\theta + \alpha)(cos\theta e_2 + sin\theta e_3)}{\sqrt{(rcos\theta+\alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2}}. \tag{7}$$

By a straightforward calculation, we get:

$$P_{\theta\theta} = -r(cos\theta e_2 + sin\theta e_3),$$

$$P_{u\theta} = -r(sin\theta e_1 + \gamma cos\theta e_2 + \gamma sin\theta e_3), \tag{8}$$

$$P_{uu} = (\alpha' - \sigma)e^1 + (\sigma' + rcos\theta + \alpha)e^2 + (\eta' + \sigma\gamma)e_3 + \gamma P_{\theta u}.$$

This leads to the coefficients e, f, and g of the second fundamental form with respect to (u, θ) as in the following:

$$e = \frac{1}{\sqrt{(rcos\theta+\alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2}} \cdot \{(rcos\theta+\alpha)[r\gamma - (\eta' + \sigma\gamma)sin\theta - (\alpha + rcos\theta + \sigma')cos\theta] + (\alpha' - \sigma - r\gamma sin\theta)(\eta sin + \sigma cos\theta)\}$$

$$f = \frac{r[-(\sigma cos\theta + \eta sin\theta)sin\theta + \gamma(rcos\theta + \alpha)]}{\sqrt{(rcos\theta+\alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2}} \tag{9}$$

$$g = \frac{r(rcos\theta+\alpha)}{\sqrt{(rcos\theta+\alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2}}.$$

Hence, for the regular points, the Gaussian curvature K and the mean curvature H are given by

$$\begin{aligned} K = \{ & ((\gamma^2 - \gamma)cos^2\theta + cos^4\theta)r^4 \\ & + 2\alpha cos^3\theta^2 cos\theta + \eta' sin\theta cos\theta \\ & + \sigma' cos^3\theta + \alpha cos^3\theta \\ & + \alpha\sigma' cos^2\theta + \eta\gamma sin^2\theta cos\theta)r^3 \\ & + (3\sigma\alpha\gamma sin\theta cos\theta + 3\alpha^2 cos^2\theta \\ & + \alpha\eta' sin\theta cos\theta + \alpha'\sigma cos^2\theta \\ & + \alpha\sigma' cos^2\theta + \alpha^2\gamma^2 - \alpha^2\gamma \\ & + 2\sigma\eta sin^3\theta cos\theta + \sigma\eta sin\theta cos\theta \\ & + \sigma^2 sin^2\theta cos^2\theta + \eta^2 sin^4\theta \\ & + \sigma^2 cos^3\theta + \alpha^2 cos\theta + \alpha\eta\gamma sin^2\theta)r^2 \\ & + (-2\gamma\sigma cos^2\theta - 2\eta\gamma sin\theta cos\theta \\ & + \alpha^2\eta' sin\theta + \alpha^2\sigma\gamma sin\theta + \alpha^3 cos\theta \\ & + \alpha^2\sigma' cos\theta - \alpha\eta\alpha' sin\theta - \alpha\sigma\alpha' cos\theta \\ & + \alpha\sigma\eta sin\theta + \alpha\sigma^2 cos\theta)r \\ & + (-2\alpha\sigma\gamma cos\theta - 2\alpha\gamma\eta sin\theta)\}/[(rcos\theta + \alpha)^2 \\ & + (\sigma cos\theta + \eta sin\theta)^2][(r\sigma cos\theta + r\eta sin\theta)^2 - (r^2 cos\theta + r\alpha)^2], \end{aligned} \tag{10}$$

and

$$\begin{aligned} H = \{ & (cos^3\theta - cos\theta + 2\gamma^2 cos\theta - \gamma cos\theta)r^4 \\ & + (-2\gamma\sigma sin\theta cos\theta + 2\eta\gamma + \eta\gamma sin^2\theta \\ & \pm 2\alpha cos^2\theta + 2\alpha\gamma^2 - \alpha + \eta' sin\theta cos\theta \\ & + \sigma' cos^2\theta - \alpha\gamma)r^3 + (-\alpha\sigma\gamma sin\theta \\ & + 2\sigma^2 sin^2\theta cos\theta + 2\eta^2 sin^2\theta cos\theta \\ & - 2\sigma\eta sin\theta cos^2\theta - 2\sigma\eta sin^3\theta \\ & + 2\alpha\gamma\eta cos\theta + \alpha\eta' sin\theta + \alpha^2 cos\theta \\ & + \alpha\sigma' cos\theta - \alpha'\eta sin\theta - \alpha'\sigma cos\theta \\ & + \sigma\eta sin\theta + \sigma^2 cos\theta)r^2\}/ \\ & 2\sqrt{(rcos\theta + \alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2} \\ & .[(r\sigma cos\theta + r\eta sin\theta)^2 - (r^2 cos\theta + r\alpha)^2], \end{aligned} \tag{11}$$

respectively.

3.1. Striction curves

In the Euclidean space, striction curves are defined as generating circles located on the normal planes of the spine curve $C(u)$ (Izumiya, 2003b; Kasap & Akyıldız, 2006; Walrave, 1995). Then, for the spacelike circular surface M, the position vector

$$\zeta(u) = C(u) + r(cos\theta e_2(u) + sin\theta e_3(u)), \qquad (12)$$

is the striction curve if $\zeta(u)$ satisfies

$$< \zeta', cos\theta e^2 + sin\theta e^3 >= 0, \forall\, u \in I \subseteq \mathbb{R},$$

and this is equivalent to

$$< e_2, C' >= 0 =< e_3, C' > \Leftrightarrow \sigma = \eta = 0. \qquad (13)$$

In this case, any curves on the surface, which transverse to generating circles satisfy the condition of striction curves. So these curves construct a surface which is called a spacelike canal (tube) surface. Therefore the class of spacelike canal surfaces is analogous to the class of cylindrical ruled surfaces. Since the lines of curvatures are geometric features, it is interesting to construct spacelike canal surface whose parametric curves are lines of curvature. Therefore, from Equations (6), (9) and (12), it follows that the parametric curves are lines of curvature if and only if

$$F = f = 0 \Leftrightarrow \gamma = 0.$$

If we substitute this into the equation (2), we obtain

$$\begin{bmatrix} e_1{}' \\ e_2{}' \\ e_3{}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \qquad (14)$$

The curve that satisfies the condition found in the equation (14) is a great circle on H_+^2. For example, a great circle can be expressed as $e_1(u) = (sinh\, u, 0, cosh\, u)$. The normal vector can be found from $e_2 = e_1{}'$ as

$$e_2(u) = (cosh\, u, 0, sinh\, u).$$

Thus e_3 has the form

$e_3(u) = (0,1,0)$. In this case, with the aid of equation (3), $C(u)$ can be expressed as

$$C(u) = \int_0^u \alpha(u)(sinh\, u, 0, cosh\, u)du.$$

Thus, we obtain the following spacelike canal surfaces family

$$P(u,\theta,\alpha(u),r) = (\int_0^u \alpha(u) \sinh u du, 0, \int_0^u \alpha(u) cosh\, u du)$$
$$+ r(cos\theta cosh\, u, sin\theta, cos\theta sinh\, u).$$

Figures 1(a), 1(b), 1(c) show members in the family with $\alpha(u) = u^2, u, 1$ and $r = 1$.

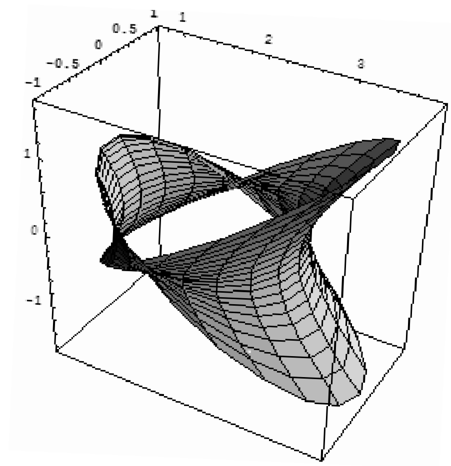

Fig. 1(a). Spacelike canal surface of $P(u,\theta,u^2,1)$

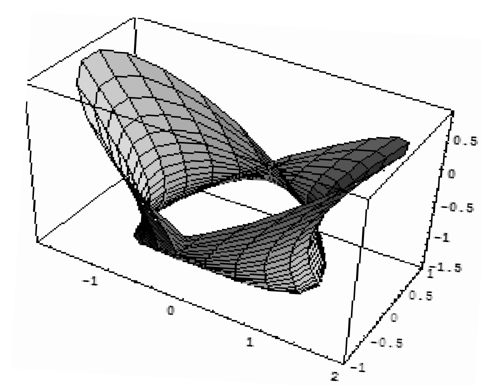

Fig. 1(b). Spacelike canal surface of $P(u,\theta,u,1)$

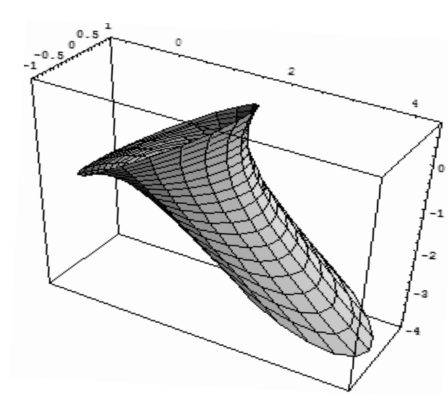

Fig. 1(c). Spacelike canal surface of $P(u,\theta,1,1)$

We also define the notion of non-canal spacelike circular surfaces analogous to that of non-cylindrical ruled surfaces:

Definition 2. A spacelike circular surface M is called a non-canal surface at $u \in \mathbb{R}$ if e_2, e_3 and C satisfy

$$\sigma = < e_2, C' > \neq 0, or\ \eta = < e_3, C' > \neq 0. \quad (15)$$

We call M a non-canal if it is non-canal at any $u \in \mathbb{R}$.

3.2. Local singularities

Singularities are essential for understanding the properties of circular surfaces and are investigated in the following: From equations (6), we can show that M has a singularity at $P(u, \theta)$ if and only if

$$\|P_u \times P_\theta\| = ((rcos\theta + \alpha)^2 + (\sigma cos\theta + \eta sin\theta)^2)^{\frac{1}{2}} = 0. \quad (16)$$

This is equivalent to the following equations

$$\sigma cos\theta + \eta sin\theta = 0, rcos\theta + \alpha = 0. \quad (17)$$

It can be seen that there are two main possible cases, as presented in the following:

Case (1) exists when $rcos\theta + \alpha = 0$. If $\sigma \neq 0$ and $\eta \neq 0$, there are two singular points occuring at

$\theta = sin^{-1}(\alpha\sigma/r\eta)$, and

$\theta = \pi + sin^{-1}(\alpha\sigma/r\eta)$.

If $\sigma \neq 0$ (resp. $\eta \neq 0$) and $\eta = 0$ (resp. $\sigma = 0$), the spacelike circular surface is a non-canal and the singular points are on the generating circles occurring at $\theta = \pm cos^{-1}(\alpha/r)$.

Case (2) exists when $\theta = -tan^{-1}(\sigma/\eta)$ and $\alpha \neq 0$. If $|\alpha| > r$, there are no singular points. If $\alpha = \pm r$, the singular points occur at $\theta = \pm\pi/2$. If $|\alpha| < r$ there are singular points on the generating circle occurring at $\theta = \pm cos^{-1}(\alpha/r)$.

For a spacelike canal circular surface to have singular points, it is necessary that $rcos\theta + \alpha = 0$ and $\sigma = \eta = 0$.

From equation (3), it follows that $C' = \alpha e_1$. The tangent vectors C' are always parallel to e^1, which is normal to the circle plane. In other words, the tangent vectors of the spine curve are always perpendicular to the circle planes. Therefore, we have the following cases:there are no singular points, if $|\alpha| > r$. The singular points occur at $\theta = \pm\pi/2$, if $\alpha = \pm r$. The singular points occur at $\theta = \pm cos^{-1}\left(\frac{\alpha}{r}\right)$, if $|\alpha| < r$.

3.3. Lines of curvature

A ruled surface is developable, if and only if its Gaussian curvature K vanishes at any regular point; this is equivalent to saying that all of rulings are lines of curvature except at umbilical points or singular points. In that case, from equations (6) and (9) it can be seen that all generating circles are lines of curvature if and only if N_θ is a multiple of P_θ. This is equivalent to

$$2r\|P_u \times P_\theta\|^2(r\eta + \alpha\eta cos\theta - \alpha\sigma sin\theta) = 0. \quad (18)$$

We will now investigate the condition (18) by assuming that it is satisfied for all u, in detail. According to the assumption of M being regular, the equation (18) is equivalent to

$$r\eta + \alpha\eta cos\theta - \alpha\sigma sin\theta = 0, \quad (19)$$

for all values of θ. It can be seen that there are three cases that equation (19) can be satisfied for all values of θ:

Case (1) if $\eta = \sigma = \alpha = 0$, then the tangent vector of the spine curve is *0*, that is, the spine curve is a fixed point. This means that the spacelike circular surfaces are hyperbolic spheres with radius r, namely, $M = \{P \in E_1^3 \mid \|P - C\|^2 = r^2\}$.

Case (2) if $\sigma = \eta = 0$, then tangent vector C' is parallel to e_1, which is normal to the circle plane. The circular surface becomes spacelike canal circular surface.

Case (3) if $\alpha = \eta = 0$, then tangent vector C' is parallel to e_2. Hence, the tangent vector of the spine curve lies on the circle plane at each point of M, namely, $C' = \sigma e_2$. When σ is constant, it follows that

$$C = C_0 + \sigma e_2, \quad (20)$$

where C_0 is a constant vector. From Equations (4) and (20) it can be found that

$$\|P - C_0\|^2 = \sigma^2 + r^2. \quad (21)$$

This means that all the circle points lie on a hyperbolic sphere of radius $\sqrt{\sigma^2 + r^2}$ with C_0 being its center point in E_1^3.

A spacelike circular surface with a non-constant σ and $\alpha = \eta = 0$ is the circular surface with spacelike spin curve $C(u)$, i.e. $< C', C' >> 0$.

In analogous with (Izumiya & Takeuchi, 2001), we call this surface as a spacelike roller coaster surface.

According to the above analysis, we obtain the following result;

Theorem 1.Besides the general spacelike circular surfaces there are two families of spacelike circular surfaces whose generating circles are lines of curvature in the Minkowski

3-space E_1^3. These two families are the spacelike roller coaster surfaces and the hyperbolic spheres with radius being larger than that of the generating circles.

4. Spacelike roller coaster surfaces

In this section we give some properties of the spacelike roller coaster surface. First, let s be arc-length parameter of the spacelike spine curve $C(u)$ and $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is the moving Frenet frame along $C(u)$, such that $\boldsymbol{t}(s), \boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. Then, we have

$$\boldsymbol{t}(s) = \frac{C'}{\|C'\|} = \boldsymbol{e}_1, \boldsymbol{n}(s) = \frac{\frac{dt}{ds}}{\left\|\frac{dt}{ds}\right\|} = \frac{\boldsymbol{e}^1 + \gamma \boldsymbol{e}^3}{\sqrt{-1+\gamma^2}},$$

$$\boldsymbol{b}(s) = \frac{\gamma e_1 + e_3}{\sqrt{-1+\gamma^2}}. \tag{22}$$

Thereby, the following relations exist:

$$\begin{bmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ sinh\varphi & 0 & cosh\varphi \\ cosh\varphi & 0 & sinh\varphi \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1(s) \\ \boldsymbol{e}_2(s) \\ \boldsymbol{e}_3(s) \end{bmatrix}, \tag{23}$$

where

$$cosh\varphi = \frac{\gamma}{\sqrt{-1+\gamma^2}}, sinh\varphi = \frac{1}{\sqrt{-1+\gamma^2}}, \gamma > 1,$$

$$\boldsymbol{t}(s) \times \boldsymbol{n}(s) = -\boldsymbol{b}(s), \boldsymbol{b}(s) \times \boldsymbol{t}(s) = \boldsymbol{n}(s),$$

$$\boldsymbol{n}(s) \times \boldsymbol{b}(s) = \boldsymbol{t}(s). \tag{24}$$

By differentiating equation (23) with respect to s and using the inverse transformation, we obtain

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} \tag{25}$$

where κ(s) and τ(s) are the natural curvature and torsion of the spine curve $C = C(s)$, respectively;

$$\kappa(s) = \frac{\sqrt{-1+\gamma^2}}{\sigma}, \tau(s) + \frac{d\varphi}{ds} = 0, \frac{d\varphi}{ds} = \frac{d\gamma/du}{\sigma(-1+\gamma^2)}. \tag{26}$$

From equation (26) it follows that if φ (resp. γ) is a constant, the torsion of the spine curve vanishes. Thus the spine curve is a spacelike planar one.

On the spacelike roller coaster surface, the singularities can be obtained as

$$\|P_u \times P_\theta\| = cos\theta\sqrt{r^2 + \sigma^2} = 0. \tag{27}$$

From this it follows that singular points occur at $\theta = \pm\pi/2$, since $r \neq 0$ and $\sigma \neq 0$. Hence there are two singular points on every generating circle. Connecting these two sets of singular points gives two striction curves that contain all the singular points of a spacelike roller coaster surface. From equation (11), it follows that the expression of these two curves is

$$\zeta^1(u) = C(u) + r\boldsymbol{e}^3(u),\ \zeta_2(u) = C(u) - r\boldsymbol{e}_3(u). \tag{28}$$

Proposition 1. Any spacelike roller coaster surface has exactly two striction curves and intersections with each generating circle which are antipodal points each other. The curvatures κ_i and torsions τ_i $(i = 1, 2)$ of the striction curves can be obtained

$$\kappa_1 = \frac{\kappa}{\sqrt{1 - r\kappa cosh\varphi}}, \tau_1 = \frac{\tau}{\sqrt{1 - r\kappa cosh\varphi}},$$

$$\kappa_2 = \frac{\kappa}{\sqrt{1 + r\kappa cosh\varphi}}, \tau_2 = \frac{\tau}{\sqrt{1 + r\kappa cosh\varphi}}. \tag{29}$$

Moreover, by substituting $\alpha = \eta = 0$ into equation (10), we have

$$K = -\frac{1}{r^2 - \sigma^2} + \frac{r\sigma'}{(r^2-\sigma^2)^2 cos\theta}. \tag{30}$$

Since every generating circle is a line of curvature for a spacelike roller coaster surface, the value of one principal curvature is

$$\lambda_1 = \frac{1}{r}. \tag{31}$$

The other principal curvature is given by

$$\lambda_2 = \frac{K}{\lambda_1} = \frac{-(r^2-\sigma^2)cos\theta + r\sigma\prime}{r(r^2-\sigma^2)^2 cos\theta}. \tag{32}$$

The results can be summarized in terms of the following theorem:

Theorem 2. If a family of spacelike roller coaster surfaces has the same radius r, the curvature function σ and its derivative σ', then the values of their Gaussian curvature and the principal curvatures are the same at the corresponding point. Furthermore, these values are independent of the geodesic curvature of the spherical indicatrix of the vector e_1.

In terms of the Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$, the parametric expression of the spacelike roller coaster surfaces can be given as follows:

$$M: P(s, \theta; r) = C(s) + rcos\theta\boldsymbol{t}(s) +$$

$$rsin\theta(cosh\varphi\boldsymbol{n}(s) - sinh\varphi\boldsymbol{b}(s)), \tag{33}$$

the partial differentiations of M with respect to s and θ are as follows

$$P_\theta = -rsin\theta t + r(cosh\varphi\boldsymbol{n} - sinh\varphi\boldsymbol{b})cos\theta$$

$$P_s = (1 - r\kappa sin\theta cosh\varphi)\boldsymbol{t} + r\kappa cos\theta\boldsymbol{n}. \quad (34)$$

Hence, the unit normal vector field N at any point is:

$$N(s,\theta) = \frac{1}{\sqrt{r^2\kappa^2 sinh^2\varphi - 1}}\{-r\kappa cos\theta sinh\varphi\boldsymbol{t}$$

$$+(1 - r\kappa sin\theta cosh\varphi)sinh\varphi\boldsymbol{n} \quad (35)$$

$$+\varphi + r\kappa sin\theta sinh^2\varphi)\boldsymbol{b}\}$$

therefore, we find the elements of the first and second fundamental forms, respectively, as follows:

$$E = (1 - r\kappa sin\theta cosh\varphi)^2 + r^2\kappa^2 cos^2\theta,$$

$$F = r(r\kappa cosh\varphi - sin\theta), G = r^2, \quad (36)$$

And

$$e = \frac{1}{\sqrt{r^2\kappa^2 sinh^2\varphi - 1}}\{-\kappa sinh\varphi$$

$$+r(\kappa^2 sin\theta sinh2\varphi - \tau\kappa cos\theta cosh\varphi) -$$

$$\dot{\kappa} cos\theta sinh\varphi - r^2\kappa^3(cos^2\varphi sinh^2\varphi)sinh\varphi\}$$

$$f = \frac{r\kappa\,(r\kappa\, cos\theta - sin\theta)sinh\varphi}{\sqrt{r^2\kappa^2 sinh^2\varphi - 1}}, \quad (37)$$

$$g = \frac{-r^2\kappa sinh\varphi}{\sqrt{r^2\kappa^2 sinh^2\varphi - 1}}.$$

Then, the Gaussian curvature K and the mean curvature H, respectively, are given by

$$K = \frac{1}{(r^2\kappa^2 sinh^2\varphi - 1)^2 cos\theta}\{\kappa[r^2\kappa^3 sinh^3\varphi cos\theta$$

$$+\kappa(sinh\varphi cos\theta + r\tau cosh\varphi) + r\dot{\kappa} sinh\varphi]sinh\varphi\} \quad (38)$$

and

$$H = -\frac{1}{2(r^2\kappa^2 sinh^2\varphi - 1)^2 cos\theta}\{2r^2\kappa^3 sinh^3\varphi cos\theta$$

$$+\kappa(2sinh\varphi cos\theta + r\tau cosh\varphi) + r\dot{\kappa} sinh\varphi\} \quad (39)$$

where $\dot{k} = \frac{dk}{ds}$. It is known that a regular surface is a flat surface if and only if its Gaussian curvature vanishes identically. Therefore, we immediately derive from (38) that

$$k[r^2k^3 sinh^3\varphi cos\theta + k(sinh\varphi cos\theta$$

$$+r\tau cosh\varphi) + r\dot{k} sinh\varphi]sinh\varphi = 0.$$

Thus in a neighborhood of any point on M with $k \neq 0$, the vanishing of the coefficients of r^0, r, and r^2 implies

$$\dot{k} sinh\varphi = 0, \tau cosh\varphi = 0,$$

and

$$sinh^3\phi cos\theta = 0.$$

This is equivalent to that the torsion τ is an identically zero function and $sinh\varphi = 0$.

A spacelike roller coaster surface whose Gaussian curvature vanishes identically is a part of spacelike plane. In the same manner,we get that M is a minimal flat surface.

We now give some examples of spacelike roller coaster surfaces. They also serve to verify the correctness of the formulae derived above.

Example 1.Given the spacelike circular helix as follows:

$$\alpha(s) = (asinh\tfrac{s}{c}, b\tfrac{s}{c}, acosh\tfrac{s}{c}),$$

$$a > 0, b \neq 0, a^2 + b^2 = c^2, -2 \leq s \leq 2.$$

It is easy to show that

$$\boldsymbol{t}(s) = \left(\tfrac{a}{c}cosh\tfrac{s}{c}, \tfrac{b}{c}, \tfrac{a}{c}sinh\tfrac{s}{c}\right),$$

$$\boldsymbol{n}(s) = \left(sinh\tfrac{s}{c}, 0, cosh\tfrac{s}{c}\right),$$

$$\boldsymbol{b}(s) = (\tfrac{b}{c}cosh\tfrac{s}{c}, -\tfrac{a}{c}, \tfrac{b}{c}sinh\tfrac{s}{c}),$$

and $\tau = \frac{b}{c^2}$, then $\varphi(s) = -\frac{b}{c^2}s$.

Thus, the spacelike roller coaster surfaces family is expressed as

$$P(s,\theta;r) = (asinh\tfrac{s}{c}, b\tfrac{s}{c}, acosh\tfrac{s}{c}) + r(cos\theta, sin\theta cosh\varphi,$$

$$-sin\theta sinh\varphi)\begin{bmatrix} \frac{a}{c}cosh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}sinh\frac{s}{c} \\ sinh\frac{s}{c} & 0 & cosh\frac{s}{c} \\ \frac{b}{c}cosh\frac{s}{c} & -\frac{a}{c} & \frac{b}{c}sinh\frac{s}{c} \end{bmatrix}$$

so, if we choose $\theta \in [0,2\pi]$, $a = 2$, $b = 1$, then for $r = 1$ the corresponding spacelike roller coaster surface is shown in Fig. 2(a).

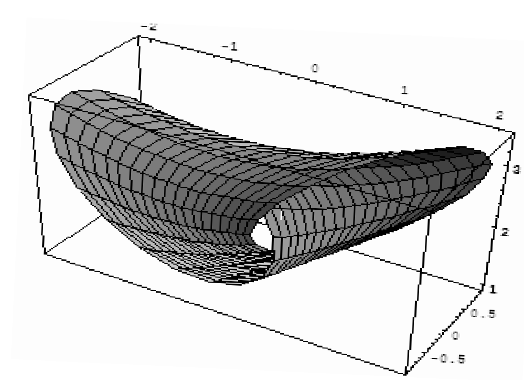

Fig. 2(a). Spacelike roller coaster surface for $\theta \in [0,2\pi], a = 2, b = 1, r = 1.$

Example 2. Suppose we are given a parametric spacelike curve

$$\alpha(s) = (0, sinh\, s, cosh\, s), \quad -2 \leq s \leq 2.$$

After simple computation, we have

$$\boldsymbol{t}(s) = (0, \cosh s, \sinh s),$$

$$\boldsymbol{n}(s) = (0, \sinh s, \cosh s),$$

$$\boldsymbol{b}(s) = (1,0,0),$$

and $\tau = 0$ which follows $\varphi(s) = \varphi_0$ is a constant.

Thus, the spacelike roller coaster surfaces family is expressed as

$$P(s, \theta; r) = (0, sinh\, s, cosh\, s) + r(cos\theta, sin\theta cosh\varphi, -sin\theta sinh\varphi)\begin{bmatrix} 0 & cosh\, s & sinh\, s \\ 0 & sinh\, s & cosh\, s \\ 1 & 0 & 0 \end{bmatrix}$$

so, if we choose $\theta \in [0,2\pi]$ and $\varphi_0 = 1.5$ then for r=1, the corresponding spacelike surface is shown in Figure 2(b).

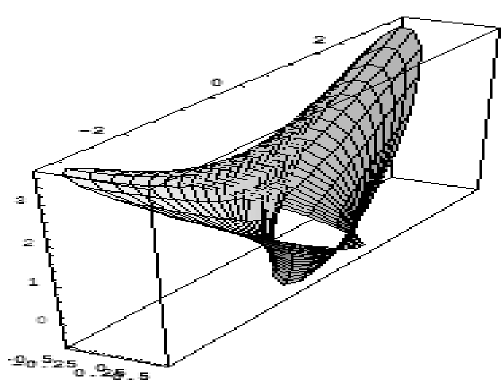

Fig. 2(b). Spacelike roller coaster surface for $\theta \in [0,2\pi]$, $\varphi_0 = 1.5$, r=1.

References

Blum, R. (1980). Circles on surfaces in the Euclidean 3-space, In Artzy, R. & Vaisman, I. (Ed.). Lecture Notes in Mathematics,Geometry and Differential Geometry, 792 pp. 213-221, Springer, Berlin.

Izumiya, S., Saji, K. & Takeuchi, N. (2007). Circular surfaces. Advances in Geometry, **7**:295-313.

Izumiya, S. & Takeuchi, N. (2001). Singularities of ruled surfaces in R^3. Mathematical Proceedings of the Cambridge Philosophical Society, **130**:1-11.

Izumiya, S. (2003a). Geometry of ruled surfaces. In: Misra. J.C. (Ed.). Applicable Mathematics in the Golden Age. 305-338. Narosa Publishing House, New Delhi, India.

Izumiya, S. (2003b). Special curves and ruled surfaces. Contributions to Algebra and Geometry, **44**:203-212.

Kasap, E. & Akyildiz, F.T. (2006). Surfaces with common geodesic in Minkowski 3-space. Applied Mathematicsand Computation, **177**(1):260-270.

Mc-Nertney, L.V. (1980). One-Parameter families of surfaces with constant curvature in Lorentz three-space. Ph.D. Thesis, Brown University, Providence, Rhode Island, USA.

O'Neill, B. (1983). Semi-Riemannian geometry with applications to relativity. Academic Press, New York. pp. 56.

Ratcliffe, J.G. (2006). Foundations of hyperbolicmanifolds.Springer Science+Business Media, LLC, New York, USA. pp. 68-72.

Walrave, J. (1995). Curves and Surfaces in Minkowski Space. Ph.D. Thesis, K.U. Leuven, Leuven, Belgium.

Xu, Z., Feng, R. & Sun, J. (2006). Analytic and algebraic properties of canal surfaces. Journal of Computational and Applied Mathematics, **195**(1-2):220-228.

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حول تقوسات سطوح دائرية مثل فضائية

[1،2]رشاد أحمد عبد الباقي، [3]ياسين أنلوتورك
[1]قسم الرياضيات – كلية العلوم للبنات – جامعة الملك عبدالعزيز – ص. بريد 126300 – جدة 21352–
المملكة العربية السعودية
[2]قسم الرياضيات – كلية العلوم – جامعة أسيوط، أسيوط 71516 جمهورية مصر العربية.
[3]قسم الرياضيات – كلية الآداب والعلوم – جامعة كيركلاريلي – 39100– كيركلاريلي– تركيا.
المؤلف البريد الالكتروني: rbaky@Live.com

خلاصة

نقدم في هذا البحث نظاماً تاماً من اللامتغيرات لدراسة السطوح الدائرية مثل الفضائية التي لها نصف قطر ثابت. و نقوم بتسهيل دراستنا عن طريق دراسة منحنين: الأول هو مبين لورنتز الكروي لنواظم الوحدة لمستويات دائرية 6 و الثاني هو منحنى المحور مثل الفضائي. نستخدم بعد ذلك المعاني الهندسية لهذه اللامتغيرات للوصول إلى خصائص السطوح الدائرية مثل الفضائية التي لها سطوح مسطرة كلاسيكية و أخيراً نعرض سطوح أفعوانيه مثل فضائية كحالات خاصة من السطوح الدائرية مثل الفضائية.