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Relations between a dual unit vector and Frenet vectors of a dual curve

Burak Şahiner*, Mehmet Önder

Dept. of Mathematics, Manisa Celal Bayar University, Muradiye, Manisa, 45140, Turkey

**Corresponding author: burak.sahiner@cbu.edu.tr*

Abstract

In this paper, we generalize the notions of helix and slant helix in dual space in a more general case such as the dual angle between a fixed dual unit vector and Frenet vectors of a dual curve is not constant. We study the motions of the lines corresponding to dual Frenet vectors and dual Frenet instantaneous rotation vector with respect to a fixed line. We obtain some characterizations for such dual curves and show that these characterizations include the definitions and characterizations of dual helix and dual slant helix in some special cases.

Key words: Dual angle; dual Darboux slant helix; dual helix; dual slant helix; motions of Frenet vectors.

1. Introduction

In Euclidean space, the special curves are an important subject of curve theory. Generally such curves are thought as the curves whose curvatures satisfy some conditions. The well-known of such curves are helices and slant helices. A helix is defined by the property that the tangent line of the curve makes a constant angle with a fixed straight line (the axis of the general helix) (Blaschke, 1945; Struik, 1988). Therefore, a general helix can be equivalently defined as one whose tangent indicatrix is a planar curve. Of course, there exist more special curves in the space. One of them is slant helix, which was first introduced by Izumiya & Takeuchi (2004) by the property that the principal normal lines of curve make a constant angle with a fixed direction in the Euclidean 3-space E^3. Later, Zıplar *et al*. (2012) introduced the Darboux helix, which is defined by the property that the Darboux vector of a space curve makes a constant angle with a fixed direction and they gave the characterizations of this new special curve.

On the other hand, the dual space is a more general space than Euclidean space and contains it. So, the concepts of helix, slant helix and Darboux helix can be considered in dual space. Dual helix and dual slant helix have been defined by Lee *et al*. (2011). They have defined these dual curves by a similar way as given in Euclidean case and obtained the characterizations of these curves in dual space. Moreover, dual vector analysis has important applications to differential geometry and kinematics (Çöken *et al.,* 2009; Kabadayı & Yaylı, 2011; Karadağ *et al*, 2014; Önder & Uğurlu, 2013).

In this paper, we consider a more general case in dual space for special curves. We take a fixed dual unit vector and a dual curve. We study the motions of dual Frenet vectors and dual Frenet instantaneous rotation vector with respect to a fixed dual unit vector. These motions correspond to the motions of lines determinate by these dual unit vectors. We obtain the relations between these dual vectors. Moreover, we obtain some new characterizations for dual helix, dual slant helix and dual Darboux helix in some special cases.

2. Preliminaries

A dual number, as introduced by W. Clifford, can be defined as an ordered pair of real numbers (a, a^*), where a is called the real part and a^* is called the dual part of the dual number. Dual numbers may be formally expressed as $\overline{a} = a + \varepsilon a^*$ where ε is the dual unit which is subjected to the following rules (Veldkamp, 1976):

$$\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0 .$$

We denote the set of dual numbers by ID, i.e.,

$$ID = \left\{ \overline{a} = a + \varepsilon a^* : \quad a, a^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}.$$

Two inner operations and equality on ID are defined as follows (Blaschke, 1945; Hacısalihoğlu, 1983):

(i) Addition :

$$(a, a^*) + (b, b^*) = (a + b,\ a^* + b^*)$$

(ii) Multiplication :

$$(a, a^*)\,(b, b^*) = (ab,\ ab^* + a^* b)$$

(iii) Equality :

$(a,a^*)=(b,b^*) \Leftrightarrow a=b,\ a^*=b^*$

Since division by pure dual numbers is not defined, the set ID of dual numbers with the above operations is a commutative ring, not a field.

A dual number $\bar{a}=a+\varepsilon a^*$ divided by a dual number $\bar{b}=b+\varepsilon b^*$, with $b\neq 0$, can be defined as (Blaschke, 1945).

$$\frac{\bar{a}}{\bar{b}}=\frac{a}{b}+\varepsilon\frac{a^*b-ab^*}{b^2}\,.$$

We can define the function $f(\bar{a})$ of a dual number $\bar{a}$ by expanding it formally in a Maclaurin series with ε as variable. Since $\varepsilon^n=0$ for $n>1$, we obtain

$$f(\bar{a})=f(a+\varepsilon a^*)=f(a)+\varepsilon a^* f'(a)\,,$$

where $f'(a)$ is derivative of $f(a)$ with respect to a (Bottema & Roth, 1979; Dimentberg, 1965).

In analogy with dual numbers, a dual vector referred to an arbitrarily chosen origin can be defined as an ordered pair of vectors $(\boldsymbol{a},\boldsymbol{a}^*)$, where $\boldsymbol{a},\boldsymbol{a}^*\in IR^3$. Also dual vectors can be expressed as $\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*$, where $\boldsymbol{a},\boldsymbol{a}^*\in IR^3$ and $\varepsilon^2=0$. We denote the set of dual vectors by ID^3, i.e.,

$$ID^3=\left\{\,\tilde{\boldsymbol{a}}=(\bar{a}_1,\bar{a}_2,\bar{a}_3):\quad \bar{a}_i\in ID,\quad i=1,2,3\,\right\}$$

ID^3 is a module over the ring ID and it is called dual space or ID-module. For any dual vectors $\tilde{\boldsymbol{a}}=\boldsymbol{a}+\varepsilon\boldsymbol{a}^*$ and $\tilde{\boldsymbol{b}}=\boldsymbol{b}+\varepsilon\boldsymbol{b}^*$ in ID^3, the scalar product and the vector product are defined by

$$\left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{b}}\right\rangle=\left\langle\boldsymbol{a},\boldsymbol{b}\right\rangle+\varepsilon\left(\left\langle\boldsymbol{a},\boldsymbol{b}^*\right\rangle+\left\langle\boldsymbol{a}^*,\boldsymbol{b}\right\rangle\right)$$

and

$$\tilde{\boldsymbol{a}}\times\tilde{\boldsymbol{b}}=\boldsymbol{a}\times\boldsymbol{b}+\varepsilon\left(\boldsymbol{a}\times\boldsymbol{b}^*+\boldsymbol{a}^*\times\boldsymbol{b}\right),$$

respectively (Veldkamp, 1976; Hacısalihoğlu, 1983).

The norm of a dual vector $\tilde{\boldsymbol{a}}$ is given by

$$\|\tilde{\boldsymbol{a}}\|=\|\boldsymbol{a}\|+\varepsilon\frac{\left\langle\boldsymbol{a},\boldsymbol{a}^*\right\rangle}{\|\boldsymbol{a}\|},$$

(Veldkamp, 1976; Hacısalihoğlu, 1983). A dual vector $\tilde{\boldsymbol{a}}$ with norm $1+\varepsilon 0$ is called dual unit vector. The set of dual unit vectors is denoted by

$$\tilde{S}^2=\left\{\tilde{\boldsymbol{a}}=(\bar{a}_1,\bar{a}_2,\bar{a}_3)\in ID^3:\quad \left\langle\tilde{\boldsymbol{a}},\tilde{\boldsymbol{a}}\right\rangle=(1,0)\right\}$$

and called dual unit sphere (Blaschke, 1945; Hacısalihoğlu, 1983). It is well known that the points of $\tilde{S}^2$ (dual unit vectors) correspond to the lines of IR^3. This correspondence is known as E. Study mapping. A dual angle, subtended by two oriented lines in space as introduced by Study in 1903, is defined as $\bar{\theta}=\theta+\varepsilon\,\theta^*$, where θ is the projected angle and θ^* is the shortest distance between the two lines (Blaschke, 1945).

$\tilde{\boldsymbol{\alpha}}(t)=\boldsymbol{\alpha}(t)+\varepsilon\boldsymbol{\alpha}^*(t)$ is a curve in dual space ID^3 and is called dual space curve where $\boldsymbol{\alpha}(t)=\left(\alpha_1(t),\alpha_2(t),\alpha_3(t)\right)$ and $\boldsymbol{\alpha}^*(t)=\left(\alpha_1^*(t),\alpha_2^*(t),\alpha_3^*(t)\right)$ are real valued curves in the real space IR^3. If the functions $\alpha_i(t)$ and $\alpha_i^*(t)$, $(1\le i\le 3)$ are differentiable then the dual space curve

$\tilde{\boldsymbol{\alpha}}: I\subset IR\to ID^3$

$$\begin{aligned}t\to\tilde{\boldsymbol{\alpha}}(t)&=\big(\alpha_1(t)+\varepsilon\alpha_1^*(t),\\ &\qquad \alpha_2(t)+\varepsilon\alpha_2^*(t),\ \alpha_3(t)+\varepsilon\alpha_3^*(t)\big)\\ &=\boldsymbol{\alpha}(t)+\varepsilon\boldsymbol{\alpha}^*(t)\end{aligned}$$

is differentiable in ID^3. The real part $\boldsymbol{\alpha}(t)$ of the dual space curve $\tilde{\boldsymbol{\alpha}}=\tilde{\boldsymbol{\alpha}}(t)$ is called indicatrix. The dual arc-length of the dual space curve $\tilde{\boldsymbol{\alpha}}(t)$ from t_1 to t is defined by

$$\bar{s}=\int_{t_1}^{t}\|\tilde{\boldsymbol{\alpha}}'(t)\|\,dt=\int_{t_1}^{t}\|\vec{\boldsymbol{\alpha}}'(t)\|\,dt\ +\varepsilon\int_{t_1}^{t}\left\langle\boldsymbol{T},\boldsymbol{\alpha}^{*\,\prime}(t)\right\rangle dt=s+\varepsilon s^*$$

where $\boldsymbol{T}$ is unit tangent vector of the indicatrix $\boldsymbol{\alpha}(t)$ (Yücesan *et al.,* 2007).

Let $\tilde{\boldsymbol{\alpha}}(\bar{s})$ be a dual space curve with dual arc length parameter $\bar{s}$. The dual unit tangent vector of $\tilde{\boldsymbol{\alpha}}$ is defined by

$$\frac{d\tilde{\boldsymbol{\alpha}}}{d\bar{s}}=\tilde{\boldsymbol{T}}\,.$$

Differentiating $\tilde{\boldsymbol{T}}$ with respect to dual arc length parameter $\bar{s}$ we have

$$\tilde{\boldsymbol{T}}'=\frac{d\tilde{\boldsymbol{T}}}{d\bar{s}}=\frac{d^2\tilde{\boldsymbol{\alpha}}}{d\bar{s}^2}=\bar{\kappa}\tilde{\boldsymbol{N}}\,,$$

where $\bar{\kappa}=\bar{\kappa}(\bar{s})$ is called dual curvature. We restrict that $\bar{\kappa}(\bar{s})$ is never a pure dual number. The dual unit vector $\tilde{\boldsymbol{N}}=(1/\bar{\kappa})\tilde{\boldsymbol{T}}'$ is called dual unit principal normal vector of $\tilde{\alpha}$. The dual unit vector $\tilde{\boldsymbol{B}}$ defined by $\tilde{\boldsymbol{B}}=\tilde{\boldsymbol{T}}\times\tilde{\boldsymbol{N}}$ is called dual unit binormal vector of $\tilde{\boldsymbol{\alpha}}$. The dual frame $\left\{\tilde{\boldsymbol{T}}(\bar{s}),\tilde{\boldsymbol{N}}(\bar{s}),\tilde{\boldsymbol{B}}(\bar{s})\right\}$ is called moving dual Frenet frame along the dual space curve $\tilde{\boldsymbol{\alpha}}(\bar{s})$ in ID^3. For the curve $\tilde{\boldsymbol{\alpha}}$, the dual Frenet derivative formulae can be given in matrix form as

$$\frac{d}{d\bar{s}}\begin{bmatrix}\tilde{\boldsymbol{T}}\\ \tilde{\boldsymbol{N}}\\ \tilde{\boldsymbol{B}}\end{bmatrix}=\begin{bmatrix}0 & \bar{\kappa} & 0\\ -\bar{\kappa} & 0 & \bar{\tau}\\ 0 & -\bar{\tau} & 0\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{T}}\\ \tilde{\boldsymbol{N}}\\ \tilde{\boldsymbol{B}}\end{bmatrix},$$

where $\bar{\tau}=\bar{\tau}(s)$ is called the dual torsion of $\tilde{\boldsymbol{\alpha}}$ (Yücesan *et al.,* 2007). Then the dual Frenet instantaneous rotation vector (or dual Darboux vector) of $\tilde{\boldsymbol{\alpha}}(\bar{s})$ is defined by $\tilde{\boldsymbol{W}}=\bar{\tau}\tilde{\boldsymbol{T}}+\bar{\kappa}\tilde{\boldsymbol{B}}$ which gives the derivative formulae as follows

$$\tilde{\boldsymbol{T}}'=\tilde{\boldsymbol{W}}\times\tilde{\boldsymbol{T}},\ \tilde{\boldsymbol{N}}'=\tilde{\boldsymbol{W}}\times\tilde{\boldsymbol{N}},\ \tilde{\boldsymbol{B}}'=\tilde{\boldsymbol{W}}\times\tilde{\boldsymbol{B}},$$

where $\tilde{\boldsymbol{T}}'=\frac{d\tilde{\boldsymbol{T}}}{d\bar{s}}$. The unit dual Frenet instantaneous rotation vector of $\tilde{\boldsymbol{\alpha}}(\bar{s})$ is defined by

$$\tilde{\boldsymbol{W}}_0=\frac{\bar{\tau}\tilde{\boldsymbol{T}}+\bar{\kappa}\tilde{\boldsymbol{B}}}{\sqrt{\bar{\tau}^2+\bar{\kappa}^2}}.$$

Let $\tilde{\boldsymbol{\alpha}}(\bar{s})$ be a unit speed dual curve. We can write this curve in another parametric representation $\tilde{\boldsymbol{\alpha}}=\tilde{\boldsymbol{\alpha}}(\bar{\theta})$, where $\bar{\theta}=\int\bar{\kappa}(\bar{s})\,d\bar{s}$, and we have new dual Frenet equations as follows:

$$\begin{bmatrix}\tilde{\boldsymbol{T}}'(\bar{\theta})\\ \tilde{\boldsymbol{N}}'(\bar{\theta})\\ \tilde{\boldsymbol{B}}'(\bar{\theta})\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & \bar{f}(\bar{\theta})\\ 0 & -\bar{f}(\bar{\theta}) & 0\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{T}}(\bar{\theta})\\ \tilde{\boldsymbol{N}}(\bar{\theta})\\ \tilde{\boldsymbol{B}}(\bar{\theta})\end{bmatrix},$$

where $\bar{f}(\bar{\theta})=\frac{\bar{\tau}(\bar{\theta})}{\bar{\kappa}(\bar{\theta})}$.

3. Relations between a unit dual vector and the tangent vector of a dual curve

In the dual space, a unit speed dual curve $\tilde{\boldsymbol{\alpha}}:I\to ID^3$ with curvature $\bar{\kappa}\neq 0$ and torsion $\bar{\tau}\neq 0$ is called a general dual helix if there is some constant dual unit vectors $\tilde{\boldsymbol{d}}$, so that $\langle\tilde{\boldsymbol{T}},\tilde{\boldsymbol{d}}\rangle=\cos\bar{\phi}$ is constant along the curve where $\bar{\phi}=\phi+\varepsilon\phi^*$ is dual constant angle between dual unit vectors $\tilde{\boldsymbol{T}},\tilde{\boldsymbol{d}}$ (Lee *et al.*, 2011). From this definition it is clear that both ϕ and ϕ^* are real constants. In this section, we will consider some more general cases such as one of the real numbers ϕ and ϕ^* is not constant, i.e., $\bar{\phi}$ is not constant.

Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve with Frenet frame $\{\tilde{\boldsymbol{T}}(s),\tilde{\boldsymbol{N}}(s),\tilde{\boldsymbol{B}}(s)\}$ and $\tilde{\boldsymbol{d}}$ be a constant unit dual vector. If $\bar{\phi}=\phi+\varepsilon\,\phi^*$ is the dual angle between unit dual vector $\tilde{\boldsymbol{d}}$ and unit dual tangent vector $\tilde{\boldsymbol{T}}$, we can write

$$\langle\tilde{\boldsymbol{T}},\tilde{\boldsymbol{d}}\rangle=\cos\bar{\phi}=\cos\phi-\varepsilon\,\phi^*\sin\phi \qquad (3.1)$$

where $\phi,\phi^*\in IR$. The unit dual vector $\tilde{\boldsymbol{d}}$ can be expressed as the linear combination of the elements of the dual Frenet trihedron as follows

$$\tilde{\boldsymbol{d}}=\bar{a}_1\tilde{\boldsymbol{T}}+\bar{a}_2\tilde{\boldsymbol{N}}+\bar{a}_3\tilde{\boldsymbol{B}} \qquad (3.2)$$

where $\bar{a}_i=\bar{a}_i(\bar{s}),\ (i=1,2,3)$ are differentiable functions. Separating (3.2) into real and dual parts, we have

$$\boldsymbol{d}=a_1\boldsymbol{T}+a_2\boldsymbol{N}+a_3\boldsymbol{B} \qquad (3.3)$$

and

$$\boldsymbol{d}^*=a_1^*\boldsymbol{T}+a_1\boldsymbol{T}^*+a_2^*\boldsymbol{N}+a_2\boldsymbol{N}^*+a_3^*\boldsymbol{B}+a_3\boldsymbol{B}^* \qquad (3.4)$$

Additionally, since $\tilde{\boldsymbol{d}}$ is a dual unit vector, we have

$$a_1^2+a_2^2+a_3^2=1 \qquad (3.5)$$

and

$$a_1a_1^*+a_2a_2^*+a_3a_3^*=0. \qquad (3.6)$$

Of course it is more complicated to study this subject under the assumption that both ϕ and ϕ^* are not constants. Then, we consider a simpler way and study two cases as follows:

Case 1. Let ϕ be a non-zero constant while ϕ^* is not constant. From E. Study mapping, it means that the lines corresponding to the vector $\tilde{\boldsymbol{T}}$ make a spatial motion in the space such that the angle ϕ between the lines corresponding to the dual unit vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$ is constant while the distance ϕ^* between these lines is not a constant. Differentiating (3.1) with respect to $\bar{\theta}$ and considering (2.6), we have

$$\langle\tilde{\boldsymbol{N}},\tilde{\boldsymbol{d}}\rangle=-\varepsilon\,\phi^{*\prime}\sin\phi. \qquad (3.7)$$

Separating (3.7) into real and dual parts, following two equations are found as

$$\langle\boldsymbol{N},\boldsymbol{d}\rangle=0,\ \langle\boldsymbol{N},\boldsymbol{d}^*\rangle+\langle\boldsymbol{N}^*,\boldsymbol{d}\rangle=-\phi^{*\prime}\sin\phi$$

respectively. With the aid of (3.3) and (3.4), we have

$$a_2=0,\ \ a_2^*=-\phi^{*\prime}\sin\phi \qquad (3.8)$$

Differentiating (3.7), we have

$$\langle-\tilde{\boldsymbol{T}}+\bar{f}\,\tilde{\boldsymbol{B}},\tilde{\boldsymbol{d}}\rangle=-\varepsilon\phi^{*\prime\prime}\sin\phi. \qquad (3.9)$$

Separating (3.9) into real and dual parts, we have

$$-\langle \boldsymbol{T},\boldsymbol{d}\rangle + f\langle \boldsymbol{B},\boldsymbol{d}\rangle = 0 \tag{3.10}$$

and

$$-\langle \boldsymbol{T},\boldsymbol{d}^*\rangle - \langle \boldsymbol{T}^*,\boldsymbol{d}\rangle + f\langle \boldsymbol{B},\boldsymbol{d}^*\rangle + f\langle \boldsymbol{B}^*,\boldsymbol{d}\rangle + f^*\langle \boldsymbol{B},\boldsymbol{d}\rangle = -\phi^{*\prime\prime}\sin\phi$$

respectively. With the aid of (3.3) and (3.4), following two equations are found as

$$a_1 = f\, a_3 \tag{3.11}$$

and

$$-a_1^* + f\, a_3^* + f^* a_3 = -\phi^{*\prime\prime}\sin\phi\,. \tag{3.12}$$

respectively. Substituting (3.11) into (3.5), we find

$$a_3 = \pm\sqrt{\frac{|1-a_2^2|}{1+f^2}}\,. \tag{3.13}$$

Differentiating (3.10) and substituting (3.13) into the result, we have

$$\frac{f'}{(1+f^2)^{3/2}} = \pm\frac{a_2}{\sqrt{|1-a_2^2|}}\,. \tag{3.14}$$

Integrating (3.14), we get

$$\frac{f}{\sqrt{1+f^2}} = \pm m\theta + c_0$$

where c_0 is a constant and $m = \dfrac{a_2}{\sqrt{|1-a_2^2|}}$. Since $a_2 = 0$, we have

$$f = c\,. \tag{3.15}$$

where c is a constant. Substituting (3.15) in (3.13) gives

$$a_3 = \frac{1}{\sqrt{1+c^2}}\,. \tag{3.16}$$

By substituting (3.16) and (3.15) into (3.11), we get

$$a_1 = \frac{c}{\sqrt{1+c^2}}\,. \tag{3.17}$$

From (3.1), we know that

$$a_1 = \cos\phi \tag{3.18}$$

and

$$a_1^* = -\phi^*\sin\phi\,. \tag{3.19}$$

By using (3.17) and (3.18), we have

$$a_1^* = -\frac{\phi^*}{\sqrt{1+c^2}}\,. \tag{3.20}$$

Substituting (3.17), (3.20), (3.8) and (3.16) into (3.6) gives

$$a_3^* = \frac{c\,\phi^*}{\sqrt{1+c^2}}\,. \tag{3.21}$$

By substituting (3.19), (3.21), (3.15) and (3.16) into (3.12), we find

$$f^* = -\phi^* - c^2\phi^* - \phi^{*\prime\prime}\,.$$

So we have

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon(-\phi^* - c^2\phi^* - \phi^{*\prime\prime})\,.$$

Now we can give the following theorem.

Theorem 3.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\phi} = \phi + \varepsilon\,\phi^*$ be the dual angle between the unit dual vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$. If ϕ is constant then*

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon(-\phi^* - c^2\phi^* - \phi^{*\prime\prime})$$

holds, where $\bar{f} = \dfrac{\bar{\tau}}{\bar{\kappa}}$.

It is known that the dual curve $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if $\dfrac{\bar{\tau}}{\bar{\kappa}}$ is constant. Then from Theorem 3.1, we can give the following corollary.

Corollary 3.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\phi} = \phi + \varepsilon\,\phi^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$, and ϕ be a constant. Then $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if the following differential equation holds*

$$-\phi^{*\prime} - c^2\phi^{*\prime} - \phi^{*\prime\prime\prime} = 0\,.$$

Case 2. Let ϕ^* be a non-zero constant while ϕ is not constant. From E. Study mapping it means that the lines corresponding to the vector $\tilde{\boldsymbol{T}}$ make a spatial motion in the space such that the angle ϕ between the lines corresponding to the dual unit vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$ is not a constant while the distance ϕ^* between these lines is constant. By differentiating (3.1), we have

$$\langle \tilde{\boldsymbol{N}},\tilde{\boldsymbol{d}}\rangle = -\phi'\sin\phi - \varepsilon\phi^*\phi'\cos\phi\,. \tag{3.22}$$

By separating (3.22) into real and dual parts and using (3.3) and (3.4), we have

$$a_2 = -\phi'\sin\phi \tag{3.23}$$

and

$$a_2^* = -\phi^*\phi'\cos\phi\,. \tag{3.24}$$

By substituting (3.18) and (3.23) into (3.5), we find

$$a_3 = \pm\sqrt{\left|1-\cos^2\phi - \phi'^2\sin^2\phi\right|}\,. \tag{3.25}$$

Substituting (3.18), (3.19), (3.23), (3.24) and (3.25) into (3.6), we find

$$a_3^* = \pm\frac{\phi^*(1-\phi'^2)\cos\phi\sin\phi}{\sqrt{\left|1-\cos^2\phi-\phi'^2\sin^2\phi\right|}}\,. \tag{3.26}$$

Differentiating (3.22), we have

$$\begin{aligned}\left\langle -\tilde{\boldsymbol{T}} + \overline{f}\,\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{d}}\right\rangle &= -\phi''\sin\phi - \phi'^2\cos\phi \\ &\quad - \varepsilon\,\phi^*(\phi''\cos\phi - \phi'^2\sin\phi)\end{aligned} \tag{3.27}$$

By separating (3.27) into real and dual parts and using (3.3) and (3.4), we have

$$-a_1 + f\,a_3 = -\phi''\sin\phi - \phi'^2\cos\phi \tag{3.28}$$

and

$$-a_1^* + f\,a_3^* + f^* a_3 = \phi^*(\phi''\cos\phi - \phi'^2\sin\phi)\,. \tag{3.29}$$

From (3.28), we find

$$f = \frac{-\phi''\sin\phi - \phi'^2\cos\phi + a_1}{a_3} \tag{3.30}$$

where a_1 and a_3 are defined as given in (3.18) and (3.25), respectively. Substituting (3.19), (3.25), (3.26) and (3.30) into (3.29), we have

$$\begin{aligned} f^* &= \frac{\phi^*(\phi''\cos\phi - \phi'^2\sin\phi) + a_1^*}{a_3} \\ &\quad + \frac{(\phi''\sin\phi + \phi'^2\cos\phi - a_1)a_3^*}{a_3^2}\end{aligned} \tag{3.31}$$

From (3.30) and (3.31), we find

$$\begin{aligned}\overline{f} = f + \varepsilon f^* &= \frac{-\phi''\sin\phi - \phi'^2\cos\phi + a_1}{a_3} \\ &\quad + \varepsilon\left(\frac{\phi^*(\phi''\cos\phi - \phi'^2\sin\phi) + a_1^*}{a_3} + \frac{(\phi''\sin\phi + \phi'^2\cos\phi - a_1)a_3^*}{a_3^2}\right)\end{aligned}$$

Now we can give the following theorem.

Theorem 3.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\overline{\phi} = \phi + \varepsilon\,\phi^*$ be the dual angle between the unit dual vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$. If ϕ^* is a constant then*

$$\begin{aligned}\overline{f} = f + \varepsilon f^* &= \frac{-\phi''\sin\phi - \phi'^2\cos\phi + a_1}{a_3} \\ &\quad + \varepsilon\left(\frac{\phi^*(\phi''\cos\phi - \phi'^2\sin\phi) + a_1^*}{a_3} + \frac{(\phi''\sin\phi + \phi'^2\cos\phi - a_1)a_3^*}{a_3^2}\right)\end{aligned}$$

holds, where $\overline{f} = \dfrac{\overline{\tau}}{\overline{\kappa}}$.

From Theorem 3.2, we can give the following corollary.

Corollary 3.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\overline{\phi} = \phi + \varepsilon\,\phi^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{T}}$, and ϕ^* be constant. Then $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if*

$$\frac{-\phi''\sin\phi - \phi'^2\cos\phi + a_1}{a_3} = \text{constant}$$

and

$$\frac{\phi^*(\phi''\cos\phi - \phi'^2\sin\phi) + a_1^*}{a_3} + \frac{(\phi''\sin\phi + \phi'^2\cos\phi - a_1)a_3^*}{a_3^2} = \text{constant}.$$

4. Relations between a unit dual vector and the principal normal vector of a dual curve

In dual space, a unit speed curve $\tilde{\boldsymbol{\alpha}}(s)$ is called a dual slant helix, if its unit dual principal normal vector $\tilde{\boldsymbol{N}}$ makes a constant angle $\overline{\varphi}$ with a fixed direction in a unit dual vector $\tilde{\boldsymbol{d}}$; that is, $\left\langle \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{d}}\right\rangle = \cos\overline{\varphi}$ is constant along the curve (Lee *et al*, 2011). From this definition it is clear that both φ and φ^* are real constants. In this section, we will consider some more general cases such as one of the real numbers φ and φ^* is not constant, i.e., $\overline{\varphi}$ is not constant.

Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\left\{\tilde{\boldsymbol{T}}(s), \tilde{\boldsymbol{N}}(s), \tilde{\boldsymbol{B}}(s)\right\}$ be the dual Frenet trihedron of $\tilde{\boldsymbol{\alpha}}(s)$. If $\overline{\varphi} = \varphi + \varepsilon\,\varphi^*$ is the dual angle between the unit dual vector $\tilde{\boldsymbol{d}}$ and the unit dual principal normal vector $\tilde{\boldsymbol{N}}$, we can write

$$\left\langle \tilde{\boldsymbol{N}}, \tilde{\boldsymbol{d}}\right\rangle = \cos\overline{\varphi} = \cos\varphi - \varepsilon\,\varphi^*\sin\varphi \tag{4.1}$$

where $\varphi, \varphi^* \in IR$. Of course it is more complicated to study this subject under the assumption that both φ and φ^* are not constants. Then, we consider a simpler way and study two cases as follows:

Case 1. Let φ be a non-zero constant while φ^* is not constant. From E. Study mapping, it means that the lines corresponding to the vector $\tilde{\boldsymbol{N}}$ make a spatial motion in the space such that the angle φ between the lines corresponding to the dual unit vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{N}}$ is constant while the distance φ^* between these lines is not a constant. Differentiating (4.1), we have

$$\left\langle -\tilde{\boldsymbol{T}} + \bar{f}\,\tilde{\boldsymbol{B}}, \tilde{\boldsymbol{d}} \right\rangle = -\varepsilon \varphi^{*\prime} \sin\varphi . \tag{4.2}$$

Separating (4.2) into real and dual parts and using (3.3) and (3.4), we have

$$-a_1 + f\, a_3 = 0 \tag{4.3}$$

and

$$-a_1^* + f\, a_3^* + f^* a_3 = -\varphi^* \sin\varphi . \tag{4.4}$$

Since (4.3) and (3.11) are the same, we find the following equation which we have found in Section 3 as

$$\frac{f'}{(1+f^2)^{3/2}} = \pm m .$$

where $m = \dfrac{a_2}{\sqrt{|1-a_2^2|}}$.

Integrating the above equation, we have

$$\frac{f}{\sqrt{1+f^2}} = \pm m(\theta + c_1)$$

where c_1 is an integration constant. The integration constant can be disappeared with a parameter change $\theta \to \theta - c_1$. So we get

$$f = \pm \frac{m\theta}{\sqrt{|1-m^2\theta^2|}} \tag{4.5}$$

where $m = \dfrac{a_2}{\sqrt{|1-a_2^2|}}$. From (4.1), we have known that

$$a_2 = \cos\varphi , \quad a_2^* = -\varphi^* \sin\varphi . \tag{4.6}$$

Substituting (4.3) and (4.6) into (3.5), we have

$$a_3 = \pm \sqrt{\frac{\sin^2\varphi}{1+f^2}} \tag{4.7}$$

and

$$a_1 = \pm f \sqrt{\frac{\sin^2\varphi}{1+f^2}} \tag{4.8}$$

where f is defined as (4.5). From (4.4), we find

$$f^* = \frac{-\varphi^{*\prime} a_3 \sin\varphi + a_1^* a_3 - a_1 a_3^*}{a_3^2} \tag{4.9}$$

where a_1 and a_3 are defined as (4.8) and (4.7), respectively. From (4.5) and (4.9), we get

$$\bar{f} = f + \varepsilon f^* = \pm \frac{m\theta}{\sqrt{|1-m^2\theta^2|}} + \varepsilon \frac{-\varphi^{*\prime} a_3 \sin\varphi + a_1^* a_3 - a_1 a_3^*}{a_3^2}$$

Now we can give the following theorem.

Theorem 4.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\varphi} = \varphi + \varepsilon\, \varphi^*$ be the dual angle between the unit dual vector $\tilde{\boldsymbol{d}}$ and the unit dual principal normal vector $\tilde{\boldsymbol{N}}$. If φ is a constant then*

$$\bar{f} = f + \varepsilon f^* = \pm \frac{m\theta}{\sqrt{|1-m^2\theta^2|}} + \varepsilon \frac{-\varphi^{*\prime} a_3 \sin\varphi + a_1^* a_3 - a_1 a_3^*}{a_3^2}$$

where $\bar{f} = \dfrac{\bar{\tau}}{\bar{\kappa}}$.

From Theorem 4.1 and definitions of dual helix and dual slant helix, we can give the following corollaries.

Corollary 4.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\varphi} = \varphi + \varepsilon\, \varphi^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{N}}$, and φ be a constant. $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if*

$$\frac{m\theta}{\sqrt{|1-m^2\theta^2|}} = \text{constant}$$

and

$$\frac{-\varphi^{*\prime} a_3 \sin\varphi + a_1^* a_3 - a_1 a_3^*}{a_3^2} = \text{constant} .$$

Corollary 4.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\varphi} = \varphi + \varepsilon\, \varphi^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{N}}$, and φ be a constant. $\tilde{\boldsymbol{\alpha}}(s)$ is a dual slant helix if and only if*

$$\bar{f} = \pm \frac{m\theta}{\sqrt{|1-m^2\theta^2|}} + \varepsilon \frac{a_1^* a_3 - a_1 a_3^*}{a_3^2} .$$

Case 2. Let φ^* be a non-zero constant while φ is not constant. From E. Study mapping, it means that the lines corresponding to the vector $\tilde{\boldsymbol{N}}$ make a spatial motion in the space such that the distance φ^* between the lines corresponding to the dual unit vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{N}}$ is constant while the angle φ between these lines is not a constant. Differentiating (4.1), we have

$$\left\langle -\tilde{T}+\bar{f}\,\tilde{B},\tilde{d}\right\rangle = -\varphi'\sin\varphi - \varepsilon\,\varphi^{*}\varphi'\cos\varphi \quad (4.10)$$

Separating (4.10) into real and dual parts and using (3.3) and (3.4), we get

$$-a_1 + f\,a_3 = -\varphi'\sin\varphi \quad (4.11)$$

and

$$-a_1^{*} + f\,a_3^{*} + f^{*}a_3 = -\varphi^{*}\varphi'\cos\varphi\,. \quad (4.12)$$

From (4.11) and (4.12), we get, respectively,

$$f = \frac{a_1 - \varphi'\sin\varphi}{a_3}$$

and

$$f^{*} = \frac{-\varphi^{*}\varphi' a_3\cos\varphi + a_1^{*}a_3 - (a_1 - \varphi'\sin\varphi)a_3^{*}}{a_3^2}$$

Now we can give the following theorem.

Theorem 4.2. *Let $\tilde{\alpha}(s)$ be a dual curve, $\tilde{d}$ be a constant unit dual vector and $\bar{\varphi} = \varphi + \varepsilon\,\varphi^{*}$ be the dual angle between the unit dual vector $\tilde{d}$ and the unit dual normal vector $\tilde{N}$. If φ^{*} is a constant then*

$$\bar{f} = f + \varepsilon f^{*} = \frac{a_1 - \varphi'\sin\varphi}{a_3} + \varepsilon\left(\frac{-\varphi^{*}\varphi'\cos\varphi + a_1^{*}}{a_3} + \frac{-(a_1 - \varphi'\sin\varphi)a_3^{*}}{a_3^2}\right)$$

where $\bar{f} = \dfrac{\bar{\tau}}{\bar{\kappa}}$.

From Theorem 4.2 and definitions of dual helix and dual slant helix, we can give the following corollaries.

Corollary 4.3. *Let $\tilde{\alpha}(s)$ be a dual curve, $\tilde{d}$ be a constant unit dual vector, $\bar{\varphi} = \varphi + \varepsilon\,\varphi^{*}$ be the dual angle between $\tilde{d}$ and $\tilde{N}$, and φ^{*} be a constant. $\tilde{\alpha}(s)$ is a dual helix if and only if*

$$\frac{a_1 - \varphi'\sin\varphi}{a_3} = \text{constant}$$

and

$$\frac{-\varphi^{*}\varphi'\cos\varphi + a_1^{*}}{a_3} + \frac{-(a_1 - \varphi'\sin\varphi)a_3^{*}}{a_3^2} = \text{constant}$$

Corollary 4.4. *Let $\tilde{\alpha}(s)$ be a dual curve, $\tilde{d}$ be a constant unit dual vector, $\bar{\varphi} = \varphi + \varepsilon\,\varphi^{*}$ be the dual angle between $\tilde{d}$ and $\tilde{N}$, and φ^{*} be a constant. $\tilde{\alpha}(s)$ is a dual slant helix if and only if*

$$\bar{f} = \frac{\bar{\tau}}{\bar{\kappa}} = \frac{\bar{a}_1}{\bar{a}_3}\,.$$

5. Relations between a unit dual vector and the binormal vector of a dual curve

In the dual space a unit speed curve $\tilde{\alpha}(s)$ is called dual $\tilde{B}$-slant helix, if its unit dual binormal vector $\tilde{B}$ makes a constant angle $\bar{\gamma}$ with a fixed unit dual direction $\tilde{d}$; that is, $\left\langle \tilde{B},\tilde{d}\right\rangle = \cos\bar{\gamma}$ is constant along the curve (Lee *et al.*, 2011). From this definition it is clear that both γ and γ^{*} are real constants. In this section, we will consider the relation between the unit dual vector $\tilde{d}$ and unit dual binormal vector $\tilde{B}$, in the case that the dual angle between them is not constant.

Let $\tilde{\alpha}(s)$ be a dual curve, $\tilde{d}$ be a constant unit dual vector and $\left\{ \tilde{T}(s), \tilde{N}(s), \tilde{B}(s) \right\}$ be the dual Frenet trihedron of $\tilde{\alpha}(s)$. If $\bar{\gamma} = \gamma + \varepsilon\,\gamma^{*}$ is the dual angle between the unit dual vectors $\tilde{d}$ and $\tilde{B}$, we can write

$$\left\langle \tilde{B},\tilde{d}\right\rangle = \cos\bar{\gamma} = \cos\gamma - \varepsilon\,\gamma^{*}\sin\gamma \quad (5.1)$$

where $\gamma, \gamma^{*} \in IR$.

Case 1. Let γ be a non-zero constant while γ^{*} is not constant. From E. Study mapping, it means that the lines corresponding to the vector $\tilde{B}$ make a spatial motion in the space such that the angle γ between the lines corresponding to the dual unit vectors $\tilde{d}$ and $\tilde{B}$ is constant while the distance γ^{*} between these lines is not a constant. Differentiating (5.1), we have

$$\left\langle -\bar{f}\,\tilde{N},\tilde{d}\right\rangle = -\varepsilon\,\gamma^{*\prime}\sin\gamma\,. \quad (5.2)$$

Separating (5.2) into real and dual parts and using (3.3) and (3.4), we get

$$-f\,a_2 = 0\,,\ f\,a_2^{*} + f^{*}a_2 = \gamma^{*\prime}\sin\gamma\,. \quad (5.3)$$

Assuming that f is not equal to zero, from (5.3), we have

$$a_2 = 0\,. \quad (5.4)$$

Differentiating (5.2) and separating the result into real and dual parts, we have

$$f'a_2 - fa_1 + f^2 a_3 = 0 \quad (5.5)$$

and

$$f'a_2^{*} + f^{*\prime}a_2 - fa_1^{*} - f^{*}a_1 + f^2 a_3^{*} + 2ff^{*}a_3 = \gamma^{*\prime\prime}\sin\gamma \quad (5.6)$$

By substituting (5.4) into (5.5), we get

$$a_1 = fa_3 \quad (5.7)$$

Since (5.7) equals to (3.11) and $a_2 = 0$, we get

$$f = c \tag{5.8}$$

where c is a constant. Using (3.13), we find

$$a_3 = \frac{1}{\sqrt{1+c^2}}. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7), we get

$$a_1 = \frac{c}{\sqrt{1+c^2}}. \tag{5.10}$$

From (5.1), we have known

$$a_3 = \cos\gamma, \quad a_3^* = -\gamma^* \sin\gamma. \tag{5.11}$$

Using (5.9) and (5.11), we find

$$a_3^* = -\frac{\gamma^* c}{\sqrt{1+c^2}}. \tag{5.12}$$

Substituting (5.4), (5.9), (5.10) and (5.12) into (3.6), we have

$$a_1^* = \frac{\gamma^*}{\sqrt{1+c^2}}. \tag{5.13}$$

Substituting (5.4), (5.8), (5.9), (5.10), (5.12) and (5.13) into (5.6) and considering that $f' = 0$, we find

$$f^* = \gamma^* + c^2\gamma^* + \frac{1}{c}\gamma^{*''}. \tag{5.14}$$

From (5.8) and (5.14), we get

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon\left(\gamma^* + c^2\gamma^* + \frac{1}{c}\gamma^{*''}\right).$$

Now we can give the following theorem.

Theorem 5.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\gamma} = \gamma + \varepsilon\gamma^*$ be the dual angle between unit dual vector $\tilde{\boldsymbol{d}}$ and unit dual binormal vector $\tilde{\boldsymbol{B}}$. If γ is a constant then*

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon\left(\gamma^* + c^2\gamma^* + \frac{1}{c}\gamma^{*''}\right).$$

holds, where $\bar{f} = \frac{\bar{\tau}}{\bar{\kappa}}$.

From Theorem 5.1, we can give the following corollary.

Corollary 5.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\gamma} = \gamma + \varepsilon\gamma^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{B}}$, and γ be a constant. Then $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if the following differential equation holds*

$$\gamma^{*'} + c^2\gamma^{*'} + \frac{1}{c}\gamma^{*'''} = 0.$$

Case 2. Let γ^* be a non-zero constant while γ is not constant. From E. Study mapping, it means that the lines corresponding to the vector $\tilde{\boldsymbol{B}}$ make a spatial motion in the space such that the distance γ^* between the lines corresponding to the dual unit vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{B}}$ is constant while the angle γ between these lines is not a constant. By differentiating (5.1), we have

$$\left\langle -\bar{f}\,\tilde{\boldsymbol{N}}, \tilde{\boldsymbol{d}} \right\rangle = -\gamma'\sin\gamma - \varepsilon\,\gamma^*\gamma'\cos\gamma. \tag{5.15}$$

Separating (5.15) into real and dual parts and using (3.3) and (3.4), we get

$$-f\,a_2 = -\gamma'\sin\gamma \tag{5.16}$$

and

$$f\,a_2^* + f^* a_2 = \gamma^*\gamma'\cos\gamma. \tag{5.17}$$

From (5.16), we find

$$f = \frac{\gamma'\sin\gamma}{a_2}. \tag{5.18}$$

By substituting (5.18) into (5.17), we have

$$f^* = \frac{-\gamma' a_2^* \sin\gamma + \gamma^*\gamma' a_2\cos\gamma}{a_2^2}. \tag{5.19}$$

From (5.18) and (5.19), we find

$$\bar{f} = f + \varepsilon f^* = \frac{\gamma'\sin\gamma}{a_2} + \varepsilon\frac{-\gamma' a_2^* \sin\gamma + \gamma^*\gamma' a_2\cos\gamma}{a_2^2}$$

Now we can give the following theorem.

Theorem 5.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\gamma} = \gamma + \varepsilon\gamma^*$ be the dual angle between the unit dual vector $\tilde{\boldsymbol{d}}$ and the unit dual binormal vector $\tilde{\boldsymbol{B}}$. If γ^* is a constant then*

$$\bar{f} = f + \varepsilon f^* = \frac{\gamma'\sin\gamma}{a_2} + \varepsilon\frac{-\gamma' a_2^* \sin\gamma + \gamma^*\gamma' a_2\cos\gamma}{a_2^2}$$

where $\bar{f} = \frac{\bar{\tau}}{\bar{\kappa}}$.

From Theorem 5.1, we can give the following corollary.

Corollary 5.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\gamma} = \gamma + \varepsilon\gamma^*$ be the dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{B}}$, and γ^* be a constant. $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if*

$$\frac{\gamma' \sin\gamma}{a_2} = \text{constant}$$

and

$$\frac{-\gamma' a_2^* \sin\gamma + \gamma^* \gamma' a_2 \cos\gamma}{a_2^2} = \text{constant} .$$

6. Relations between a unit dual vector and the dual Frenet instantaneous rotation vector of a dual curve

In this section, we will consider the relation between a fixed unit dual vector $\tilde{\boldsymbol{d}}$ and dual unit Frenet instantaneous rotation vector $\tilde{\boldsymbol{W}}_0$. First, we give the following definition.

Definition 6.1. In the dual space a unit speed curve $\tilde{\boldsymbol{\alpha}}(s)$ is called a dual Darboux-slant helix if its unit dual Darboux vector (dual Frenet instantaneous rotation vector) $\tilde{\boldsymbol{W}}_0$ makes a constant dual angle $\bar{\sigma} = \sigma + \varepsilon\, \sigma^*$ with a fixed unit dual direction $\tilde{\boldsymbol{d}}$; that is, $\langle \tilde{\boldsymbol{W}}_0, \tilde{\boldsymbol{d}} \rangle = \cos\bar{\sigma}$ is constant along the curve. Then we have

$$\langle \tilde{\boldsymbol{W}}_0, \tilde{\boldsymbol{d}} \rangle = \cos\bar{\sigma} = \cos\sigma - \varepsilon\, \sigma^* \sin\sigma$$

where $\sigma, \sigma^* \in IR$ are constants.

From this definition it is clear that both σ and σ^* are real constants. Now, we will consider a more general case that the dual angle between these dual unit vectors is not constant. Then

$$\langle \tilde{\boldsymbol{W}}_0, \tilde{\boldsymbol{d}} \rangle = \cos\bar{\sigma} = \cos\sigma - \varepsilon\, \sigma^* \sin\sigma \tag{6.1}$$

is not a constant. Separating (6.1) into real and dual parts, we have

$$f\, a_1 + a_3 = \cos\sigma \tag{6.2}$$

and

$$f^* a_1 + f\, a_1^* + a_3 = -\sigma^* \sin\sigma . \tag{6.3}$$

From (6.2) and (6.3), we find

$$f = \frac{\cos\sigma - a_3}{a_1}$$

and

$$f^* = \frac{-\sigma^* a_1 \sin\sigma - a_1 a_3 - (\cos\sigma - a_3) a_1^*}{a_1^2} .$$

Thus, we get

$$\bar{f} = f + \varepsilon f^* = \frac{\cos\sigma - a_3}{a_1} - \varepsilon \frac{\sigma^* a_1 \sin\sigma + a_1 a_3 + (\cos\sigma - a_3) a_1^*}{a_1^2}$$

Theorem 6.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\sigma} = \sigma + \varepsilon\, \sigma^*$ be non-constant dual angle between unit dual vector $\tilde{\boldsymbol{d}}$ and the unit dual Frenet instantaneous rotation vector $\tilde{\boldsymbol{W}}_0$, then*

$$\bar{f} = f + \varepsilon f^* = \frac{\cos\sigma - a_3}{a_1} - \varepsilon \frac{\sigma^* a_1 \sin\sigma + a_1 a_3 + (\cos\sigma - a_3) a_1^*}{a_1^2}$$

holds, where $\bar{f} = \dfrac{\bar{\tau}}{\bar{\kappa}}$.

From Theorem 6.1, we can give the following corollary.

Corollary 6.1. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\sigma} = \sigma + \varepsilon\, \sigma^*$ be dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{W}}_0$. Then, $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if*

$$\frac{\cos\sigma - a_3}{a_1} = \text{constant}$$

and

$$\frac{-\sigma^* a_1 \sin\sigma - a_1 a_3 - (\cos\sigma - a_3) a_1^*}{a_1^2} = \text{constant}$$

Let now consider the special case that σ is a constant. Differentiating (6.2), we have

$$f' a_1 = 0 .$$

Since f^* can not be found in the case of $a_1 = 0$, we will assume that $f' = 0$. So, we get

$$f = c \tag{6.4}$$

where c is a constant. By substituting (6.4) into (6.3), we find

$$f^* = \frac{-\sigma^* \sin\sigma - c a_1^* - a_3^*}{a_1} . \tag{6.5}$$

From (6.4) and (6.5), we have

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon \frac{-\sigma^* \sin\sigma - c a_1^* - a_3^*}{a_1} .$$

Now we can give the following theorem.

Theorem 6.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector and $\bar{\sigma} = \sigma + \varepsilon\, \sigma^*$ be dual angle between unit dual vectors $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{W}}_0$. If σ is a constant then*

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon \frac{-\sigma^* \sin\sigma - c a_1^* - a_3^*}{a_1}$$

holds, where c is a constant.

From Theorem 6.2, we can give the following corollary.

Corollary 6.2. *Let $\tilde{\boldsymbol{\alpha}}(s)$ be a dual curve, $\tilde{\boldsymbol{d}}$ be a constant unit dual vector, $\bar{\sigma} = \sigma + \varepsilon\,\sigma^*$ be dual angle between $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{W}}_0$ and σ be a constant. Then, $\tilde{\boldsymbol{\alpha}}(s)$ is a dual helix if and only if*

$$\frac{-\sigma^* \sin\sigma - ca_1^* - a_3^*}{a_1} = \text{constant}$$

where c is a constant.

Furthermore, if σ^* is constant in Theorem 6.2, we have that $\tilde{\boldsymbol{\alpha}}(s)$ is a dual Darboux slant helix. Then we have the following corollary.

Corollary 6.3. *If $\tilde{\boldsymbol{\alpha}}(s)$ is a dual Darboux slant helix, then*

$$\bar{f} = f + \varepsilon f^* = c + \varepsilon \frac{n - ca_1^* - a_3^*}{a_1}$$

holds, where $n = -\sigma^* \sin\sigma$ is a real constant.

7. Conclusions

The study of motion of a line in space is an important research area in kinematics. This study can be achieved more efficiently by using dual vector algebra and obtained results can be interpreted. This paper is an example of such studies of special motions. In this paper, the motions of two lines corresponding to a fixed unit dual vector and dual Frenet vectors of a dual curve are studied in detail. A general case of this motion such that the dual angle between dual unit vectors is not constant is studied. Later, some special cases are considered and characterizations for dual helix, dual slant helix and dual Darboux slant helix are obtained.

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علاقات بين متجه الوحدة الثنوي و متجهات فرينية لمنحى ثنوي

[1]براك ساهينر ، [2]مهمت أندر
جامعة جلال بايار– كلية الآداب والعلوم – قسم الرياضيات – 45140 مرادية – مانيسا – تركيا.
البريد الإلكتروني: burak.sahiner@cbu.edu.tr ،mehmet.onder@cbu.edu.tr

خلاصة

نقوم في هذا البحث بتعميم مفاهيم اللولب و اللولب الجانبي في فضاء ثنوي إلى الحالة العامة حيث الزاوية الثنوية بين متجه وحدة ثنوي ثابت و متجهات فرينية لمنحنى ثنوي غير ثابتة . كما ندرس حركة الخطوط المقابلة لمتجهات فرينية الثنوية و متجهات فرينية الدورانية اللحظية بالنسبة إلى خط ثابت. و نحصل بذلك على بعض خصائص هذه المنحنيات الثنوية و نثبت أن هذه الخصائص تنطبق على حالة اللولب الثنوي . واللولب الثنوي الجانبي في بعض الحالات الخاصة.