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Finite simple groups with some abelian Sylow subgroups

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Abstract

In this paper, we classify the finite simple groups with an abelian sylow subgroup.

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1. Introduction

Sylow subgroups are very important subgroups of finite groups. Some well-known theorems on the structure of groups link their Sylow subgroups. In particular, groups with some abelian Sylow subgroups have been widely researched. At the same time, some problems related to abelian Sylow subgroups are put forward. Since the condition of abelian Sylow subgroups inherits to subgroups and quotient groups, some problem will reduce to simple having some abelian Sylow subgroups.

Walter (1969) classified finite non-abelian simple groups with abelian Sylow 2-subgroups, which are $L_2(q)$, where $q = 2^f$ ($f \geq 2$) or $q \equiv 3,5$ (mod 8), J_1 or ${}^2G_2(q)$, where $q = 3^{2m+1}$ and $m \geq 1$. Also the structure of Sylow subgroups is studied by Weir (1955). He proved that the Sylow r-subgroups of the classical groups over $GF(q)$ with q prime to r and r odd are expressible as direct products of groups defined inductively by $G_i := G_{i-1}\ wr\ C_p$, where G_0 is an abelian p-subgroup of the classical group and C_p a cyclic group of order p.

In this paper, we use Artin (1959) invariant of the cyclotomic factorization (see Artin (1959)) to classify the finite simple groups with an abelian Sylow subgroup.

Theorem. Let S be a finite non-abelian simple group. Suppose that S has an abelian Sylow r-subgroup and r an odd prime. Then S is one of the following groups:

(1) an alternating group A_n with $n < r^2$,

(2) a linear simple group $PSL_2(q)$ with $r \mid q$,

(3) $PSL_3(q)$, $q \equiv 4,7 \pmod 9$ with $r = 3$,

(4) $PSU_3(q^2)$, $2 < q \equiv 2,5 \pmod 9$ with $r = 3$,

(5) a simple group of Lie type over the Galois field $GF(q)$ and $e_L(mr_m) = 0$ except the above groups of (3) and (4), where $r = r_m$ is a primitive prime of $q^m - 1$, the function $e_L(x)$ is defined in Table 1 and Table 2,

(6) a sporadic simple groups listed in Table 4.

From above theorem, we can get the following result.

Corollary. Let S be a finite simple group. Suppose that S has an abelian Sylow r-subgroup R. Then R is isomorphic to some direct products of copies of a cyclic r-group.

Afterwards we consider the classification of finite simple groups having an elementary abelian Sylow subgroup. In the following, all considered groups are finite. Recall that a section of a group G is a quotient of a subgroup of G. Denote by G^k the group of $G \times \cdots \times G$ with k times. The notations $G.K$ and $G:K$ mean an extension and a split extension of the group G by K, respectively. We denote $p^s \parallel n$ if $p^s \mid n$ but p^{s+1} not divide n for a prime p and natural number n.

2. Alternating groups

Theorem 2.1. Let r be an odd prime. Then the alternating group A_n has a Sylow abelian r-subgroup if and only if $n > r^2$.

Proof. Since r is an odd prime, Sylow r-subgroups of A_n are ones of the symmetric group $Sym(n)$. We use the well known structure of Sylow subgroups of symmetric groups. Now let r^s be the largest number such that $r^s \le n$. If $[\frac{n}{r^s}] \ge 1$ and $s \ge 2$, then there exists a subgroup $C_r \ wr \ C_r$, which is not abelian. So Sylow r-subgroups of A_n are not abelian. Next we assume that $s = 1$ and $[\frac{n}{s}] \le r - 1$. Then Sylow r-subgroups of A_n is $C_r^{[\frac{n}{r}]}$, which is an elementary abelian group.

Next we give a result on the alternating groups having a Sylow r-subgroup isomorphic to $C_r \times C_r$.

Corollary 2.2. Let r be an odd prime. Then the alternating group $A_n (n \ge 5)$ has a Sylow r-subgroup $C_r \times C_r$ if and only if $2r \le n < 3r$.

3. Simple groups of Lie type

In this section, we will discuss simple groups of Lie type over the Galois field $GF(q)$. For any natural number m let $\Phi_m(x)$ denote the mth cyclotomic polynomial. So the degree of $\Phi_m(x)$ is the Euler function $\phi(x)$ and $x^m-1=\prod_{d|m}\Phi_d(m)$ by the book of Ribenboin (1989). Recall that a primitive prime divisor (or Zsigmondy prime) r_m of q^m-1 is a prime such that $r_m\,|\,(q^m-1)$ but $r^m\,|\,(q^i-1)$ for $1\le i\le m-1$. A well-known theorem by Zsigmondy(1892) asserts that primitive primes of q^m-1 exist except if $(q,m)=(2,6)$ or $m=2$ and $q=2^k-1$.

Lemma 3.1. Let q and m be natural numbers. Then $\Phi_1(q)\equiv -1$ (mod q) and $\Phi_m(q)\equiv 1$ (mod q) for $m\ge 2$.

Proof. Clearly the result is true for $m=1$ and 2. Next we prove the remaining case by induction on m. Since $q^m-1=\Phi_1(q)\Phi_m(q)\prod_{d|m,d\ne 1,m}\Phi_d(q)$, we have $q^m-1\equiv\Phi_1(q)\Phi_m(q)\equiv -\Phi_m(q)$ (mod q) by induction, and hence $\Phi_m(q)\equiv 1-q^m\equiv 1$ (mod q) whenever $m\ge 2$.

Lemma 3.2 (Ribenboin (1989)). Let q, m be positive integers with $q\ge 2$ and r_m a primitive prime of q^m-1. Then r_m divides $\Phi_m(q)$.

Lemma 3.3 (Malle *et al.* (2006)). Let r_m be an odd primitive prime of q^m-1. Then $r_m\,|\,\Phi_m(q)$ if and only if $n=mr_m^j$ for some $j\ge 0$.

Lemma 3.4. Let r_m be an odd primitive prime of q^m-1 and $j\ge 1$. Then $r_2^j\,\|\,\Phi_{2r_2^j}(q)$ and $r_m\,\|\,\Phi_{mr_m^j}(q)$ for $m\ne 2$.

Proof. The proof of the case $m\ge 3$ is given by Lemma 2.1 of Feit (1988) or Lemma 1 of Artin (1959). Next we prove the cases $m=1$ and 2.

First, we consider the case $m=1$. Now we set $q-1\equiv k$ (mod r_1^2) with $0\le k\le r_1^2-1$. Since $r_1\,|\,(q-1)$, we have $r_1\,|\,k$, and let $k=lr_1$. Then $q\equiv lr_1+1$ (mod r_1^2) where $0\le l\le r_1-1$. Moreover

$$\Phi_{r_1^j(q)}=\frac{q^{r_1^j-1}}{q^{r_1^{j-1}-1}}=q^{r_1^j-r_1^{j-1}}+q^{r_1^j-2r_1^{j-1}}+\ldots+q^{r_1^j-(r_1-1)r_1^{j-1}}+1$$

so that $\Phi_{r_1^j}(q)\equiv r_1$ (mod r_1^2), and then $r_1\,\|\,\Phi_{r_1^j}(q)$.

We next discuss the case $m=2$. Similarly, we let $q+1\equiv k$ (mod r_2^{j+1}) with $0\le k\le r_2^{j+1}-1$. Since $r_2\,|\,(q+1)$, we have $r_2\,|\,k$, and let $k=lr_2$. Then $q\equiv lr_2-1$ (mod r_2^{j+1}) where $0\le l\le r_2^j-1$. Since $\Phi_{2r_2^j}(q)=\sum_{i=0}^{r_2^j-1}(-1)^i q^i$, it follows that

$$\Phi_{2r_2^j}(q)\equiv\sum_{i=0}^{r_2^j-1}(-1)^i(lr_2-1)^i=\frac{(lr_2-1)^{r_2^j}+1}{lr_2}=\sum_{i=1}^{r_2^j}(-1)^{i-1}\binom{r_2^j}{i}(lr_2)^{i-1}\equiv r_2^j\ (\text{mod}\ r_2^{j+1})$$

and thus $r_2^j\,\|\,\Phi_{2r_2^j}(q)$.

In the following, let $L(q)$ be a simple group of Lie type of charactersic p. The order of $L(q)$ has the cyclotomic factorization in terms of q:

$$|L(q)| = \frac{1}{d} q^h \prod_m \Phi_m(q)^{e_L(m)} \tag{1}$$

where d is the denominator and h the exponent given for $L(q)$ in Table 1 and Table 2, where $\Phi_m(q)$ is the cyclotomic polynomial for the primitive mth roots of unity, and where the $e_L(m)$ are the exponents deducible from the factors. The function $e_L(x)$ is defined in Table 1 and Table 2 by Kimmerle *et al.* (1990).

Table 1. Cyclotomic factorisation: classical groups of Lie type

L	d	h	$e_L(x)$ where x is an integer
$PSL_n(q)$	gcd(n,$q-1$)	$n(n-1)/2$	$n-1$ if $x=1$ $\left[\frac{n}{x}\right]$ if $x>1$
$PSU_n(q^2)$	gcd(n,$q+1$)	$n(n-1)/2$	$\left[\frac{n}{lcm(2,x)}\right]$ if $x \not\equiv 2 \pmod 4$ $n-1$ if $x=2$ $\left[\frac{2n}{x}\right]$ if $2<x\equiv 2 \pmod 4$
$PSp_{2n}(q)$ $P\Omega_{2n+1}(q)$	gcd(2,$q-1$)	n^2	$\left[\frac{2n}{lcm(2,x)}\right]$
$P\Omega^+_{2n}(q)$	gcd(4,q^n-1)	$n(n-1)$	$\left[\frac{2n}{lcm(2,x)}\right]$ if $x \mid n$ or $x \nmid 2n$ $\frac{2n}{x}-1$ if $x \nmid n$ and $x \mid 2n$
$P\Omega^-_{2n}(q)$	gcd(4,q^n+1)	$n(n-1)$	$\left[\frac{2n}{lcm(2,x)}\right]-1$ if $x \mid n$ $\left[\frac{2n}{lcm(2,x)}\right]$ if $x \nmid n$

Table 2. Cyclotomic factorisation: exceptional groups of Lie type

m	${}^2B_2$	${}^3D_4$	G_2	${}^2G_2$	F_4	${}^2F_4$	E_6	${}^2E_6$	E_7	E_8
1	1	2	2	1	4	2	6	4	7	8
2		2	2	1	4	2	4	6	7	8
3		2	1		2		3	2	3	4
4	1				2	2	2	2	2	4
5							1		1	2
6		2	1	1	2	1	2	3	3	4
7									1	1
8					1		1	1	1	2
9							1		1	1
10								1	1	2
12		1			1	1	1	1	1	2
14									1	1
15										1
18								1	1	1
20										1
24										1
30										1
d	1	1	1	1	1	1	$\gcd(3, q-1)$	$\gcd(3, q+1)$	$\gcd(2, q-1)$	1
h	2	12	6	3	24	12	36	36	63	120

Note that for Table 2, the numbers in the row headings are those integers m for which a cyclotomic polynomial $\Phi_m(q)$ enters into the cyclotomic factorisation in terms of q of one of the exceptional groups $L(q)$ of Lie type according to the formula (1). The types L are the headings, and the entry in the m-row and the L-column is $e_L(m)$ if non-zero and blank otherwise.

Moreover, let R_m be the set of all primitive primes of $q^m - 1$. Denote by $Z_m(q) = \prod_{r_m \in R_m} |\Phi_m(q)|_{r_m}$, where $|\Phi_m(q)|_{r_m}$ means the r_m's maximum prime power divisor of $|\Phi_m(q)|$. Then we have

$$|L(q)| = \frac{1}{d} q^h \prod_m \prod_{r_m \in R_m} Z_m(q)^{e_L(m)} r_m^{\tilde{e}_L(m)} \tag{2}$$

where $\widetilde{e}_L(m)$ is the number of divisor r_m of all possible $\Phi_{m'}(q)$ such that $m' > m$. Note that we define $Z_6(2) = 1$ and $Z_2(r) = 1$, where r is a Mersenne prime.

Lemma 3.5. Keep the above notations. Suppose that r_m is an odd prime. Then $\tilde{e}_L(m) = \sum_{j=1}^{\infty} e_L(mr_m^j)$, if $m = 1$ or $m \geq 3$; $\widetilde{e}_L(m) = \sum_{j=1}^{\infty} je_L(mr_m^j)$, if $m = 2$ for the simple groups of Lie type, where the function $e_L(x)$ is given by Table 1 and Table 2.

Proof. We consider the general term $\Phi_m(q)^{e_L(m)}$ to give a proof. By Lemma 3.3 and Lemma 3.4, it follows that the r_m-part of $\Phi_{mr_m^j}(q)$ is r_m whenever $m \neq 2$, and r_m^j for $m = 2$. So we have $\Phi_{mr_m^j}(q)^{e_L(mr_m^j)} = (r_m Z_{mr_m^j}(q))^{e_L(mr_m^j)}$ if $m \neq 2$, and equals $(r_m^j Z_{mr_m^j}(q))^{e_L(mr_m^j)}$ if $m = 2$. Thus $\widetilde{e}_L(m) = \sum_{j=1}^{\infty} e_L(mr_m^j)$ for $m \neq 2$, and $\tilde{e}_L(m) = \sum_{j=1}^{\infty} je_L(mr_m^j)$, for $m = 2$.

We compute directly in term of Lemma 3.1 and the formulas of Lemma 3.5, the following result can be obtained.

Lemma 3.6. Keep the above notations. Suppose that r_m is an odd primitive prime of $q^m - 1$. Then $\widetilde{e}_L(m) = 0$ for the exceptional simple groups of Lie type except the cases listed in Table 3.

Table 3. The function $\widetilde{e}_L(m)$ for exceptional groups of Lie type

r_m	${}^3D_4$	G_2	F_4	${}^2F_4$	E_6	${}^2E_6$	E_7	E_8
$r_1 = 3$	2	1	2		4	2	4	5
$r_1 = 5$					1		1	2
$r_1 = 7$							1	1
$r_2 = 3$	2		2	1	2	4	4	5
$r_2 = 5$						1	1	2
$r_2 = 7$							1	1
$r_4 = 5$								1

Next we classify the simple groups of Lie type of characteristic p which have an abelian Sylow p-subgroup.

Theorem 3.7. Let L be a simple group of Lie type over the finite field $GF(q)$ of characteristic p. Then S has an abelian Sylow p-group if and only if L is isomorphic to $PSL_2(q)$.

Proof. It is known that the Sylow p-subgroup of L is a maximal unipotent subgroup. We use the notations of Vdovin (2001), let $a(\Phi, p)$ be the maximum of orders of abelian p-subsets of the root system Φ^+ by Mal'tsev (1945). By hypothesis, the Sylow p-subgroup is abelian, so we have $q^h < q^{a(\Phi,p)}$ for classical simple groups (where $a(\Phi, p)$ is listed in Mal'tsev (1945)), and $q^h < q^{a(U)}$ for exceptional simple groups (where $a(U)$ in Table 4 of Vdovin (2001)). Therefore, L has an abelian Sylow p-group if and only if L is isomorphic to $PSL_2(q)$.

Next we give a necessary and sufficient condition for a simple group of Lie type to have an abelian Sylow r-subgroups and r is prime to the characteristic.

Theorem 3.8. Let L be a simple group of Lie type over the Galois field $GF(q)$ and r_m an odd primitive prime of $q^m - 1$. Then L has an abelian Sylow r_m-subgroup if and only if $e_L(mr_m) = 0$ except the following two cases:

(1) $r_1 = 3$, $L = PSL_3(q)$ and $q \equiv 4,7 \pmod 9$;

(2) $r_2 = 3$, $L = PSU_3(q^2)$ and $2 < q \equiv 2,5 \pmod 9$.

Proof. We use Propositions 7-10 of Carter (1972,1981)'s results on maximal tori of simple groups of Lie type to prove. We divide into several cases.

Case 1: $L = PSL_n(q)$ with $n \geq 2$. Then every maximal torus T of $PSL_n(q)$ has order $\frac{1}{\gcd(n,q-1)(q-1)}(q^{n_1}-1)(q^{n_2}-1)...(q^{n_k}-1)$ for appropriate partition $n = n_1 + n_2 + \cdots + n_k$.

We first suppose the $m \geq 2$. Then there exists a maximal torus has the order $\frac{1}{\gcd(n,q-1)(q-1)}(q^m-1)^{\left[\frac{n}{m}\right]}$, in the terms of partition $n = m + m + \cdots + m + l$, where $0 \leq l < m$ so that the largest order of abelian r_m-subgroups is the r_m-part of $(q^m-1)^{\left[\frac{n}{m}\right]}$. Thus Sylow r_m-subgroups are abelian if and only if $\widetilde{e}_L(m) = 0$, which is equivalent to the condition $e_L(mr_m) = 0$ in terms of Lemma 3.5. Next we assume $m = 1$, that is $r_1 \mid (q-1)$. We split it into two cases, r_1 does not divide n, and then r_1 divides n, as well.

Subcase 1: r_1 does not divide n. Then there exists a maximal torus of order $\frac{1}{\gcd(n,q-1)}(q-1)^{n-1}$ whose r_1-part is the maximum order of the abelian r_1-subgroup. Thus Sylow r_1-subgroups are abelian if and only if $\widetilde{e}_L(r_1) = 0$.

Subcase 2: r_1 divide n. Let $r_1^s \parallel \gcd(n, q-1)$ and $r_1^t \parallel (q-1)$. We first consider the group $GL_n(q)$. The Sylow r_1-subgroup of $GL_n(q)$ is isomorphic to one of the group M of monomial matrices which have one nonzero entry in each row and column, so they are the product of a permutation matrix and a diagonal matrix, that is $M \cong D : Sym(n)$, where D is the group of the set diagonal matrices. Thus a Sylow r_1-subgroup of $SL_n(q)$ is a Sylow subgroup of $M_1 = D_1 : Sym(n)$, where D_1 is the set all diagonal matrices of determinant 1. Therefore, a Sylow r_1-subgroup of $M_1 / Z(SL_n(q))$ is a Sylow r_1-subgroup of $PSL_n(q)$. Moreover, we note that $M_1 / Z(SL_n(q)) \cong D_1 / Z(SL_n(q)) : Sym(n)$. Obviously, $D_1 / Z(SL_n(q))$ is an abelian group of order $\dfrac{(q-1)^{n-1}}{\gcd(n,q-1)}$, so that the Sylow r_1-subgroup of $PSL_n(q)$ is abelian if and only if the r_1-parts of $|D_1 / Z(SL_n(q)) : Sym(n)|$ and $n!$ equals 1 and $r_1^i (i = 1,2)$, or r_1 and r_1 respectively. Then $n = r_1$, $t = 1$ and $r_1 = 3$. Thus the group $PSL_3(q)$ and $3 \| (q-1)$, that is $PSL_3(q)$ and $q \equiv 4,7 \pmod 9$.

Case 2: $L \cong PSU_n(q^2)$ with $n \ge 3$. Then every maximal torus T of $PSU_n(q^2)$ has the order $\dfrac{1}{\gcd(n,q+1)(q+1)}(q^{n_1} - (-1)^{n_1})(q^{n_2} - (-1)^{n_2})...(q^{n_k} - (-1)^{n_k})$ for appropriate partition $n = n_1 + n_2 + ... + n_k$. Now we first assume that $m \ne 2$, if m is an odd number, then there exists a maximal torus whose order is $\dfrac{1}{\gcd(n,q+1)(q+1)}(q^{2m} - 1)^{\left[\frac{n}{m}\right]}$ by the partition $n = 2m + 2m + ... + 2m + l$, where $0 \le l < 2m$. Furthermore, the r_m-part of this order is one of the largest order of abelian r_m-subgroup. Since $e_{PSU_n(q)}(m) = \left[\dfrac{n}{2m}\right]$, it follows that Sylow r_m-subgroups are abelian if and only if $\widetilde{e}_L(m) = 0$, that is the condition $e_L(mr_m) = 0$. If $4 | m$, we choose the partition $n = m + m + \cdots + m + l$ and $0 \le l < m$, then the order of corresponding torus is $\dfrac{1}{\gcd(n,q+1)(q+1)}(q^m + 1)^{\left[\frac{n}{m}\right]}$. If $2 < m \equiv 2 \pmod 4$, then there is the order of a torus $\dfrac{1}{\gcd(n,q+1)(q+1)}(q^{\frac{m}{2}} + 1)^{\left[\frac{2n}{m}\right]}$ by the partition $n = m/2 + m/2 + \cdots + m/2 + l$, where $0 \le l < m/2$. So the largest order of abelian r_m-subgroup is $e_L(m)$, and then the Sylow r_m-subgroups are abelian if and only if $e_L(mr_m) = 0$.

Next we let $m = 2$. If r_2 does not divide n, then there exists a maximal torus has order $\dfrac{1}{\gcd(n,q+1)}(q+1)^{n-1}$ by the partition $n = 1 + 1 + \cdots + 1$. Since $e_L(2) = n - 1$, it follows that Sylow r_2-subgroups are abelian if and only if $\widetilde{e}_L(2) = 0$, that is $e_L(2r_2)$. When $r_2 | n$, since Sylow r_2-subgroup of $SU_n(q^2)$ is one of $SL_n(q^2)$, the Sylow r_2-subgroup of $PSU_n(q^2)$ is one of the group $\overline{M} : Sym(n)$, where $\overline{M}$ is a abelian group of order $\dfrac{(q^2-1)^{n-1}}{\gcd(n,q+1)}$. Assume that $r_2^t \| (q+1)$. Then Sylow r_1 subgroups are abelian if and only if $n = r_2 = 3$ and $t = 1$, that is the group $PSU_3(q^2)$ and $q \equiv 2,5 \pmod 9$.

Case 3: $L \cong PSp_{2n}(q)$ $(n \geq 3)$, $P\Omega_{2n+1}(q)$, $(n \geq 2)$. Every maximal torus has order $\frac{1}{\gcd(2,q-1)}(q^{n_1}-1)(q^{n_2}-1)...(q^{n_k}-1)(q^{l_1}+1)(q^{l_2}+1)...(q^{l_t}+1)$ for appropriate partition $n = n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_t$. Now we choose the partition $n = m + m + \cdots + m + l$ if m is odd and $n = m/2 + m/2 + \cdots + m/2 + l$ if m is even, where $0 \leq l < \frac{m}{\gcd(2,m)}$. So the largest order of abelian r_m-subgroup is a divisor of $\Phi_m(q)^{e_L(m)}$, and then the Sylow r_m-subgroups are abelian if and only if $e_L(mr_m) = 0$.

Case 4: $L \cong P\Omega_{2n}^{\varepsilon}(q)$, $\varepsilon \in \{+,-\}$. Then every maximal torus has the order $\frac{1}{\gcd(2,q^n-\varepsilon 1)}(q^{n_1}-1)(q^{n_2}-1)...(q^{n_k}-1)(q^{l_1}+1)(q^{l_2}+1)...(q^{l_t}+1)$ for appropriate partition $n = n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_t$ of n, where t is even if $\varepsilon = +$, and t is odd if $\varepsilon = -$. So whenever $r_m \neq 2$, it see that Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$ by the previous method.

Case 5: $L \cong {}^2B_2(q)$, $q = 2^{2n+1}$. Since there exist cyclic subgroups of order $q-1, 2^{2n+1}+2^{n+1}+1$ and $2^{2n+1}-2^{n+1}+1$ (see Theorem 4.2 of Wilson (2009)), we have Sylow r-subgroup are cyclic except $r = 2$. On the other hand, $e_L(mr_m) = 0$ by Lemma 3.6.

Case 6: $L \cong {}^3D_4(q)$, By Theorem 4.1 of Wilson (2009), the simple group ${}^3D_4(q)$ has subgroups $SL_2(q^3), SL_3(q), SU_3(q^2)$, $(C_{q^2+q+1} \times C_{q^2+q+1}) : SL_2(3)$, $(C_{q^2-q+1} \times C_{q^2-q+1}) : SL_2(3)$ and $C_{q^4-q^2+1}$. Let $r_m^t \parallel (q^m - 1)$. If $m = 1$ and $r_1 \neq 3$. then there is abelian subgroup $C_{q-1} \times C_{q-1}$ of $SL_3(q)$, and so Sylow r_1-subgroup of ${}^3D_4(q)$ is abelian. When $r_1 = 3$, the Sylow r_1-subgroup is not abelian. If $m = 2$ and $r_2 \neq 3$, then Sylow 3-subgroup is not abelian by listed subgroups. If $m = 3,6,12$, then Sylow subgroups are $C_{r_3^t} \times C_{r_3^t}, C_{r_6^t} \times C_{r_6^t}$ and $C_{r_{12}^t}$, respectively. These are all abelian. Thus the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$.

Case 7: $L \cong G_2(q)$, By the section 4.2.6 of Wilson (2009) there exist subgroups $GL_2(q), SL_3(q)$ and $SU_3(q)$ of $G_2(q)$, and then if $r \neq r_1 (= 3)$, then Sylow r-subgroup is abelian. Now if $r_1 = 3$, then Sylow 3-subgroup of $G_2(q)$ is not abelian. So Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$.

Case 8: $L \cong {}^2G_2(q)$ and $q = 3^{2n+1} \geq 27$. In terms of Table 3, we know that $e_L(mr_m) = 0$ for any m. Moreover, Sylow r_m-subgroups are cyclic by Theorem 4.3 of Wilson (2009), and so Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$.

Case 9: $L \cong F_4(q)$. Since ${}^3D_4(q)$ is a subgroup of $F_4(q)$ (see Section 4.8 of Wilson (2009)), we have Sylow r_6 and r_{12}-subgroup of $F_4(q)$ are ones of ${}^3D_4(q)$ (see Table

1 and Table 2), which are abelian by the above Case 6. Furthermore, $F_4(q)$ has a subgroup $C_2^2.P\Omega_8^+(q)$ (see Section 4.8 of Wilson (2009)), then Sylow r_4 and r_8-subgroups of $P\Omega_8^+(q)$ are ones of $F_4(q)$, which are abelian by the above Case 4. Since $F_4(q)$ has the maximal torus $C_{q\pm1}^4$ and $C_{q^2-q+1}^2$ (also see Section 4.8 of Wilson (2009)), Sylow r_m-subgroup is abelian if and only if $e_L(mr_m)=0$ for $m=1,2,3$.

Case 10: $L\cong{}^2F_4(q)$ and $q=2^{2n+1}\geq 8$. Since there exists a subgroup $SU_3(q)$ (see Theorem 4.4, Wilson (2009)) and Sylow r_2, r_6-subgroups are ones of ${}^2F_4(q)$ (see Table 1 and Table 2), by the discussion of Case 2, we have these Sylow subgroups are abelian if and only if $e_L(mr_m)=0$. Moreover, ${}^2F_4(q)$ has subgroups $Sz(q)\times C_{q-1}, C^2_{2^{2n+1}\pm 2^{n+1}+1}$ and $C_{q^2+q+1\pm 2^{n+1}(q+1)}$ (also see Theorem 4.4, Wilson (2009)), let $r^m \,\|\, (q^m-1)$, and then the Sylow r_i-subgroup is $C^2_{r_i^t}$ for $i=1,4$, and Sylow r_{12}-subgroup is cyclic. So Sylow r_m-subgroup is abelian if and only if $e_L(mr_m)=0$.

Case 11: $L\cong E_6(q)$. Since $F_4(q)$ is a subgroup of $E_6(q)$ (see Section 4.6.4, Wilson (2009)), we have Sylow r_i-subgroups of $E_6(q)$ are ones of $F_4(q)$ for $i=2,4,6,8,12$ (see Table 1 and Table 2). By the proof of Case 9, it follows that the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m)=0$ for $m=2,4,6,8,12$. Also the group $E_6(q)$ has subgroups $PSL_2(q)\times PSL_5(q)\times C_{q-1}$ if $r_1\neq 3,5$ and $PSL_3(q)^3$ (also see Section 4.6.4, Wilson (2009)). So in this case there exist abelian subgroups $C^6_{r_1^t}, C^3_{r_3^t}$ and $C_{r_5^t}$ provided $r_i^t \,\|\, (q^i-1)$ for $i=1,2,5$, but the order of the maximal torus whose r_1, r_3 and r_5-part are the largest are $(q-1)^6, (q^2+q+1)^3$ and q^5-1, respectively. Then the Sylow $r_1(\neq 3,5)$ and r_3-subgroup is abelian if and only if $e_L(r_1)=0$. Also there exists a cyclic subgroup of order q^6+q^3+1, so Sylow r_9-subgroups are cyclic. If $r_1=5$, since $E_6(q)$ has the subgroup $P\Omega_{10}^+(q)$ whose Sylow 5-subgroups are non-abelian (see Case 4), then the Sylow $r_1(=5)$-subgroup is abelian if and only if $e_L(r_1)=0$. Next we consider $r_1=3$, and use the same notation of the section 4.6.1 of Wilson (2009) to denote by $GE_6(q)$ the group of matrices which multiply the determinant by a scalar, so that $GE_6(q)\cong C_3.(C_{\frac{q-1}{3}}.E_6(q)).C_3$. Then Sylow 3-subgroup of $E_6(q)$ is not abelian.

Case 12: $L\cong{}^2E_6(q)$. Since $F_4(q)$ is a subgroup of ${}^2E_6(q)$ (see Section 4.11 of Wilson (2009)), we have Sylow r_i-subgroups of ${}^2E_6(q)$ is ones of $F_4(q)$ for $i=1,3,4,8,12$. Similarly, by the proof of Case 9, it follows that the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m)=0$ for $m=1,3,4,8,12$. It is known that $PSU_6(q)<{}^2E_6(q)$, we have Sylow r_{10}-subgroups are cyclic. Obviously, Sylow r_{18}-subgroups are also cyclic. Since there is a maximal torus $C^3_{q^2-q+1}$, we have Sylow r_6-subgroups are abelian. Next if $r_2\neq 3$, then the order of a maximal torus whose r_2-part is the largest is $(q+1)^6$, and so that Sylow r_2-subgroups are not abelian. If $r_2=3$, similar to the case $E_6(q)$, then Sylow 3-subgroup is not abelian.

Case 13: $L \cong E_7(q)$. Since $E_6(q)$ is a subgroup of $E_7(q)$ (see Section 4.12 of Wilson (2009)), Sylow r_m-subgroups of $E_6(q)$ are ones of $E_7(q)$ for $m = 3,4,5,8,9,12,18$ (see Table 1 and Table 2). So by Case 11, it follows that the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$ for $m = 3,4,5,8,9,12,18$. Also there is a section of subgroups of $E_7(q)$ which is isomorphic to ${}^2E_6(q)$ (also see Section 4.12 of Wilson (2009)), but Sylow r_m-subgroups for $m = 6,10$ of ${}^2E_6(q)$ are ones of $E_7(q)$, and then the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$ for $m = 6,10$. Since there exist sections $PSU_8(q)$ and $PSL_7(q)$ of subgroups of $E_7(q)$, so the Sylow r_m-subgroup is abelian if and only if $e_L(mr_m) = 0$ for $m = 7,14$. Next we consider $m = 1,2$. Since $r_1 = 2$, by the structure of maximal torus we have the largest r_1 and r_2-part of abelian subgroups are ones of $(q-1)^7$ and $(q+1)^7$, and so Sylow r_1 and r_2-subgroups are abelian if and only if $e_L(mr_m) = 0$.

Case 14: $L \cong E_8(q)$. Since $E_7(q)$ and $P\Omega_{16}^+$ are subgroups of $E_8(q)$ (see Section 4.12 of Wilson (2009)), the Sylow r_m-subgroup of $E_8(q)$ is abelian if and only if $e_L(mr_m) = 0$ for $m = 7,8,9,14,18$. Also ${}^3D_4(q) \times {}^3D_4(q)$ is a subgroup of $E_8(q)$ (also see Section 4.12 of Wilson (2009)), then the result is true for $m = 3,6,12$. By the structure of maximal torus we have Sylow r_m-subgroups for $m = 15,20,24,30$ are cyclic, and Sylow r_5, r_{10}-subgroups are ones of $C^2_{q^4+q^3+q^2+q+1}$ and $C^2_{q^4-q^3+q^2-q+1}$ respectively. So Sylow r_5, r_{10}-subgroups of $E_8(q)$ are ablelian. For remaining cases of $m = 1,2,4$, we use the maximal torus. Since a maximal torus whose m-part is the largest for $m = 1,2,4$ are the groups $C^4_{q\pm1}$ and $C^4_{q^2+1}$, so Sylow r_m-subgroups are abelian if and only if $e_L(mr_m) = 0$.

From the above proof, we can get the following corollary.

Corollary 3.9. Let L be a simple group of Lie type over the Galois field $GF(q)$ and r_m an odd primitive prime of $q^m - 1$ and $r_m^t \parallel (q^m - 1)$. Suppose that L has an abelian Sylow r_m-subgroup R. Then $R \cong C^{e_L(m)}_{r_m^t}$, except the following cases:

(1) $r_1 = 3, L \cong PSL_3(q)$ and $q \equiv 4,7 \pmod 9$;

(2) $r_2 = 3, L \cong PSU_3(q^2)$ and $2 < q \equiv 2,5 \pmod 9$.

Next we discuss simple groups of Lie type which have an elementary abelian Sylow subgroup. In terms of Corollary 3.9, it is sufficient to satisfy with the condition $r_m \parallel (q^m - 1)$. First, we give a lemma.

Lemma 3.10. Let r_m be a primitive prime of $q^m - 1$ and e a primitive root of r_m. Let $e_i \equiv e^{\frac{(r_m-1)i}{m}} \pmod{r_m}$, where $\gcd(i,m) = 1, 1 \le i \le m$ and $1 \le e_i \le r_m$. Then

(1) $r_1 \parallel (q-1)$ if and only if $q \equiv 1 + r_1, 1 + 2r_1, \ldots, 1 + (r_1 - 1)r_1 \pmod{r_1^2}$;

(2) for $m \geq 2$, $r_m \| (q^m - 1)$ if and only if $q \equiv e_i, e_i + r_m, e_i + 2r_m, \ldots, e_i + (r_m - 1)r_m \pmod{r_m^2}$.

Proof. Let $q \equiv x \pmod{r_m^2}$ so that $1 \leq x \leq r_m^2$. Then we have $q^m - 1 \equiv x^m - 1 \pmod{r_m^2}$. Hence $r_m \| (q^m - 1)$ if and only if the order of the element x in the cyclic group $C_{r_m - 1} \cong \langle e \rangle$ is m. But the set of elements x of order m in $\langle e \rangle$ is $\{e_i \mid e_i \equiv e^{\frac{(r_m-1)i}{m}} \pmod{r_m}$, $\gcd(i, m) = 1$, $1 \leq i \leq m, 1 \leq e_i \leq r_m\}$, and then $q \equiv e_i, e_i + r_m, e_i + 2r_m, \ldots, e_i + (r_m - 1)r_m$, if $m \geq 2$. For the case of $m = 1$, clearly, $e_1 = 1$, and if $q \equiv 1 \pmod{r_1^2}$, then it contradicts $r_1 \| (q-1)$. So $q \equiv 1 + r_m, 1 + 2r_m, \ldots, 1 + (r_1 - 1)r_1 \pmod{r_1^2}$.

Theorem 3.11. Let L be a simple group of Lie type over the Galois field $GF(q)$, r_m an odd primitive prime of $q^m - 1$ and e a primitive root of r_m. Set $e_i \equiv e^{\frac{(r_m-1)i}{m}} \pmod{r_m}$, where $\gcd(i, m) = 1$, $1 \leq i \leq m$ and $1 \leq e_i \leq r_m$. We have the following conditions:

(1) if $(L, r_1) \neq (PSL_3(q), 3)$, then L has an elementary abelian Sylow r_1-subgroup if and only if $q \equiv 1 + r_1, 1 + 2r_1, \ldots, 1 + (r_1 - 1)r_1 \pmod{r_1^2}$, and

(2) for $m \geq 2$, if $(L, r_m) \neq (PSU_3(q^2), 3)$, then L has an elementary abelian Sylow r_m-subgroup if and only if $q \equiv e_i, e_i + r_m, e_i + 2r_m, \ldots, e_i + (r_m - 1)r_m \pmod{r_m^2}$.

Finally, we give an example of classification on simple groups having a Sylow subgroup $C_5 \times C_5$.

Example 3.12. Let L be a simple group of Lie type over finite field $GF(q)$. Suppose that S has a Sylow subgroup $C_5 \times C_5$. Then L is one of following groups:

(1) $PSL_2(25)$;

(2) $PSL_3(q), PSU_4(q^2), PSp_4(q), P\Omega_5(q), {}^3D_4(q), G_2(q)$ where $q \equiv 6, 11, 16, 21 \pmod{25}$;

(3) $PSL_4(q), PSL_5(q), PSU_3(q^2), P\Omega_5(q), {}^3D_4(q), G_2(q)$ where $q \equiv 4, 9, 14, 19, 24 \pmod{25}$;

(4) $PSL_8(q), PSL_9(q), PSL_{10}(q), PSL_{11}(q), PSU_8(q^2), PSU_9(q^2), PSU_{10}(q^2), PSU_{11}(q^2)$, $P\Omega_8^+(q), P\Omega_{10}^+(q), P\Omega_{12}^+(q), PSp_8(q), PSp_{10}(q), P\Omega_9(q), P\Omega_{11}(q), P\Omega_8^-(q), P\Omega_{10}^-(q), P\Omega_{12}^-(q)$, $F_4(q), {}^2F_4(q), (q = 2^{2n+1} \geq 8), E_6(q), {}^2E_6(q), E_7(q)$, where $q \equiv 2, 3, 7, 8, 12, 13, 17, 18, 22, 23 \pmod{25}$.

4. Sporadic simple groups

Next we give Table 4 to list the information of Sylow subgroups of sporadic simple groups. Note that we denote by "+" the simple group S having abelian Sylow r-subgroup if "+" in the position of S-row and r-column, the blank otherwise. By the

Atlas of finite groups written by Conway *et al.* (1985), we know that if $r \geq 17$ and $r \| S |$, then $r \| S |$, and so Sylow r-subgroups are cyclic. We omit the prime divisors which are greater than or equal to 17 in Table 4.

Table 4. The abelian Sylow r-subgroup of Sporadic Simple Groups

	3	5	7	11	13
M_{11}	+	+	+		
M_{12}		+	+		
J_1	+	+	+	+	
M_{22}	+	+	+	+	
J_2		+	+		
M_{23}	+	+	+	+	
HS	+		+	+	
J_3		+			
M_{24}		+	+	+	
M^cL			+	+	
He		+			
Ru			+	+	
Suz			+	+	+
O'N		+		+	
Co_3			+	+	
Co_2			+	+	
Fi_{22}		+	+	+	
HN			+	+	
Ly			+	+	
Th			+		+
Fi_{23}		+	+	+	
Co_1			+	+	+
J_4		+	+		
Fi'_{24}		+		+	+
B			+	+	+
M				+	

Moreover, we can know that the abelian Sylow subgroups must be an elementary abelian group for 26 sporadic simple groups.

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خلاصة

نقوم في هذا البحث بتصنيف الزمر المنتهية البسيطة التي لها زمر جزئية سايلو أبيلية.