*Kuwait J. Sci.* **42 (1)** pp. 1-15, 2015

# On a subclass of $p$-valent analytic functions involving fractional $q$-calculus operators

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## ABSTRACT

In this paper, we consider a new subclass of $p$-valent analytic functions in the open unit disk involving a fractional $q$-differeintegral operator. For this subclass of functions, we derive the coefficient inequality and some distortion theorems. Special cases of the main results are also mentioned.

**Keywords:** Analytic functions; $p$-valent functions; fractional $q$-calculus operators; coefficient bounds; distortion theorems.

## INTRODUCTION

The theory of $q$-analysis has been applied in many areas of applied mathematics, see the various references cited in Purohit & Raina (2011). Recently, authors in Purohit & Raina (2011), Purohit & Raina (2013), Purohit & Raina (2014) have used the fractional $q$-calculus operators in investigating certain classes of functions which are analytic in the open unit disk. Purohit (2012) also studied similar work and considered new classes of multivalently analytic functions in the open unit disk.

In the present paper, we consider a new subclass of functions defined by applying the fractional $q$-calculus operators which are $p$-valent and analytic in the open unit disk. Among the results derived include, the coefficient inequalities and distortion theorems for the subclasses defined and introduced below. Special cases of the results are also pointed out in the concluding section of this paper.

## PRELIMINARIES AND DEFINITIONS

To make this paper reasonably self-contained, we give here some basic definitions and related details of the $q$-calculus.

The $q$-shifted factorial (see Gasper & Rahman, 1990) is defined for $\alpha, q \in \mathbb{C}$ as a product of $n$ factors by

$$(\alpha;q)_n = \begin{cases} 1 & ; \quad n=0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & ; \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

and in terms of the basic analogue of the gamma function by

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n>0), \tag{2}$$

where the $q$-gamma function is defined by (Gasper & Rahman, 1990 [p. 16, Equation (1.10.1)])

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \, (0<q<1). \tag{3}$$

If $|q|<1$, the definition (1) remains meaningful for $\infty$, as a convergent infinite product given by

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

We recall here the following $q$-analogue definitions given by Gasper & Rahman (1990).

The recurrence relation for $q$-gamma function is given by

$$\Gamma_q(x+1) = \frac{(1-q^x)\Gamma_q(x)}{1-q} \tag{4}$$

and the $q$-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{5}$$

It may be noted that the $q$-Gauss hypergeometric function (Gasper & Rahman, 1990 [p.3, Equation (1.2.14)]) is defined by

$${}_2\Phi_1\left[\alpha,\beta;\gamma;q,z\right] = \sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n} z^n \qquad (|q|<1,|z|<1), \tag{6}$$

and as a special case of the above series for $\gamma=\beta$, we have

$${}_1\Phi_0\left[\alpha;-;q,z\right] = \sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n} z^n \qquad (|q|<1,|z|<1). \tag{7}$$

Also, the Jackson's $q$-derivative and $q$-integral of a function $f$ defined on a subset of $\mathbb{C}$ are, respectively, given by (Gasper & Rahman, 1990 [pp. 19, 22])

$$D_{q,z}f(z) = \frac{f(z)-f(zq)}{z(1-q)} \qquad (z\neq 0, q\neq 1) \tag{8}$$

and

$$\int_0^z f(t)d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \tag{9}$$

By noting the relation that

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{10}$$

we observe that the $q$-shifted factorial (1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)\ (n\in\mathbb{N})$.

The fractional $q$-calculus operators of a complex-valued function $f(z)$ (see Purohit and Raina, 2011) are contained in the following definitions.

Definition 1 (Fractional $q$-Integral Operator). The fractional $q$-integral operator $I_{q,z}^{\alpha} f(z)$ of a function of order is defined by

$$I_{q,z}^{\alpha} f(z) \equiv D_{q,z}^{-\alpha} f(z) = \frac{1}{\Gamma_q(\alpha)}\int_0^z (z-tq)_{\alpha-1} f(t)d_q t \quad (\alpha>0), \tag{11}$$

where $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin. In view of the relation (5), the $q$-binomial function $(z-tq)_{\alpha-1}$ can be expressed as

$$(z-tq)_{\alpha-1} = z^{\alpha-1}\,{}_1\Phi_0[q^{-\alpha+1};-;q,tq^\alpha/z]. \tag{12}$$

The series ${}_1\Phi_0[\alpha;-;q,z]$ is obviously single-valued when $|\arg(z)|<\pi$ and $|z|<1$, (see for details Gasper & Rahman, 1990 [pp. 104-106]), therefore, in view of the representation of the integral defined by (11), it may be noted that the function $(z-tq)_{\alpha-1}$ in (11) is single-valued when $|\arg(-tq^\alpha/z)|<\pi, |tq^\alpha/z)|<1$ and $|\arg z|<\pi$. Thus, for suitably selected function $f(z)$ (which ensures its convergence), the operator (11) is well defined.

**Definition 2** (Fractional $q$-Derivative Operator). The fractional $q$-derivative operator $D_{q,z}^{\alpha} f(z)$ of a function $f(z)$ of order $\alpha$ is defined by

$$D_{q,z}^{\alpha} f(z) = D_{q,z}\, I_{q,z}^{1-\alpha} f(z) = \frac{1}{\Gamma_q(1-\alpha)} D_{q,z}\int_0^z (z-tq)_{-\alpha} f(t)d_q t \quad (0\le\alpha<1), \tag{13}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 1 above.

**Definition 3.** (Extended Fractional $q$-Derivative Operator). Under the hypotheses of Definition 2, the fractional $q$-derivative for a function $f(z)$ of order $\alpha$ is defined by

$$D_{q,z}^{\alpha} f(z) = D_{q,z}^{m} I_{q,z}^{m-\alpha} f(z) \quad (m-1 \leq \alpha < m; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{14}$$

where $\mathbb{N}$ denotes the set of natural numbers.

We note for $\alpha = 1$ in (13) that

$$D_{q,z}^{1} f(z) = D_{q,z} f(z), \tag{15}$$

where the operator $D_{q,z}$ is given by (8).

In the sequel, we shall be using the following image formulas (Purohit & Raina, 2011 [pp. 58-59]) which are easily derivable, respectively, from the operators (11) and (14):

$$I_{q,z}^{\alpha} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda+\alpha)} z^{\lambda+\alpha} \quad (\alpha > 0, \lambda > -1) \tag{16}$$

and

$$D_{q,z}^{\alpha} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-\alpha)} z^{\lambda-\alpha} \quad (\alpha \geq 0, \lambda > -1). \tag{17}$$

By adopting the fractional $q$-integral and fractional $q$-derivative operators defined, respectively, by (11) and (13), we first introduce a fractional $q$-differintegral operator and use it to construct a class of $p$-valent analytic functions $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$ in the open unit disk $\mathbb{U}$ (defined below by (25)). Besides finding the coefficient bound inequality, we also obtain some distortion inequalities for functions belonging to this class of functions. Our investigations provide the $q$-analogues to the classes studied earlier in Owa & Srivastava (1985), Raina & Nahar (2002) and Shukla & Dashrath (1984). More precise related details describing the usefulness of investigating such a subclass of functions are stated briefly in the concluding section of this paper.

## A CLASS OF ANALYTIC FUNCTIONS

Let $\mathcal{A}_p(n)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \qquad (n, p \in \mathbb{N}), \tag{18}$$

which are analytic and $p$-valent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, let $\mathcal{A}_p^{-}(n)$ denote the subclass of $\mathcal{A}_p(n)$ consisting of $p$-valent analytic functions given by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0, n, p \in \mathbb{N}). \tag{19}$$

For the purpose of this paper, we define a fractional $q$-differintegral operator $\Omega_{q,z}^{\alpha,p}$ of a function $f(z)$ by

$$\Omega_{q,z}^{\alpha,p} f(z) = \frac{\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)} z^{\alpha-p} D_{q,z}^{\alpha} f(z) \tag{20}$$

$$(\alpha < p+1; p \in \mathbb{N}; 0 < q < 1; z \in \mathbb{U}),$$

where $D_{q,z}^{\alpha} f(z)$ in (20) represents, respectively, a fractional $q$-integral of $f(z)$ of order $\alpha$ when $-\infty < \alpha < 0$, and a fractional $q$-derivative of $f(z)$ of order $\alpha$ when $\alpha \geq 0$. Thus, in view of (19), the operator (20) has the series representation given by

$$\Omega_{q,z}^{\alpha,p} f(z) = 1 - \sum_{k=n+p}^{\infty} \Delta_k(\alpha,p,q) a_k z^{k-p} \qquad (-\infty < \alpha < p+1), \tag{21}$$

where

$$\Delta_k(\alpha,p,q) = \frac{\Gamma_q(1+p-\alpha)\Gamma_q(1+k)}{\Gamma_q(1+p)\Gamma_q(1+k-\alpha)}. \tag{22}$$

By the application of Cauchy-Hadamard formula, the radius of convergence of the series in (21) can be calculated by the formula that

$$R = \underset{k\to\infty}{Lim} \left| a_k \Delta_k(\alpha,p,q) \right|^{1/k}.$$

As the series (19) is analytic in unit disk, so it implies that

$$\underset{k\to\infty}{Lim} \left| a_k \right|^{1/k} \geq 1.$$

Now

$$\underset{k\to\infty}{Lim} \left| \Delta_k(\alpha,p,q) \right|^{1/k} = \underset{k\to\infty}{Lim} \left| \frac{\Gamma_q(1+p-\alpha)\Gamma_q(1+k)}{\Gamma_q(1+p)\Gamma_q(1+k-\alpha)} \right|^{1/k}$$

$$= \underset{k\to\infty}{Lim} \left| \frac{\Gamma_q(1+p-\alpha)}{\Gamma_q(1+p))} \right|^{1/k} \underset{k\to\infty}{Lim} \left| \frac{\Gamma_q(1+k)}{\Gamma_q(1+k-\alpha)} \right|^{1/k}.$$

Upon using the familiar asymptotic formula (Ismail & Muldoon, 1994) given by

$$\Gamma_q(x) \approx (1-q)^{1-x} \prod_{n=0}^{\infty} (1-q^{n+1}) \quad (x \to \infty, 0 < q < 1),$$

it can be verified that

$$\underset{k\to\infty}{Lim} \left| \Delta_k(\alpha,p,q) \right|^{1/k} = \underset{k\to\infty}{Lim} \left| (1-q)^{\alpha} \right|^{1/k} \underset{k\to\infty}{Lim} \left| \frac{1}{(1-q)^{\alpha}} \right|^{1/k} = 1.$$

It readily follows that $\mathbf{R} \geq 1$, and hence, the series representation of $\Omega_{q,z}^{\alpha,p} f(z)$ given by the series (21) is analytic in the unit disk.

**Remark.** In view of (17), we express (21) as follows:

$$L_{q,z}^{\alpha,p} f(z) = 1 - \sum_{k=n+p}^{\infty} \Delta_k(\alpha,p,q) a_k z^{k-p} \quad (0 \leq \alpha < p+1), \tag{23}$$

where $\Delta_k(\alpha,p,q)$ is given by the Equation (22).

On the other hand, on using (16), we write

$$M_{q,z}^{\alpha,p} f(z) = 1 - \sum_{k=n+p}^{\infty} \Delta_k(-\alpha,p,q) a_k z^{k-p} \quad (\alpha > 0), \tag{24}$$

where $\Delta_k(-\alpha,p,q)$ is given by the same Equation (22).

Evidently, it follows that

$$\Omega_{q,z}^{\alpha,p} f(z) = L_{q,z}^{\alpha,p} f(z) \cup M_{q,z}^{\alpha,p} f(z),\ z \in \mathbb{U}.$$

Let $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$ represent the class of functions $f(z) \in \bar{\mathcal{A}}_p(n)$, which satisfy the condition that

$$\left| \frac{\Omega_{q,z}^{\alpha,p} f(z) - 1}{b\,\Omega_{q,z}^{\alpha,p} f(z) - a} \right| < \sigma \tag{25}$$

$$(-\infty < \alpha < p+1, -1 \leq a < b \leq 1, 0 < b \leq 1, 0 < \sigma \leq 1, 0 < q < 1, z \in \mathbb{U}),$$

where the operator $\Omega_{q,z}^{\alpha,p}$ is given by (20).

Further, in view of the relationships in (15) and (4), we find from (20) that

$$\Omega_{q,z}^{1,p} f(z) = \left( \frac{1-q}{1-q^p} \right) z^{1-p} D_{q,z} f(z) \tag{26}$$

$$(p \in \mathbb{N}; 0 < q < 1; z \in \mathbb{U}).$$

Thus, when $\alpha = \sigma = 1$, the condition (25) reduce to the inequality:

$$\left| \left( \frac{D_{q,z} f(z)}{z^{p-1}} - [p]_q \right) \left( \frac{b\,D_{q,z} f(z)}{z^{p-1}} - a\,[p]_q \right)^{-1} \right| < 1 \tag{27}$$

$$(p \in \mathbb{N}, -1 \leq a < b \leq 1, 0 < b \leq 1, 0 < q < 1, z \in \mathbb{U}),$$

where the $q$-natural number is expressed as

$$[n]_q = \frac{1-q^n}{1-q} \quad (0<q<1), \tag{28}$$

and we have

$$\mathcal{H}^1_{q,p}(a,b,1) = \mathcal{G}_{q,p}(a,b), \tag{29}$$

where $\mathcal{G}_{q,p}(a,b)$ is a subclass of $p$-valent analytic functions which satisfy the condition (27).

We now obtain the following coefficient bounds for functions of the form (19) belonging to the classes $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$ and $\mathcal{G}_{q,p}(a,b)$ (defined above).

**Theorem 1.** A function *f* of the form (19) belongs to the class $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$ if and only if

$$\sum_{k=p+n}^{\infty} \Delta_k(\alpha,p,q)(1+b\sigma)a_k \le \sigma(b-a), \tag{30}$$

where $\Delta_k(\alpha,p,q)$ is given by (22). The result is sharp.

**Proof.** Assume that the inequality (30) holds true, and let $|z|=1$, then on using (19) and (21), we obtain

$$\begin{aligned}\left|\Omega^{\alpha,p}_{q,z} f(z)-1\right| - \sigma\left|b\,\Omega^{\alpha,p}_{q,z} f(z)-a\right| &= \left|-\sum_{k=p+n}^{\infty}\Delta_k(\alpha,p,q)a_k z^{k-p}\right| \\ &\quad -\sigma\left|b-a-\sum_{k=p+n}^{\infty}\Delta_k(\alpha,p,q)a_k z^{k-p}\right| \\ &\le \sum_{k=p+n}^{\infty}\Delta_k(\alpha,p,q)(1+b\sigma)a_k - \sigma(b-a) \\ &\le 0,\end{aligned}$$

by our hypothesis. This implies that $f(z)\in \mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$.

To prove the converse, let $f(z)$ defined by (19) be in the class $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$, then it follows that

$$\begin{aligned}\left|\frac{\Omega^{\alpha,p}_{q,z} f(z)-1}{b\,\Omega^{\alpha,p}_{q,z} f(z)-a}\right| &= \left|-\sum_{k=p+n}^{\infty}\Delta_k(\alpha,p,q)a_k z^{k-p}\right| \\ &\quad \times\left|b-a-\sum_{k=p+n}^{\infty}\Delta_k(\alpha,p,q)a_k z^{k-p}\right|^{-1} < \sigma.\end{aligned} \tag{31}$$

Since $|\Re(z)| \le |z|$ for any $z$, therefore, choosing values of $z$ on the real axis so that $\Omega_{q,z}^{\alpha,p} f(z)$ is real, and letting $z \to 1^-$ through real values, we obtain from (31) the following inequality:

$$\sum_{k=p+n}^{\infty} \Delta_k(\alpha, p, q) a_k \le \sigma(b-a) - \sigma \sum_{k=p+n}^{\infty} \Delta_k(\alpha, p, q) a_k, \tag{32}$$

which yields the desired result (30).

Finally, we note that the assertion (30) of Theorem 1 is sharp and the extremal function is given by

$$f(z) = z^p - \frac{\sigma(b-a)}{(1+b\sigma)\,\Delta_{p+n}(\alpha,p,q)} z^{p+n} \quad (n \in \mathbb{N}),$$

or

$$f(z) = z^p - \frac{\sigma(b-a)\,(q^{1+p-\alpha};q)_n}{(1+b\sigma)\,(q^{1+p};q)_n} z^{p+n} \quad (n \in \mathbb{N}). \tag{33}$$

We consider here giving few examples of the functions that belong to the new subclass of the $p$-valent functions by suitably choosing the bounded sequence $a_k$, so that the sum function (19) is convergent. To this end, let us choose

$$a_k = \frac{(q^\lambda;q)_k}{(q;q)_k} \quad (\lambda \ge 0),$$

then from (19), we have

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{(q^\lambda;q)_k}{(q;q)_k} z^k \quad (\lambda \ge 0, n, p \in \mathbb{N}),$$

which on changing the summation index: $k-n-p=j$ and using (2) yields

$$f(z) = z^p - \frac{\Gamma_q(\lambda+n+p)}{\Gamma_q(\lambda)\Gamma_q(1+n+p)} z^{n+p} \sum_{j=0}^{\infty} \frac{(q^{\lambda+n+p};q)_j\,(q;q)_j}{(q^{1+n+p};q)_j\,(q;q)_j} z^j \quad (\lambda \ge 0, n, p \in \mathbb{N}).$$

Now, on making use of (6), we get

$$f(z) = z^p - \frac{\Gamma_q(\lambda+n+p)}{\Gamma_q(\lambda)\Gamma_q(1+n+p)} z^{n+p} {}_2\Phi_1[q^{\lambda+n+p}, q; q^{1+n+p}; q, z] \quad (\lambda \ge 0, n, p \in \mathbb{N}). \tag{34}$$

Thus, for the aforementioned sequence $a_k = \dfrac{(q^\lambda;q)_k}{(q;q)_k}$, and by making use of (34) and Theorem 1, we obtain the following result:

**Example 1.** A function *f* of the form (34) belongs to the class $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$ if and only if

$$\sum_{k=p+n}^{\infty} \frac{\Delta_k(\alpha,p,q)(1+b\sigma)(q^{\lambda};q)_k}{(q;q)_k} \leq \sigma(b-a), \tag{35}$$

where $\Delta_k(\alpha,p,q)$ is given by (22). The result is sharp.

An obvious consequence of Example 1 would be a simple case which occurs when $a_k = 1$ (that is $\lambda = 1$ in Equation (34)), we would readily get the following result for the function:

$$f(z) = z^p - z^{n+p} \, {}_1\Phi_0[q;-;q,z] \quad (n,p \in \mathbb{N}). \tag{36}$$

**Example 2.** The function $z^p - z^{n+p} \, {}_1\Phi_0[q;-;q,z]$ belongs to the class $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$ if and only if

$$\sum_{k=p+n}^{\infty} \Delta_k(\alpha,p,q)(1+b\sigma) \leq \sigma(b-a), \tag{37}$$

where $\Delta_k(\alpha,p,q)$ is given by (22). The result is sharp.

Let us set $\alpha = \sigma = 1$ and make use of the relation (29), then Theorem 1 yields the following coefficient inequality for the class $\mathcal{G}_{q,p}(a,b)$.

**Corollary 1.** A function *f* of the form (19) belongs to the class $\mathcal{G}_{q,p}(a,b)$ if and only if

$$\sum_{k=p+n}^{\infty} [k]_q (1+b) a_k \leq (b-a)[p]_q. \tag{38}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{(b-a)(1-q^p)}{(1+b)(1-q^{p+n})} z^{p+n} \quad (n \in \mathbb{N}). \tag{39}$$

## DISTORTION THEOREMS

In this section, we establish certain distortion theorems for the function classes defined above involving the fractional *q*-calculus operators.

**Theorem 2.** Corresponding to the inequalities:

$$-\infty < \alpha < p+1, p,n \in \mathbb{N}; -1 \leq a < b \leq 1, 0 < b \leq 1; 0 < \sigma \leq 1; 0 < q < 1,$$

let the function $f(z)$ defined by (19) be in the class $\mathcal{H}^{\alpha}_{q,p}(a,b,\sigma)$, then

$$\left| |f(z)| - |z|^p \right| \leq \frac{\sigma(b-a)\,(q^{1+p-\alpha};q)_n}{(1+b\sigma)\,(q^{1+p};q)_n} |z|^{p+n} \qquad (z \in \mathbb{U}). \tag{40}$$

Furthermore,

$$\left| |\mathcal{M}_{q,z}^{\alpha,p} f(z)| - |z|^p \right| \leq \frac{\sigma(b-a)}{1+b\sigma} |z|^{p+n} \qquad (z \in \mathbb{U}), \tag{41}$$

where the normalized fractional $q$-calculus operator $\mathcal{M}_{q,z}^{\alpha,p}$ is given by

$$\mathcal{M}_{q,z}^{\alpha,p} f(z) = z^p\, \Omega_{q,z}^{\alpha,p} f(z) = \frac{\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)} z^{\alpha} D_{q,z}^{\alpha} f(z). \tag{42}$$

**Proof.** Since $f(z) \in \mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$, then in view of Theorem 1, we first show that the function

$$\varphi(k) = \Delta_k(\alpha,p,q) = \frac{\Gamma_q(k+1)\Gamma_q(p+1-\alpha)}{\Gamma_q(p+1)\Gamma_q(k+1-\alpha)} \quad (k \geq p+1;\, p \in \mathbb{N})$$

is a non-increasing function of $k$ for $-\infty < \alpha < p+1, 0 < q < 1$.

Now, we have

$$\frac{\varphi(k+1)}{\varphi(k)} = \frac{\Gamma_q(k+2)\Gamma_q(k+1-\alpha)}{\Gamma_q(k+1)\Gamma_q(k+2-\alpha)} = \frac{1-q^{k+1}}{1-q^{k+1-\alpha}} \quad (k \geq p+1;\, p \in \mathbb{N}),$$

and it is sufficient to consider here the value $k = p+1$, so that on using (4), we get

$$\frac{\varphi(p+2)}{\varphi(p+1)} = \frac{1-q^{p+2}}{1-q^{p+2-\alpha}} \quad (0 < q < 1).$$

The function $\phi(k)$ is a decreasing function of $k$ if $\frac{\varphi(p+2)}{\varphi(p+1)} \leq 1\ (p \in \mathbb{N})$, and this gives

$$\frac{1-q^{p+2}}{1-q^{p+2-\alpha}} \leq 1 \quad (0 < q < 1).$$

Multiplying the above inequality both sides by $1-q^{p+2-\alpha}$ (provided that $\alpha < p+1$), and we are at once lead to the inequality $\alpha \leq 0$. Thus, $\varphi(k)\ (k \geq p+1;\, p \in \mathbb{N})$ is a decreasing function of $k$ for $-\infty < \alpha < p+1, 0 < q < 1$.

Consequently from (30), we get the following inequality:

$$\Delta_{p+n}(\alpha,p,q) \sum_{k=p+n}^{\infty} a_k \leq \sum_{k=p+n}^{\infty} \Delta_k(\alpha,p,q) a_k \leq \frac{\sigma(b-a)}{1+b\sigma)},$$

which in view of Equation (4) implies that

$$\sum_{k=p+n}^{\infty} a_k \le \frac{\sigma(b-a)\,(q^{p+1-\alpha};q)_n}{(1+b\sigma)\,(q^{p+1};q)_n},$$

and this last inequality in conjunction with the inequality (easily obtainable from (19)):

$$|z|^p - |z|^{p+n} \sum_{k=p+n}^{\infty} a_k \le |f(z)| \le |z|^p + |z|^{p+n} \sum_{k=p+n}^{\infty} a_k, \tag{43}$$

yields the assertions (40) of Theorem 2.

Next, on using (21) and (42), we observe that

$$\left|\mathcal{M}_{q,z}^{\alpha,p} f(z)\right| \ge |z|^p - \sum_{k=p+n}^{\infty} \Delta_k(\alpha,p,q) a_k |z|^k$$

$$\ge |z|^p - |z|^{p+n} \sum_{k=p+n}^{\infty} \Delta_k(\alpha,p,q) a_k,$$

which on using Theorem 1 gives

$$\left|\mathcal{M}_{q,z}^{\alpha,p} f(z)\right| \ge |z|^p - \frac{\sigma(b-a)}{1+b\sigma}|z|^{p+n}. \tag{44}$$

Similarly, it follows that

$$\left|\mathcal{M}_{q,z}^{\alpha,p} f(z)\right| \le |z|^p - \frac{\sigma(b-a)}{1+b\sigma}|z|^{p+n}, \tag{45}$$

which establishes the assertion (41) of Theorem 2.

In view of Equations (20) and (42), the assertion (41) of Theorem 2 on using (23) gives the following distortion inequality for the function $\mathbb{L}_{q,z}^{\alpha,p} f(z) = z^p L_{q,z}^{\alpha,p} f(z) \in \mathcal{A}_p^-(n)$ involving the fractional $q$-derivative operator $D_{q,z}^{\alpha}$ defined by (14).

**Corollary 2.** Let the function $f(z)$ defined by (19) be in the class $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$, then

$$\left|\left|\mathbb{L}_{q,z}^{\alpha,p} f(z)\right| - |z|^p\right| \le \frac{\sigma(b-a)}{1+b\sigma}|z|^{p+n}, \tag{46}$$

where $0 \le \alpha < p+1$, $z \in \mathbb{U}$.

Also, in view of (20) and (42), the assertion (41) of Theorem 2 on using (24) gives the following inequality for the function $\mathbb{M}_{q,z}^{\alpha,p} f(z) = z^p M_{q,z}^{\alpha,p} f(z) \in \mathcal{A}_p^-(n)$ involving the fractional $q$-integral operator $I_{q,z}^{\alpha}$ defined by (11).

**Corollary 3.** Let the function $f(z)$ be in the class $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$, then

$$\left| \left| \mathbb{M}_{q,z}^{\alpha,p} f(z) \right| - |z|^{p} \right| \leq \frac{\sigma(b-a)}{1+b\sigma} |z|^{p+n}, \tag{47}$$

where $\alpha > 0$, $z \in \mathbb{U}$.

Next, on setting $\alpha = \sigma = 1$ and making use of relationship (29), Theorem 2 yields the following distortion inequality for the function $f(z) \in \bar{\mathcal{A}}_p$ involving the $q$-derivative operator $D_{q,z}$:

**Corollary 4.** Let $p, n \in \mathbb{N}$ satisfy the inequalities:

$$-1 \leq a < b \leq 1, 0 < b \leq 1; 0 < q < 1,$$

and let the function $f(z)$ defined by (19) be in the class $\mathcal{G}_{q,p}(a,b)$, then

$$\left| |f(z)| - |z|^{p} \right| \leq \frac{(b-a)(1-q^{p})}{(1+b)(1-q^{p+n})} |z|^{p+n} \qquad (z \in \mathbb{U}). \tag{48}$$

Furthermore,

$$\left| \left| D_{q,z} f(z) \right| - [p]_q \, |z|^{p-1} \right| \leq \frac{b-a}{1+b} [p]_q \, |z|^{p+n-1}. \tag{49}$$

## CONCLUDING OBSERVATIONS AND REMARKS

We now briefly consider some consequences of the results derived in the preceding sections. If we let $q \to 1^{-}$, and make use of the limit formula (10), we observe that the function class $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$, and the inequality (30) of Theorem 1, and also the inequalities (40) and (41) of Theorem 2 provide, respectively, the $q$-extensions of the main class and related inequalities due to Owa & Srivastava (1985) and Shukla & Dashrath (1984). Similarly, for the function class $\mathcal{G}_{q,p}(a,b)$ defined by (29), Corollary 1 and the distortion inequalities (48) and (49) of Corollary 4 are also the $q$-extensions of the corresponding known function class and related results due to Shukla & Dashrath (1984). Further, it may be noted that the function class $\mathcal{H}_{q,p}^{\alpha}(a,b,\sigma)$ is the $q$-extension of the known class due to Raina & Nahar (2002).

We also observe that if we set $a = 2\delta - 1\,(0 < \delta \leq 1)$, $b = 1$ and replace $\sigma$ by $\beta$, then the condition (25) reduces to the inequality:

$$\left| \frac{\Omega_{q,z}^{\alpha,p} f(z) - 1}{\Omega_{q,z}^{\alpha,p} f(z) - 2\delta + 1} \right| < \beta \tag{50}$$

$$(\alpha < p+1, 0 \leq \delta < 1, 0 < \beta \leq 1, 0 < q < 1, z \in \mathbb{U}),$$

and we have

$$\mathcal{H}_{q,p}^{\alpha}(2\delta-1,1,\beta)=\mathcal{J}_{q,p}(\delta,\beta,\alpha), \tag{51}$$

where $\mathcal{J}_{q,p}(\delta,\beta,\alpha)$ is precisely the subclass of analytic and multivalent functions studied recently by Purohit (2012). Hence, if we set $a=2\delta-1\,(0<\delta\leq 1)$, $b=1$ and $\sigma=\beta$ and use (51), then Theorems 1 and 2 yield (for $n=1$) the coefficient inequality and distortion theorems obtained recently by Purohit, 2012 [p. 132, Theorem 1 and p.134, Theorem 2].

Lastly, we note that if we put $p=1$, then the operator (20) reduces to

$$\Omega_{q,z}^{\alpha,1}f(z)\equiv\Omega_{q,z}^{\alpha}f(z)=\frac{\Gamma_q(2-\alpha)}{\Gamma_q(2)}z^{\alpha-1}D_{q,z}^{\alpha}f(z), \tag{52}$$

and the condition (50) reduces to the inequality:

$$\left|\frac{\Omega_{q,z}^{\alpha}f(z)-1}{\Omega_{q,z}^{\alpha}f(z)-2\delta+1}\right|<\beta \tag{53}$$

$$(\alpha<2, 0\leq\delta<1, 0<\beta\leq 1, 0<q<1, z\in\mathbb{U}).$$

Thus, we have

$$\mathcal{J}_{q,1}(\delta,\beta,\alpha)=\mathcal{J}_{q,\delta}^{\alpha}, \tag{54}$$

where $\mathcal{J}_{q,\delta}^{\alpha}$ is precisely the subclass of analytic and univalent functions studied recently by Purohit & Raina (2011).

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***Submitted*** ***:*** 02/09/2013

***Revised*** ***:*** 28/01/2014

***Accepted*** ***:*** 11/02/2014

# حول صنف جزئي لدوال تحليلية لها مؤثرات كسرية حسبانية

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## خلاصة

ندرس في هذا البحث صنف جزئي جديد لدوال تحليلية لها q قيم على قرص الوحدة المفتوح، وتشتمل على مؤثر حسبا تكاملي. ونحسب لهذا الصنف من الدوال معاملات المتباينة وبعض مبرهنات التحريف. نقوم كذلك بدراسة بعض الحالات الخاصة لنتائجنا الأساسية.

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صدر العدد الأول في
يناير ١٩٧٧


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## المراسلات


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مجلة الحقوق - جامعة الكويت ص.ب: ٦٤٩٨٥ الشويخ - ب 70460 الكويت
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عنوان المجلة في شبكة الإنترنت http://www.pubcouncil.kuniv.edu.kw/jol

ISSN 1029 - 6069