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An optimization problem for some nonlinear elliptic equation

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Abstract

In this paper we prove existence and uniqueness of the optimal solution for an optimization problem related to a nonlinear elliptic equation. We use the concept *tangent cones* to derive the optimality condition satisfied by optimal solution.

Keywords: Existence; optimality condition; optimization; tangent cones; uniqueness

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1. Introduction

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$. For $f \in L^2(\Omega)$ and $\lambda > 0$, we consider the following boundary value problem, which appears as a steady state in a model of chemical reaction phenomenon, see (Pao, 1997):

$$\begin{cases} -\Delta u + \lambda |u|^{p-1} u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $n \geq 3$ and $1 < p < (n+2)/(n-2)$. In the next section, we show that the boundary value problem (1) has a unique nonnegative solution. The reader can see the well known paper by Brézis & Nirenberg (1983), which deals $\Delta u + g(x,u) + u^{\frac{n+2}{n-2}} = 0$, $u > 0$. They explain what happens when $p = (n+2)/(n-2)$ and the loss of solutions if $\lambda < 0$.

It is well known that $u_f \in H_0^1(\Omega)$ is a weak solution of (1) whenever u_f is a critical point of the functional $J(f,.) : H_0^1(\Omega) \to \mathrm{R}$ defined by

$$J(f,u) := (p+1)\int_\Omega fu \, \mathrm{d}x - \lambda \int_\Omega |u|^{p+1} \, \mathrm{d}x - \frac{p+1}{2}\int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

We define the functional $\Phi : L^2(\Omega) \to \mathrm{R}$ by

$$\Phi(f) := J(f,u_f) = \max_{u \in H_0^1(\Omega)} J(f,u), \tag{2}$$

We are interested in the following optimization problem

$$\inf_{f \in A} \Phi(f), \tag{3}$$

where

$$A = \left\{ f \in L^\infty(\Omega) : 0 \le f \le 1, \int_\Omega f \, dx = \alpha \right\}.$$

We'll prove that there exists a unique function (optimal solution) $\hat{f} \in A$ such that $\Phi(\hat{f}) = \min_{f \in A} \Phi(f)$. Also, we derive the optimality condition satisfied by optimal solution. The main mathematical tool that we will use in the last section, is the notion of *tangent cones*.

In Emamizadeh & Zivari-Rezapour (2007), we have investigated the optimization problem (3) when A be a rearrangement class of a fixed function in $L^2(\Omega)$. In Kurata *et al*. (2004), the authors have investigated the minimization problem (3) when $f = \chi_D$, D is a measurable subset of Ω, in (1), $0 < \alpha \le |\Omega|$ and $A = \{D \subset \Omega : |D| = \alpha\}$. Here $|E|$ denotes the n-dimensional Lebesgue measure of $E \subseteq \Omega$. They proved that $\hat{f} = \hat{D} = \{x \in \Omega : u_{\hat{f}}(x) \ge t\}$ for some $t > 0$.

Finally, we mention that this paper is motivated by a very interesting paper authored by Henrot & Maillot (2001). They used the concept tangent of cones to formulate the optimality conditions satisfied by optimal solutions.

2. Existence and uniqueness

First we show that the boundary value problem (1) has a unique nonnegative solution.

Since the function $x \to |x|^n$, $x \in \mathbb{R}^n$, is strictly convex, we deduce $-J(f,.)$ is strictly convex. By Hölder and Poincaré inequalities we have

$$-J(f,u) \ge C\frac{p+1}{2}\|u\|^2 - (p+1)\|f\|_2\|u\|_2 \ge C\frac{p+1}{2}\|u\|^2 - C'\|u\|,$$

for some positive constants C and C'. Here and henceforth $\|.\| = (\int_\Omega |\nabla .|^2 \, dx)^{\frac{1}{2}}$ and $\|.\|_2 := (\int_\Omega |.|^2 \, dx)^{\frac{1}{2}}$ are the standard norms on $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively. Thus $-J(f,.)$ is coercive. As the functional is coercive, convex and continuous, it has a global minimum point, which is a critical point. Since $-J(f,.)$ is strictly convex, the critical point is unique. Therefore, for every $f \in L^2(\Omega)$ the boundary value problem (1) has a unique (weak) solution denoted by u_f. Moreover, when f is nonnegative, by the weak maximum principle, see Theorem 1.1 of (Damascelli, 1998), we have $u_f \ge 0$ in Ω.

We know that u_f satisfies the following integral equation

$$\int_\Omega \nabla u_f \cdot \nabla v \, dx + \lambda \int_\Omega |u_f|^{p-1} u_f v \, dx = \int_\Omega f v \, dx, \quad \forall\, v \in H_0^1(\Omega). \tag{4}$$

Now we prove some auxiliary lemmas:

Lemma 1. *The functional* $\Phi : A \to \mathbb{R}$ *is bounded.*

Proof. By (4) we infer that

$$\Phi(f) = J(f, u_f) = \frac{p+1}{2}\int_\Omega |\nabla u_f|^2 \, dx + \lambda p \int_\Omega |u_f|^{p+1} \, dx \geq 0.$$

So, Φ is nonnegative. From Hölder's inequality we infer that

$$\Phi(f) \leq (p+1)\|f\|_2 \|u_f\|_2 - \lambda \int_\Omega |u_f|^{p+1} \, dx - \frac{p+1}{2}\int_\Omega |\nabla u_f|^2 \, dx.$$

Now for any $\varepsilon > 0$, by Young's inequality we have

$$\Phi(f) \leq (p+1)\left(\frac{1}{2\varepsilon}\|f\|_2^2 + \frac{\varepsilon}{2}\|u_f\|_2^2\right) - \lambda \int_\Omega |u_f|^{p+1} \, dx - \frac{p+1}{2}\int_\Omega |\nabla u_f|^2 \, dx.$$

By Poincaré's inequality, there exists a constant $0 < C_0 < 1$ such that

$$\int_\Omega |\nabla u_f|^2 \, dx \geq C_0 \int_\Omega |u_f|^2 \, dx.$$

Thus

$$\Phi(f) \leq \frac{p+1}{2\varepsilon}\|f\|_2^2 + \frac{p+1}{2}\left(\frac{\varepsilon}{C_0} - 1\right)\int_\Omega |\nabla u_f|^2 \, dx.$$

Therefore, By setting $\varepsilon = C_0$, we obtain

$$0 \leq \Phi(f) \leq \frac{p+1}{2C_0}|\Omega|, \quad \forall\, f \in A.$$

Lemma 2. *The functional* $\Phi : A \to \mathbb{R}$ *is continuous with respect to the weak* topology in* $L^\infty(\Omega)$.

Proof. Let $\{f_i\} \subset A$ is a sequence that $f_i \rightharpoonup^* f$ in $L^\infty(\Omega)$. For simplicity set $u_i := u_{f_i}$.

We have

$$\Phi(f)+(p+1)\int_\Omega (f_i-f)u_f \,\mathrm{d}x$$

$$=(p+1)\int_\Omega f_i u_f \,\mathrm{d}x-\lambda\int_\Omega |u_f|^{p+1}\,\mathrm{d}x-\frac{p+1}{2}\int_\Omega |\nabla u_f|^2\,\mathrm{d}x$$

$$\le J(f_i,u_i)=\Phi(f_i)$$

$$=(p+1)\int_\Omega f u_i \,\mathrm{d}x-\lambda\int_\Omega |u_i|^{p+1}\,\mathrm{d}x-\frac{p+1}{2}\int_\Omega |\nabla u_i|^2\,\mathrm{d}x+(p+1)\int_\Omega (f_i-f)u_i\,\mathrm{d}x$$

$$\le J(f,u_f)+(p+1)\int_\Omega (f_i-f)u_i\,\mathrm{d}x$$

$$=\Phi(f)+(p+1)\int_\Omega (f_i-f)u_i\,\mathrm{d}x.$$

Therefore

$$\Phi(f)+(p+1)\int_\Omega (f_i-f)u_f\,\mathrm{d}x\le\Phi(f_i)\le\Phi(f)+(p+1)\int_\Omega (f_i-f)u_i\,\mathrm{d}x. \quad (5)$$

Since $f_i \rightharpoonup^* f$ in $L^\infty(\Omega)$, we infer that

$$\lim_{i\to\infty}\int_\Omega (f_i-f)u_f\,\mathrm{d}x=0. \quad (6)$$

We now prove that

$$\lim_{i\to\infty}\int_\Omega (f_i-f)u_i\,\mathrm{d}x=0. \quad (7)$$

From (4), for $v=u_i$ we have

$$\int_\Omega |\nabla u_i|^2\,\mathrm{d}x+\lambda\int_\Omega |u_i|^{p+1}\,\mathrm{d}x=\int_\Omega f_i u_i\,\mathrm{d}x.$$

By Hölder's inequality and then Poincaré's inequality for some positive constant C we deduce

$$\|\nabla u_i\|_2^2\le\int_\Omega f_i u_i\,\mathrm{d}x\le\|f_i\|_2\|u_i\|_2\le C\|f_i\|_2\|\nabla u_i\|_2 .$$

Thus, since $f_i\in A$, we infer that

$$\|\nabla u_i\|_2\le C\sqrt{|\Omega|},\quad \forall\, i\in\mathbb{N}.$$

Therefore $\{u_i\}$ is a bounded sequence in $H_0^1(\Omega)$. So there exists a subsequence of it, still denoted by $\{u_i\}$, that converges weakly in $H_0^1(\Omega)$ and strongly in $L^{p+1}(\Omega)$ to some function $\hat{u} \in H_0^1(\Omega)$ (note that $p+1 < 2n/(n-2)$). Thus we deduce that $\int_\Omega (f_i - f)u_i \, dx$ tends to zero as $i \to \infty$. Therefore from (5), (6) and (7) we obtain that Φ is continuous with respect to the weak* topology in $L^\infty(\Omega)$.

Remark 1. *We claim that, the function $\hat{u}$ that mentioned in Lemma 2 is equal to u_f almost every where in Ω. We know that*

$$\Phi(f_i) = (p+1)\int_\Omega f_i u_i \, dx - \lambda \int_\Omega |u_i|^{p+1} \, dx - \frac{p+1}{2}\int_\Omega |\nabla u_i|^2 \, dx.$$

Also

$$\lim_{i\to\infty}\int_\Omega f_i u_i \, dx = \int_\Omega f\hat{u} \, dx,$$

$$\lim_{i\to\infty}\int_\Omega |u_i|^{p+1} \, dx = \int_\Omega |\hat{u}|^{p+1} \, dx,$$

and

$$\liminf_{i\to\infty}\int_\Omega |\nabla u_i|^2 \, dx \geq \int_\Omega |\nabla \hat{u}|^2 \, dx.$$

Thus

$$\Phi(f) \leq (p+1)\int_\Omega f\hat{u} \, dx - \lambda \int_\Omega |\hat{u}|^{p+1} \, dx - \frac{p+1}{2}\int_\Omega |\nabla \hat{u}|^2 \, dx$$

$$= J(f,\hat{u}) \leq J(f,u_f) = \Phi(f).$$

By the uniqueness of the maximizer of $J(f,.)$ we yield $\hat{u} = u_f$ almost every where in Ω.

Lemma 3. *The functional Φ have the following properties:*

1. *The functional $\Phi : L^\infty(\Omega) \to \mathrm{R}$ is strictly convex.*

2. *The functional $\Phi : A \to \mathrm{R}$ is Gâteaux differentiable with derivative*

$$\langle \Phi'(f), g \rangle = (p+1)\int_\Omega g u_f \, dx,$$

for all $g \in L^\infty(\Omega)$.

Proof. (*i*) Let $t \in [0,1]$ and $f, g \in L^\infty(\Omega)$. We have

$$J(tf + (1-t)g, u) = tJ(f, u) + (1-t)J(g, u) \quad \text{for all } u \in H_0^1(\Omega).$$

Thus

$$\sup_{u \in H_0^1(\Omega)} J(tf + (1-t)g, u) \le t \sup_{u \in H_0^1(\Omega)} J(f, u) + (1-t) \sup_{u \in H_0^1(\Omega)} J(g, u).$$

Hence

$$\Phi(tf + (1-t)g) \le t\Phi(f) + (1-t)\Phi(g).$$

Therefore Φ is convex. Now we prove strict convexity. Assume there exists $t \in (0,1)$ such that

$$\Phi(tf + (1-t)g) = t\Phi(f) + (1-t)\Phi(g).$$

Thus

$$J(h, u_h) = tJ(f, u_f) + (1-t)J(g, u_g),$$

where $h = tf + (1-t)g$. So

$$tJ(f, u_h) + (1-t)J(g, u_h) = tJ(f, u_f) + (1-t)J(g, u_g).$$

By uniqueness of the maximizer we infer that $u_h = u_f = u_g$ almost every where in Ω. From this fact and the equation (4), for all $v \in H_0^1(\Omega)$ we obtain

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x + \lambda \int_\Omega |u_h|^{p-1} u_h v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x,$$

and

$$\int_\Omega \nabla u_h \cdot \nabla v \, \mathrm{d}x + \lambda \int_\Omega |u_h|^{p-1} u_h v \, \mathrm{d}x = \int_\Omega gv \, \mathrm{d}x.$$

Therefore $f = g$ almost every where in Ω, so Φ is strictly convex.

(*ii*) Let $\{t_i\}$ be a sequence of positive numbers that tends to zero. Let $f, g \in L^\infty(\Omega)$ and $f_i := f + t_i(g - f)$, $i \in \mathrm{N}$. By (5) we have

$$\begin{aligned}
&\Phi(f) + (p+1)t_i \int_\Omega (g - f)u_f \, \mathrm{d}x \\
&\le \Phi(f_i) \\
&\le \Phi(f) + (p+1)t_i \int_\Omega (g - f)u_{f_i} \, \mathrm{d}x.
\end{aligned}$$

By $\lim_{i\to\infty} f_i = f$ and *Remark* 1 we deduce that u_{f_i} tends to u_f in $L^\infty(\Omega)$ as $i \to \infty$. Therefore

$$\lim_{i\to\infty} \frac{\Phi(f + t_i(g - f)) - \Phi(f)}{t_i} = (p+1)\int_\Omega (g - f)u_f \,\mathrm{d}x.$$

Thus the proof of the part (*ii*) follows.

We are now ready to prove the main result;

Theorem 4. *The optimization problem (3) has a unique solution.*

Proof. A is closed for the weak* topology in $L^\infty(\Omega)$, see Henrot & Pierre (2005), and convex. Hence A is weak* compact, see Hille & Phillips (1957). Since Φ is bounded below, see Lemma 1, and weak* continuous we infer that (3) is solvable. The solution is unique since Φ is strictly convex.

3. The first order optimality condition

We begin with the definition of tangent cones;

Definition 1. *Let X be a normed linear space and $\hat{a}$ be an element of the closure of a nonempty subset C of X. The inner tangent cone of C at $\hat{a}$ is denoted by $T'_C(\hat{a})$; moreover, $v \in T'_C(\hat{a})$ whenever for each sequence $\{t_i\}$ of positive real numbers converging to zero, there is a sequence $\{v_i\}$ in X satisfying* $\lim_{i\to\infty} v_i = v$ *and $\hat{a} + t_i v_i \in C$ for all $i \in \mathrm{N}$.*

In the following lemma we state the first order optimality condition satisfied by optimal solution.

Lemma 5. *The function $\hat{f}$ is the minimizer of Φ in A if and only if*

$$\langle \Phi'(\hat{f}), h\rangle \geq 0, \ \forall\, h \in T'_A(\hat{f}).$$

Proof. Assume $\hat{f}$ is the minimizer of Φ in A. Let $h \in T'_A(\hat{f})$ and $\{t_i\}$ be a sequence of positive real numbers that tends to zero. Thus there exists a sequence $\{f_i\}$ in $L^\infty(\Omega)$ such that $f_i \to h$ as $i \to \infty$ and $\hat{f} + t_i f_i \in A$ for all $i \in \mathrm{N}$. Hence

$$\langle \Phi'(\hat{f}), h\rangle = \lim_{i\to\infty} \frac{\Phi(\hat{f} + t_i h) - \Phi(\hat{f})}{t_i} = \lim_{i\to\infty} \frac{\Phi(\hat{f} + t_i f_i) - \Phi(\hat{f})}{t_i} \geq 0,$$

because $\hat{f}$ is minimizer of Φ in A.

Conversely, assume $\hat{f} \in A$ and $\langle \Phi'(\hat{f}), h\rangle \geq 0$ for all $h \in T'_A(\hat{f})$. We show that $\Phi(\hat{f}) \leq \Phi(f)$ for all $f \in A$. To derive a contradiction we assume there is $f \in A$ such that $\Phi(\hat{f}) > \Phi(f)$. Since A is convex we observe that $f - \hat{f} \in T'_A(\hat{f})$. Let $\{t_i\}$ be a sequence of real numbers in (0,1] converging to zero. We set $v_i = f - \hat{f}$ for all $i \in \mathbb{N}$. It's clear that $\lim_{i\to\infty} v_i = f - \hat{f}$ and $\hat{f} + t_i(f - \hat{f}) \in A$. Hence

$$\langle \Phi'(\hat{f}), f - \hat{f} \rangle = \lim_{i\to\infty} \frac{\Phi(\hat{f} + t_i(f - \hat{f})) - \Phi(\hat{f})}{t_i}$$

$$\leq \lim_{i\to\infty} \frac{(1-t_i)\Phi(\hat{f}) + t_i\Phi(f) - \Phi(\hat{f})}{t_i}$$

$$= \Phi(f) - \Phi(\hat{f}) < 0,$$

which is a contradiction. Therefore $\hat{f}$ minimizes Φ.

Definition 2**.** *For any function $f \in A$, we define*

1. $\Omega_0(f) := \{x \in \Omega : f(x) = 0\}$,
2. $\Omega^*(f) := \{x \in \Omega : 0 < f(x) < 1\}$,
3. $\Omega_1(f) := \{x \in \Omega : f(x) = 1\}$.

Let $f \in A$. The following lemma characterize the elements of $T'_A(f)$.

Lemma 6. *If $f \in A$, then the tangent cone $T'_A(f)$ is the set of every function $h \in L^\infty(\Omega)$ such that*

1. $\int_\Omega h(x)\, dx = 0$,
2. $\lim_{i\to\infty} \| \chi_{Q_i^0(f)} h^- \|_\infty = 0$,
3. $\lim_{i\to\infty} \| \chi_{Q_i^1(f)} h^+ \|_\infty = 0$,

where h^+ (resp. h^-) is the positive (resp. negative) part of h, $Q_i^0(f) = \{x \in \Omega : f(x) \leq 1/i\}$ and $Q_i^1(f) = \{x \in \Omega : f(x) \geq 1 - 1/i\}$.

Proof. See Proposition 2.1 in Bednarczuk *et al*. (2000) and Proposition 4.5 in Cominetti & Penot (1997).

Corollary 7. *Let $f \in A$. If $h \in T'_A(f)$, then*

$$h(x) \geq 0 \text{ a.e. in } \Omega_0(f), \quad h(x) \leq 0 \text{ a.e. in } \Omega_1(f).$$

Proof. Since $\Omega_0(f) \subset Q_i^0(f)$ and $\Omega_1(f) \subset Q_i^1(f)$ for all $i \in \mathbb{N}$, the assertion readily follows from Lemma 6.

We are now ready to state the main result of this section;

Theorem 8. *The function* $\hat{f}$ *minimizes* Φ *on* A *if and only if*

1. $\hat{u} := u_{\hat{f}}$ *is constant on* $\Omega^*(\hat{f})$ *(as soon as* $|\Omega^*(\hat{f})| > 0$ *);*

2. $\hat{u}(x_0) \geq \hat{u}(x^*) \geq \hat{u}(x_1)$, $\forall\ (x_0, x^*, x_1) \in \Omega_0(\hat{f}) \times \Omega^*(\hat{f}) \times \Omega_1(\hat{f})$.

Proof. Let $\hat{f}$ minimizes Φ on A. Set $\Omega_i^*(\hat{f}) = \{x \in \Omega : 1/i \leq \hat{f} \leq 1 - 1/i\}$. Since $\Omega^*(\hat{f}) = \bigcup_{i=1}^{\infty} \Omega_i^*(\hat{f})$, it is enough to prove that $\hat{u}$ is constant on $\Omega_i^*(\hat{f})$ for all i. To derive a contradiction, suppose $\hat{u}$ is not constant on $\Omega_i^*(\hat{f})$ for some i. Thus, there exist two measurable sets ω_1 and ω_2 in $\Omega_i^*(\hat{f})$ such that

$$|\omega_1| = |\omega_2| \quad \text{and} \quad \int_{\omega_1} \hat{u}\, \mathrm{d}x < \int_{\omega_2} \hat{u}\, \mathrm{d}x.$$

We define

$$h(x) := \begin{cases} 1 & x \in \omega_1, \\ -1 & x \in \omega_2, \\ 0 & x \in (\omega_1 \cup \omega_2)^c. \end{cases}$$

By Lemma 6 we infer that $h \in T_A'(\hat{f})$. Therefore

$$\langle \Phi'(\hat{f}), h \rangle = (p+1) \int_{\Omega} h\hat{u}\, \mathrm{d}x = (p+1) \left(\int_{\omega_1} \hat{u}\, \mathrm{d}x - \int_{\omega_2} \hat{u}\, dx \right) < 0.$$

This inequality is a contradiction by Lemma 5. Thus $\hat{u}$ is constant on $\Omega^*(\hat{f})$.

The second point is proved in a similar way. To derive a contradiction, suppose there exist two measurable sets $\omega_0 \subset \Omega_0(\hat{f})$ and $\omega^* \subset \Omega^*(\hat{f})$ such that

$$|\omega_0| = |\omega^*| \quad \text{and} \quad \int_{\omega_0} \hat{u}\, \mathrm{d}x < \int_{\omega^*} \hat{u}\, \mathrm{d}x.$$

We define

$$h(x) := \begin{cases} 1 & x \in \omega_0, \\ -1 & x \in \omega^*, \\ 0 & x \in (\omega_0 \cup \omega^*)^c, \end{cases}$$

which belongs to $T_A'(\hat{f})$. Thus

$$\langle \Phi'(\hat{f}), h \rangle = (p+1) \left(\int_{\omega_0} \hat{u}\, \mathrm{d}x - \int_{\omega^*} \hat{u}\, dx \right) < 0.$$

This is a contradiction. Hence $\hat{u}|_{\Omega_0(\hat{f})} \geq \hat{u}|_{\Omega^*(\hat{f})}$. We prove in the same way that $\hat{u}|_{\Omega^*(\hat{f})} \geq \hat{u}|_{\Omega_1(\hat{f})}$.

Conversely, let us assume that $\hat{f}$ and $\hat{u}$ satisfy (*i*) and (*ii*). Let $\hat{u}|_{\Omega^*(\hat{f})} = c^*$ and $h \in T_A'(\hat{f})$. According to Corollary 7, h is nonnegative on $\Omega_0(\hat{f})$ and nonpositive on $\Omega_1(\hat{f})$. Thus

$$\langle \Phi'(\hat{f}), h\rangle = (p+1)\int_\Omega h\hat{u}\,\mathrm{d}x$$

$$= (p+1)\left(\int_{\Omega_0(\hat{f})} h\hat{u}\,\mathrm{d}x + \int_{\Omega^*(\hat{f})} h\hat{u}\,\mathrm{d}x + \int_{\Omega_1(\hat{f})} h\hat{u}\,\mathrm{d}x\right)$$

$$\geq (p+1)\left(\int_{\Omega_0(\hat{f})} hc^*\,\mathrm{d}x + \int_{\Omega^*(\hat{f})} hc^*\,\mathrm{d}x + \int_{\Omega_1(\hat{f})} hc^*\,\mathrm{d}x\right)$$

$$= (p+1)c^*\int_\Omega h\,\mathrm{d}x = 0.$$

Therefore, by Lemma 5 we infer that $\hat{f}$ minimizes Φ.

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مسألة إمثال لمعادلة ناقصيه غير خطية

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خلاصة

نقوم في هذا البحث بإثبات الوجود و الوحدانية للحل الأمثل لمسألة إمثال مرتبطة بمعادلة ناقصيه غير خطية. و نستخدم مفهوم المخروطات المماسة للحصول على شرط الأمثلة الذي يحققه الأمثل.