Kuwait J. Sci. **43 (2)** pp. 16-30, 2016

Some general properties of a fractional Sumudu transform in the class of Boehmians

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Abstract

In literature, there are several works on the theory and applications of integral transforms of Boehmian spaces, but fractional integral transforms of Boehmians have not yet been reported. In this paper, we investigate a fractional Sumudu transform of an arbitrary order on some space of integrable Boehmians. The fractional Sumudu transform of an integrable Boehmian is well-defined, linear and sequentially complete in the space of continuous functions. Two types of convergence are also discussed in details.

Mathematics Subject Classification: 44A15; 44A35.

Keywords: Boehmian; fractional Sumudu transform; fractional derivative; Mittag-Leffler function; Sumudu transform.

1. Introduction

The Sumudu transform of a one variable function $\vartheta(t)$ was proposed originally by Watugala (1993) as a new integral transform given by

$$S(\vartheta(t))(y) = \frac{1}{y}\int_0^\infty \vartheta(t)\exp\left(\frac{-t}{y}\right)\mathrm{d}t, \tag{1.1}$$

where $y \in (-\tau_1, \tau_2)$, $\vartheta(t)$ is a function of power series form and is a member of the set A, where

$$A = \left\{\vartheta(t) : \exists M, \tau_1 \text{ and/or } \tau_2 > 0, \text{ such that } |\vartheta(t)| < Me^{\frac{t}{\tau_j}}, t \in (-1)^j \times [0,\infty)\right\}. \tag{1.2}$$

For a given function ϑ in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist.

While we are in agreement with most of the claims expounded by Watugala, the transform is not so new as proclaimed; the Sumudu transform is connected to the s-multiplied Laplace transform. But, this, however, in no way diminishes its importance or usefulness; Belgasem *et al.* (2003).

Let ϑ_1 and ϑ_2 be integrable functions and $*$ is their usual convolution product

$$(\vartheta_1 \circledast \vartheta_2)(x) = \int_0^\infty \vartheta_1 (x-t)\, \vartheta_2 (t)\, \mathrm{d}t,$$

then the Sumudu transform of the product $*$ is given by Al-Omari & Kilicman (2013)

$$S(\vartheta_1 * \vartheta_2)(y) = yS(\vartheta_1 (t))(y)\, S(\vartheta_2 (t))(y). \tag{1.3}$$

Some properties of Sumudu transforms and their applications are given in Weerakoon (1994, 1998); Kadem (2005); Watugala (1993, 1998, 2002); Belgacem & Karaballi (2006); Asiru (2002); Zhang (2007); Eltayeb & Kilicman (to appear) and many others. The Sumudu transform was applied to distributions in Eltayeb *et al.* (2010) and to Boehmians by Al-Omari & Kilicman (2013):

The purpose of the present contribution is exactly to provide a possible generalization of the fractional Sumudu transform. For the convenience of the reader, this work is organized as follows: In Section 2, we firstly give a brief background on the fractional calculus and establish the convolution theorem of the fractional Sumudu transform by aid of a certain convolution product of fractional order. In Section 3, we recall the abstract contruction of Boehmians. In Section 4, we shall define the generalized Sumudu transform of fractional order in the context of Boehmians and derive some of its main basic properties.

2. Sumudu transform of of fractional order

The term fractional calculus; fractional derivatives, fractional integrals, and fractional differential equation, is as old as the calculus of differentiation and goes back to times when Leibnitz, Gauss, and Newton invented this kind of calculation. Fractional calculus is a generalization of the ordinary differentiation and integration to non-integer order.

Fractional transforms play an important role in information processing, computing technology, optical fibers, electronic computing, paraxial diffraction in free space, quadratic refractive index medium and resolution of the non-stationary Schrödinger equation in quantum mechanics, phase retrieval, and so forth. Number of known fractional integral transforms were studied by different mathematicians. Among those we invoke here are the fractional linear and radial canonical transform; Torre (2003); the fractional Fourier transform; Cai & Lin (2003); Ozaktas *et al.* (2001); the fractional of Hankel transform; Namias (1980); the fractional Hilbert transform Daviset *et al.* (1998); the fractional cosine, sine, and Hartley transforms Pei & Ding (2002), and many others to mention but a few.

Advantages of fractional integral transforms comes over the ordinary ones because they naturally arise under the consideration of different problems and, because fractionalization gives us a new degree of freedom, which can be used for more complete characterization of an object or as an additional encoding parameter.

Let α be the parameter of fractionalization. Then, the fractional integral transform operator $F_rT_\alpha\left(f\right)$ is defined by Alieva *et al.* (2005)

$$F_rT_\alpha\left(f\right)\left(u\right)=\int_{-\infty}^{\infty}\omega\left(\alpha,x,u\right)\vartheta\left(x\right)\mathtt{d}x, \tag{2.1}$$

where $\omega\left(\alpha,x,u\right)$ is the transform kernel.

Fractional integral transforms are additive with respect to the real parameter α,

$$F_rT_{\alpha_1+\alpha_2}=\left(F_rT_{\alpha_1}\right)\left(F_rT_{\alpha_2}\right) \tag{2.2}$$

and produce the ordinary transform and powers of it for integer values of α.

The integral of a continuous function $\vartheta\left(x\right)$ with respect to $\left(\mathtt{d}t\right)^\alpha$ is defined as the solution of the fractional differential equation (Gupta *et al.* (2010))

$$\mathtt{d}y=\vartheta\left(x\right)\left(\mathtt{d}x\right)^\alpha, x\geq 0, y\left(0\right)=0, \tag{2.3}$$

and given as

$$y=\int_0^x\vartheta\left(\xi\right)\left(\mathtt{d}\xi\right)^\alpha=\alpha\int_0^x\left(x-\xi\right)^{\alpha-1}\vartheta\left(\xi\right)\mathtt{d}\xi, \tag{2.4}$$

$0<\alpha<1$.

Definition 2.1. (Gupta *et al.* (2010), (2.1)). Let $\vartheta\left(t\right)$ denote a function that vanishes for negative values of t. Then, the Sumudu transform of $\vartheta\left(t\right)$ of order α is defined by

$$S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right):=\int_0^\infty\varepsilon_\alpha\left(-t^\alpha\right)\vartheta\left(yt\right)\left(\mathtt{d}t\right)^\alpha:=\lim_{M\uparrow\infty}\int_0^M\varepsilon_\alpha\left(-t^\alpha\right)\vartheta\left(yt\right)\left(\mathtt{d}t\right)^\alpha, \tag{2.5}$$

where $y\in\mathbb{C}$ and $\varepsilon_\alpha\left(t\right)$ is the Mittag-Leffler function of one variable

$$\varepsilon_\alpha\left(t\right)=\sum_{k=0}^{\infty}\frac{t^k}{(\alpha k)!}. \tag{2.6}$$

The fractional Laplace-Sumudu transform duality of order α is given as

$$S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right)=\frac{1}{y^\alpha}\mathfrak{L}_\alpha\left(\vartheta\left(t\right)\right)\left(\frac{1}{y}\right), \tag{2.7}$$

where $0<\alpha<1$, $\mathfrak{L}_\alpha\left(\vartheta\left(t\right)\right)\left(y\right)$ is the fractional Laplace transform of order α.

Some properties of the fractional Sumudu transform S_α are given as given by (Gupta *et al.* (2010), (2.7))

(i) $S_\alpha\left(\vartheta\left(at\right)\right)\left(y\right)=S_\alpha\left(\vartheta\left(t\right)\right)\left(ay\right)$,

(ii) $S_\alpha\left(\vartheta\left(t-b\right)\right)\left(y\right)=\varepsilon_\alpha\left(-b^\alpha\right)S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right)$,

(iii) $S_\alpha\varepsilon_\alpha\left(-c^\alpha t^\alpha\right)\vartheta\left(t\right)\left(y\right)=\frac{1}{(1+cy)^\alpha}S_\alpha\vartheta\left(t\right)\left(\frac{y}{1+cy}\right)$.

The fractional Sumudu transform was also given by (Gupta *et al.* (2010), (3.5))

$$S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right):=\int_0^\infty\varepsilon_\alpha\left(-t^\alpha\right)\vartheta\left(yt\right)\left(\mathrm{d}t\right)^\alpha,$$

which by change of variables can be written as

$$S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right)=\int_0^\infty\frac{\varepsilon_\alpha\left(-\left(\frac{t}{y}\right)^\alpha\right)}{y^\alpha}\vartheta\left(t\right)\left(\mathrm{d}t\right)^\alpha, \tag{2.8}$$

where $0<\alpha<1$.

The inversion formula is then recovered from (2.8), by aid of (Gupta *et al.* (2010), (3.6)), giving

$$\vartheta\left(t\right)=\frac{1}{\left(m_\alpha\right)^\alpha}\int_{-i\infty}^{i\infty}\frac{\varepsilon_\alpha\left(\left(ty\right)^\alpha\right)}{y^\alpha}S_\alpha\left(\vartheta\left(t\right)\right)\left(\left(ty\right)^\alpha\right)\left(\mathrm{d}y\right)^\alpha, \tag{2.9}$$

where m_α is the period of the Mittag-Leffler function (2.6).

Definition 2.2. The convolution product of fractional order of two functions ϑ and g is defined by

$$\left(\vartheta\left(t\right)*^\alpha g\left(t\right)\right):=\int_0^t\vartheta\left(t-\tau\right)g\left(\tau\right)\left(\mathrm{d}\tau\right)^\alpha. \tag{2.10}$$

We establish now the convolution theorem in the fractional sense.

Theorem 2.3. Let $\vartheta\left(t\right)$ and $g\left(t\right)$ be functions that vanish for negative values of t, then we have

$$S_\alpha\left(\vartheta\left(t\right)*^\alpha g\left(t\right)\right)\left(y\right)=y^\alpha S_\alpha\left(\vartheta\left(t\right)\right)\left(y\right)S_\alpha\left(g\left(t\right)\right)\left(y\right). \tag{2.11}$$

Proof By applying (2.8) for (2.10) we get

$$S_\alpha\left(\vartheta(t)*^\alpha g(t)\right)(y) = \int_0^\infty \frac{\varepsilon_\alpha\left(-\left(\frac{t}{y}\right)^\alpha\right)}{y^\alpha}\left(\vartheta(t)*^\alpha g(t)\right)(\mathtt{d}t)^\alpha$$

$$\text{i.e.} \qquad = \int_0^\infty \frac{\varepsilon_\alpha\left(-\left(\frac{t}{y}\right)^\alpha\right)}{y^\alpha}\int_0^t \vartheta(t-\tau)\,g(\tau)\,(\mathtt{d}\tau)^\alpha(\mathtt{d}t)^\alpha .$$

By the fact (Jumarie (2009), (3.10))

$$\varepsilon_\alpha\left(\lambda(x+y)^\alpha\right) = \varepsilon_\alpha\left(\lambda x^\alpha\right)\varepsilon_\alpha\left(\lambda y^\alpha\right),$$

we write that

$$S_\alpha\left(\vartheta(t)*^\alpha g(t)\right)(y) = \int_0^\infty \frac{\varepsilon_\alpha\left(-\left(\frac{t-\tau+\tau}{y}\right)^\alpha\right)}{y^\alpha}\int_0^t \vartheta(t-\tau)\,g(\tau)\,(\mathtt{d}\tau)^\alpha(\mathtt{d}t)^\alpha$$

$$\text{i.e.} \qquad = y^\alpha\int_0^\infty \frac{\varepsilon_\alpha\left(-\left(\frac{t-\tau}{y}\right)^\alpha\right)}{y^\alpha}\frac{\varepsilon_\alpha\left(-\left(\frac{\tau}{y}\right)^\alpha\right)}{y^\alpha}$$

$$\int_0^t \vartheta(t-\tau)\,g(\tau)\,(\mathtt{d}\tau)^\alpha(\mathtt{d}t)^\alpha .$$

By change of variables, we get

$$S_\alpha\left(\vartheta(t)*^\alpha g(t)\right)(y) = y^\alpha\int_0^\infty \varepsilon_\alpha\left(-\left(\frac{\xi}{y}\right)^\alpha\right)\varepsilon_\alpha\left(-\left(\frac{\tau}{y}\right)^\alpha\right)$$

$$\int_0^t \vartheta(\xi)\,g(\tau)\,(\mathtt{d}\tau)^\alpha(\mathtt{d}\xi)^\alpha$$

$$\text{i.e.} \qquad = y^\alpha\int_0^\infty \varepsilon_\alpha\left(-\left(\frac{\xi}{y}\right)^\alpha\right)\vartheta(\xi)\,(\mathtt{d}\xi)^\alpha$$

$$\int_0^t \varepsilon_\alpha\left(-\left(\frac{\tau}{y}\right)^\alpha\right)g(\tau)\,(\mathtt{d}\tau)^\alpha .$$

Hence, the theorem has been completely proved.

For the convenience of the reader we usually recall the abstract construction of Boehmian spaces.

3. Abstract construction of Boehmians and essential results

Boehmians are used for all objects defined by an algebraic construction similar to that of field of quotients. A minimal structure necessary for the construction of Boehmians consists of the following elements :

(1) A nonempty set $\mathfrak{a}$;

(2) A commutative semigroup $(\mathfrak{b}, \star)$;

(3) An operation $\star : \mathfrak{a} \times \mathfrak{b} \to \mathfrak{a}$ such that for each $x \in \mathfrak{a}$ and $s_1, s_2, \in \mathfrak{b}$, $x \star (s_1 \bullet s_2) = (x \star s_1) \star s_2$;

(4) A collection $\Delta \subset \mathfrak{b}$ such that :

(*i*) If $x, y \in \mathfrak{a}$, $(s_n) \in \Delta$, $x \bullet s_n = y \bullet s_n$ for all n, then $x = y$;

(*ii*) If $(s_n), (t_n) \in \Delta$, then $(s_n \bullet t_n) \in \Delta$.

Elements of Δ are called delta sequences.

Consider

$$Q = \{(x_n, s_n) : x_n \in \mathfrak{a}, (s_n) \in \Delta, x_n \star s_m = x_m \star s_n, \forall m, n \in \mathbb{N}\}.$$

If $(x_n, s_n), (y_n, t_n) \in Q$, $x_n \star t_m = y_m \star s_n$, $\forall m, n \in \mathbb{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in Q. The space of equivalence clases in Q is denoted by $\mathbf{m}$. Elements of $\mathbf{m}$ are called Boehmians.

Between $\mathfrak{a}$ and $\mathbf{m}$ there is a canonical embedding expressed as

$$x \leftrightsquigarrow \left[\frac{x \star (s_n)}{(s_n)}\right].$$

The operation $\star$ can be extended to $\mathbf{m} \times \mathfrak{a}$ by

$$\left[\frac{(x_n)}{(s_n)}\right] \star t = \left[\frac{(x_n) \star t}{(s_n)}\right].$$

The relationship between the notion of convergence and the product $\star$ is given by:

(*i*) If $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and, $\phi \in \mathfrak{b}$ is any fixed element, then $f_n \star \phi \to f \star \phi$ in $\mathfrak{a}$ (as $n \to \infty$);

(*ii*) If $f_n \to f$ as $n \to \infty$ in $\mathfrak{a}$ and $(\delta_n) \in \Delta$, then $f_n \star \delta_n \to f$ in $\mathfrak{a}$ (as $n \to \infty$).

The operation $\star$ is extended to $\mathbf{m} \times \mathfrak{b}$ as follows: If $\left[\frac{(x_n)}{(s_n)}\right] \in \mathbf{m}$ and $\phi \in \mathfrak{b}$, then

$$\left[\frac{(x_n)}{(s_n)}\right] \star \phi = \left[\frac{(x_n) \star \phi}{(s_n)}\right].$$

Convergence in $\mathbf{m}$ is defined as follows:

(1) A sequence (h_n) in $\mathbf{m}$ is said to be δ convergent to h in $\mathbf{m}$, $h_n \overset{\delta}{\to} h$, if there exists $(s_n) \in \Delta$ such that $(h_n \star s_n), (h \star s_n) \in \mathfrak{a}$, $\forall k, n \in \mathbb{N}$, and

$$(h_n \star s_k) \to (h \star s_k)$$

as $n \to \infty$, in $\mathfrak{a}$, for every $k \in \mathbb{N}$.

(2) A sequence (h_n) in $\mathbf{m}$ is said to be Δ convergent to h in $\mathbf{m}$, $h_n \overset{\Delta}{\to} h$, if there exists a $(s_n) \in \Delta$ such that $(h_n - h) \star s_n \in \mathfrak{a}$, $\forall n \in \mathbb{N}$, and

$$(h_n - h) \star s_n \to 0$$

as $n \to \infty$ in $\mathfrak{a}$.

More about the abstract construction of Boehmians we refer to Mikusinski (1987, 1995). For integral transforms of Boehmians we refer to Al-Omari *et al.* (2008); Al-Omari & Kilicman (2012a, 2013, 2012b, 2014); Al-Omari (2015); Rajendran & Roopkumar (2014). Let $\boldsymbol{l}^1_+$ denote the space of Lebesgue integrable functions defined on $\mathbb{R}_+, \mathbb{R}_+ = (0, \infty)$. Following Al-Omari & Kilicman (2012a), the space of Lebesgue integrable Boehmians restricted to $\mathbb{R}_+$ is denoted by $\mathbf{m}_{\boldsymbol{l}^1_+}$.

Definition 3.1. By r_+ we denote the Schwartz' space of test functions of bounded supports over $\mathbb{R}_+$ and by $\Delta_+^{r_+}$ the subset of r_+ of those delta sequences (δ_n^+) that satisfy

$$\int_{\mathbb{R}_+} \delta_n^+ (t) \, (\mathrm{d}t)^\alpha = 1, n \in \mathbb{N}, \tag{3.1}$$

$$\int_{\mathbb{R}_+} \left|\delta_n^+ (t)\right| (\mathrm{d}t)^\alpha < M, \text{ where } M \text{ is a positive real number, } n \in \mathbb{N}. \tag{3.2}$$

$$\int_{|t|>\varepsilon} \left|\delta_n^+ (t)\right| (\mathrm{d}t)^\alpha \to 0 \text{ as } n \to \infty, \varepsilon > 0. \tag{3.3}$$

The integrable space $\mathbf{m}_{\boldsymbol{l}^1_+}$ with $\Delta_+^{r_+}$ is denoted by $\mathbf{m}^\alpha_{\boldsymbol{l}^1_+}$ which is a convolution algebra with the pointwise operations

$$\text{(i) } \lambda \left[\frac{(\vartheta_n)}{(\delta_n^+)}\right] = \left[\frac{\lambda(\vartheta_n)}{(\delta_n^+)}\right],$$

$$\text{(ii) } \left[\frac{(\vartheta_n)}{(\delta_n^+)}\right] + \left[\frac{(g_n)}{(\alpha_n^+)}\right] = \left[\frac{(\vartheta_n *^\alpha \alpha_n^+) + (g_n *^\alpha \delta_n^+)}{(\delta_n^+ *^\alpha \alpha_n^+)}\right].$$

Convergence in $\mathbf{m}_{\boldsymbol{l}^1_+}$ is defined as:

(i) A sequence $(h_n) \in \mathbf{m}_{\boldsymbol{l}^1_+}$ is said to be δ convergent to $h \in \mathbf{m}_{\boldsymbol{l}^1_+}$, $h_n \xrightarrow{\delta} h$, if there exists $(\delta_n^+) \in \Delta$ such that $(h_n *^\alpha \delta_n^+), (h *^\alpha \delta_n^+) \in \boldsymbol{l}^1_+, \forall k, n \in \mathbb{N}$, and

$$\left(h_n *^\alpha \delta_k^+\right) \to \left(h *^\alpha \delta_k^+\right)$$

as $n \to \infty$, in $\boldsymbol{l}^1_+$, for every $k \in \mathbb{N}$.

(ii) A sequence $(h_n) \in \mathbf{m}_{\boldsymbol{l}^1_+}$ is said to be Δ convergent to $h \in \mathbf{m}_{\boldsymbol{l}^1_+}$, $h_n \xrightarrow{\Delta} h$, if there exists a $(\delta_n^+) \in \Delta$ such that $(h_n - h) *^\alpha \delta_n^+ \in \boldsymbol{l}^1_+, \forall n \in \mathbb{N}$, and

$$(h_n - h) *^\alpha \delta_n^+ \to 0$$

as $n \to \infty$ in $\boldsymbol{l}^1_+$.

Lemma 3.2. Let $(\delta_n^+) \in \Delta_+^{r+}$, then $S_\alpha\left(\delta_n^+(t)\right)(y) \to \dfrac{1}{y^\alpha}$ as $n \to \infty$.

Proof By considering the fractional Sumudu transform (2.8) and using (2.6) we get that

$$S_\alpha\left(\delta_n^+(t)\right)(y) = \int_0^\infty \frac{\boldsymbol{\varepsilon}_\alpha\left(-\left(\frac{t}{y}\right)^\alpha\right)}{y^\alpha}\delta_n^+(t)\,(\mathrm{d}t)^\alpha \to \frac{1}{y^\alpha} \text{ as } n \to \infty.$$

This completes the proof of the theorem.

Theorem 3.3. Let $\left[\dfrac{(\vartheta_n)}{(\delta_n^+)}\right] \in \mathbf{m}^\alpha_{\boldsymbol{l}^1_+}$. Then, the sequence

$$S_\alpha\left(\vartheta_n(t)\right)(y) = \int_0^\infty \frac{\boldsymbol{\varepsilon}_\alpha\left(-\left(\frac{t}{y}\right)^\alpha\right)}{y^\alpha}\vartheta_n(t)\,(\mathrm{d}t)^\alpha$$

converges uniformly on every compact subset of $\mathbb{R}_+$.

Proof Let K be a compact subset of $\mathbb{R}_+$, then, by Lemma 3.2, we get that

$$S_\alpha\left(\vartheta_n(t)\right)(y) = S_\alpha\left(\vartheta_n(t)\right)(y)\,\frac{S_\alpha\left(\delta_n^+(t)\right)(y)}{S_\alpha\left(\delta_n^+(t)\right)(y)}\ .$$

By (2.4) we have

$$S_\alpha\left(\vartheta_n(t)\right)(y) = y^{-\alpha}\frac{S_\alpha\left(\vartheta_n(t) *^\alpha \delta_n^+(t)\right)(y)}{S_\alpha\left(\delta_n^+(t)\right)(y)}\ .$$

By the concept of quotionts of sequences of $\mathbf{m}^\alpha_{\boldsymbol{l}^1_+}$ we from the above equation obtain

$$S_\alpha(\vartheta_n(t))(y) = y^{-\alpha}\frac{S_\alpha(\vartheta_k(t) *^\alpha \delta_n^+(t))(y)}{S_\alpha(\delta_k^+(t))(y)} \ .$$

Hence, Lemma 3.2 gives

$$S_\alpha(\vartheta_n(t))(y) \to \frac{S_\alpha(\vartheta_k(t))(y)}{S_\alpha(\delta_k^+(t))(y)} y^{-2\alpha} \ ,$$

as $n \to \infty$ on compact subsets of $\mathbb{R}_+$.

This completes the proof of the theorem.

4. Sumudu transform of Boehmians of fractional order

We start this section by the definition of the fractional Sumudu transform in the context of the space $\mathbf{m}_{\boldsymbol{l}_+^1}^\alpha$ then we obtain some general properties.

Definition 4.1. Let β be a Boehmians in the Lebesgue space $\mathbf{m}_{\boldsymbol{l}_+^1}^\alpha$ defined by $\beta = \left[\frac{(\vartheta_n)}{(\delta_n)}\right]$.

Then, by aid of Theorem 6, we define the fractional Sumudu transform of β as the limit of the sequence $(S_\alpha \vartheta_n)$ in the space of continuous functions, where the limit ranges over all compact subsets of $\mathbb{R}_+$.

In notations, this definition is interpreted to mean

$$S_\alpha^\beta\left[\frac{(\vartheta_n)}{(\delta_n^+)}\right] = \lim_{n\to\infty} S_\alpha(\vartheta_n(t))(y) \,, \tag{4.1}$$

on compact subsets of $\mathbb{R}_+$.

We claim that our definition is well-defined. For, let $\beta_1, \beta_2 \in \mathbf{m}_{\boldsymbol{l}_+^1}^\alpha$, $\beta_1 = \beta_2$, where $\beta_1 = \left[\frac{(\vartheta_n)}{(\delta_n^+)}\right]$, $\beta_2 = \left[\frac{(g_n)}{(\epsilon_n^+)}\right]$, $(\vartheta_n), (g_n) \in \boldsymbol{l}_+^1$ and $(\delta_n^+), (\epsilon_n^+) \in \Delta_+^{r+}$; then by the concept of equivalence classes of $\mathbf{m}_{\boldsymbol{l}_+^1}^\alpha$ we write

$$\vartheta_n(t) *^\alpha \epsilon_m^+(t) = g_m(t) *^\alpha \delta_n^+(t) \,, m, n \in \mathbb{N}, \tag{4.2}$$

and, in particular, for $m = n$, we have

$$\vartheta_n(t) *^\alpha \epsilon_n^+(t) = g_n(t) *^\alpha \delta_n^+(t) \,, n \in \mathbb{N}. \tag{4.3}$$

Applying the fractional Sumudu transform on both sides of (4.3) gives

$$y^{\alpha}S_{\alpha}\left(\vartheta_{n}\left(t\right)\right)\left(y\right)S_{\alpha}\left(\epsilon_{n}^{+}\left(t\right)\right)\left(y\right)=y^{\alpha}S_{\alpha}\left(g_{n}\left(t\right)\right)\left(y\right)S_{\alpha}\left(\delta_{n}^{+}\left(t\right)\right)\left(y\right).$$

Therefore, considering the limit, as $n\rightarrow\infty$, on compact subsets of $\mathbb{R}_{+}$ and using Lemma 3.2 yield

$$\lim_{n\rightarrow\infty}S_{\alpha}\left(\vartheta_{n}\left(t\right)\right)\left(y\right)=\lim_{n\rightarrow\infty}S_{\alpha}\left(g_{n}\left(t\right)\right)\left(y\right),\forall n\in\mathbb{N}.$$

Hence, from (4.1), we obtain

$$S_{\alpha}^{\beta}\left[\frac{\left(\vartheta_{n}\right)}{\left(\delta_{n}^{+}\right)}\right]=S_{\alpha}^{\beta}\left[\frac{\left(g_{n}\right)}{\left(\epsilon_{n}^{+}\right)}\right].$$

This completes the proof of our claim.

Theorem 4.2. The operator S_{α}^{β} is linear .

Proof Let $\beta_{1},\beta_{2}\in\mathbf{m}_{\boldsymbol{l}_{+}^{1}}^{\alpha}$ such that $\beta_{1}=\left[\frac{\left(\vartheta_{n}\right)}{\left(\delta_{n}^{+}\right)}\right],\beta_{2}=\left[\frac{\left(g_{n}\right)}{\left(\epsilon_{n}^{+}\right)}\right]$. Then, addition of Boehmians and Equation 4.1 imply

$$\begin{aligned}S_{\alpha}^{\beta}\left(\beta_{1}+\beta_{2}\right) &= S_{\alpha}^{\beta}\left(\left[\frac{\left(\vartheta_{n}*^{\alpha}\epsilon_{n}^{+}\right)+\left(g_{n}*^{\alpha}\delta_{n}^{+}\right)}{\left(\delta_{n}^{+}*^{\alpha}\epsilon_{n}^{+}\right)}\right]\right)\\ &= \lim_{n\rightarrow\infty}S_{\alpha}\left(\vartheta_{n}\left(t\right)*^{\alpha}\epsilon_{n}^{+}\left(t\right)+g_{n}\left(t\right)*^{\alpha}\delta_{n}^{+}\left(t\right)\right)\left(y\right).\end{aligned}$$

Applying the limit for the left hand side of the above equation and using Lemma 3.2 yield

$$S_{\alpha}^{\beta}\left(\beta_{1}+\beta_{2}\right)=\lim_{n\rightarrow\infty}S_{\alpha}\left(\vartheta_{n}\left(t\right)\right)\left(y\right)+\lim_{n\rightarrow\infty}S_{\alpha}\left(g_{n}\left(t\right)\right)\left(y\right).$$

That is,

$$S_{\alpha}^{\beta}\left(\beta_{1}+\beta_{2}\right)=S_{\alpha}^{\beta}\beta_{1}+S_{\alpha}^{\beta}\beta_{2}.$$

Over and above, we have

$$S_{\alpha}^{\beta}\left(\gamma\beta_{1}\right)=S_{\alpha}^{\beta}\left(\gamma\left[\frac{\left(\vartheta_{n}\right)}{\left(\delta_{n}^{+}\right)}\right]\right)=S_{\alpha}^{\beta}\left(\left[\frac{\gamma\left(\vartheta_{n}\right)}{\left(\delta_{n}^{+}\right)}\right]\right)=\gamma S_{\alpha}^{\beta}\beta_{1},\gamma\in\mathbb{C}.$$

This completes the proof of the theorem.

Theorem 4.3. If $\beta\in\mathbf{m}_{\boldsymbol{l}_{+}^{1}}^{\alpha}$ and $\beta=0$; then $S_{\alpha}^{\beta}\beta=0$.

Proof of this theorem is straightforward. Details are thus avoided.

Theorem 4.4. S_α^β is sequentially complete.

Proof Let $\beta_n \to \beta$ as $n \to \infty$, in $\mathbf{m}^{\alpha}_{\boldsymbol{l}^1_+}$, then, indeed, $S_\alpha^\beta \beta_{nm} \to S_\alpha^\beta \beta$ as $n \to \infty$ on compact subsets of $\mathbb{R}_+$.

Hence the theorem is proved.

Theorem 4.5. The operator S_α^β is continuous with respect to the convergence of type δ. Proof Given $\beta_n \overset{\delta}{\to} \beta$ in $\mathbf{m}^{\alpha}_{\boldsymbol{l}^1_+}$ as $n \to \infty$. Then, we can find $\vartheta_{n,k}$ and ϑ_k, δ_k such that

$$\beta_n = \left[\frac{(\vartheta_{n,k})}{(\delta_k)}\right], \left[\frac{(\vartheta_k)}{(\delta_k)}\right] \text{ and } \vartheta_{n,k} \to \vartheta_k \text{ as } n \to \infty, k \in \mathbb{N}.$$

Hence, applying S_α gives $S_\alpha \vartheta_{n,k} \to S_\alpha \vartheta_k$ as $n \to \infty$. That is,

$$S_\alpha \vartheta_{n,k} \to S_\alpha \vartheta_k$$

as $n \to \infty$. This leads to

$$S_\alpha^\beta \beta_n \overset{\delta}{\to} S_\alpha^\beta \beta$$

as $n \to \infty$.

The proof is, therefore, completed.

Theorem 4.6. The fractional Sumudu transform S_α^β is continuous with respect to convergence of type Δ.

Proof Let $\beta_n \overset{\Delta}{\to} \beta$ as $n \to \infty$, then we can find $\vartheta_n \in \boldsymbol{l}^1_+$ such that $(\beta_n - \beta)$ $*^\alpha \delta_n^+ = \left[\frac{\vartheta_n *^\alpha \delta_n^+}{\left(\delta_n^+\right)}\right], \vartheta_n \to 0 \text{ as } n \to \infty.$

Therefore, by (4.1) and Lemma 3.2, we get

$$\begin{aligned} S_\alpha^\beta \left((\beta_n - \beta) *^\alpha \delta_n^+\right) &= S_\alpha^\beta \left[\frac{\left(\vartheta_n *^\alpha \delta_n^+\right)}{\left(\delta_n^+\right)}\right] \\ &\to \lim_{n\to\infty} S_\alpha \left(\vartheta_n (t) *^\alpha \delta_n^+ (t)\right) (y) \\ &\to \left(\frac{\alpha}{y^\alpha}\right) \lim_{n\to\infty} S_\alpha \left(\vartheta_n (t)\right) (y) \\ &\to 0 \end{aligned}$$

as $n \to \infty$, since $S_\alpha \vartheta_n \to 0$ as $n \to \infty$.

Hence the theorem is completely proved.

Theorem 4.7. Let $\beta_1 = \left[\frac{(\vartheta_n)}{(\delta_n)}\right] \in \mathbf{m}^{\alpha}_{\boldsymbol{l}^1_+}$ and $\beta_2 = \left[\frac{(g_n)}{(\epsilon_n)}\right] \in \mathbf{m}^{\alpha}_{\boldsymbol{l}^1_+}$; then

$$S^{\beta}_{\alpha}\left(\beta_1 *^{\alpha} \beta_2\right) = y^{\alpha} S^{\beta}_{\alpha}\beta_1 S^{\beta}_{\alpha}\beta_2.$$

Proof Assume the requirements of the theorem are satisfied for some β_1 and β_2 $\in \mathbf{m}^{\alpha}_{\boldsymbol{l}^1_+}$; then the operation $*^{\alpha}$ of Boehmians gives

$$S^{\beta}_{\alpha}\left(\beta_1 *^{\alpha} \beta_2\right) = S^{\beta}_{\alpha}\left(\left[\frac{(\vartheta_n) *^{\alpha} (g_n)}{(\delta_n) *^{\alpha} (\epsilon_n)}\right]\right).$$

By Equation 4.1, we write

$$S^{\beta}_{\alpha}\left(\beta_1 *^{\alpha} \beta_2\right) = \lim_{n\to\infty} S_{\alpha}\left(\vartheta_n(t) *^{\alpha} g_n(t)\right)(y).$$

By Theorem 2.3, we obtain

$$S^{\beta}_{\alpha}\left(\beta_1 *^{\alpha} \beta_2\right) = y^{\alpha}\lim_{n\to\infty} S_{\alpha}\left(\vartheta_n(t)\right)(y) \lim_{n\to\infty} S_{\alpha}\left(g_n(t)\right)(y).$$

Therefore, Equation 4.1, gives

$$S^{\beta}_{\alpha}\left(\beta_1 *^{\alpha} \beta_2\right) = y^{\alpha} S^{\beta}_{\alpha}\beta_1 S^{\beta}_{\alpha}\beta_2.$$

This completes the proof of the theorem.

5. Conclusion

Prior to this work, no approach of extending fractional integral transforms to spaces of Boehmians has yet been given. In this paper, the fractional Sumudu transform has been extended to a context of Boehmians and some of its properties are also obtained. The technique we follow in this paper can be applied to various fractional integral transforms of Boehmians and, hence, to distributions and ultra distributions as well.

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Submitted **:** 25/10/2014

Revised **:** 30/01/2015

Accepted **:** 08/02/2015

بعض الخصائص العامة لتحويل سومودو الكسري في صنف البوهيميات

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خلاصة

يوجد في المنشورات دراسات عديدة حول نظرية التحويلات التكاملية لفضاءات بوهيمية و تطبيقاتها، لكننا لا نجد أية دراسات حول التحويلات التكاملية الكسرية لفضاءات بوهيمية. نقوم في هذا البحث بدراسة تحويل سومودو الكسري من أية رتبة على فضاءات بوهيمية قابلة للتكامل. و نثبت أن هذه التحويلات هي حسنة التعريف، خطية و تامة تتابعياً في فضاء الدوال المستمرة. و نعطي نقاشاً تفصيلياً لنوعين من أنواع التقارب.