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Evaluation of some reciprocal trigonometric sums via partial fraction decomposition

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Abstract

Recently, Melham computed some finite sums in which the denominator of the summand includes products of 'sine' or 'cosine'. In this paper, generalizations of the sums, which he studied in 2016, are presented, by allowing arbitrary factors in the denominator of the summand. Our approach uses the elementary technique of partial fraction decomposition. Furthermore, some of the sums, which he studied in 2017, are treated in the same style.

Keywords: Partial fraction decomposition; reciprocal sum identities; trigonometric functions.

Mathematical Subject Classifications 2010: 11B99

1. Introduction

Recently, some finite reciprocal sums were studied (Kılıç *et al*. 2017; Melham 2016). In Kılıç *et al*. (2017), the authors computed finite reciprocal sums in which the denominator of the summand is a product of some recursive sequences, by using the elementary partial fraction decomposition method.

Melham (2016) considered the sums in which the denominator of the summand is a finite product of the trigonometric functions 'sine' or 'cosine'. He confined himself to some special summation formulæ where the denominator of the summand includes at most five distinct factors in arithmetic progression. We cite here some of the sums he studied (for non-negative integer d):

$$\sum_{j=1}^{n-1}\frac{1}{\sin j\sin(j+d)},$$

$$\sum_{j=1}^{n-1}\frac{\cos(j+d)}{\sin j\sin(j+d)\sin(j+2d)},$$

$$\sum_{j=1}^{n-1}\frac{\sin(j+d)\sin(j+2d)}{\cos j\cos(j+d)\cos(j+2d)\cos(j+3d)},$$

$$\sum_{j=1}^{n-1}\frac{\cos(3j+6d)}{\sin j\sin(j+d)\sin(j+2d)\sin(j+3d)\sin(j+4d)}.$$

In Melham 2017, the author continued the same theme. Here he considered finite reciprocal trigonometric sums such that at least one of the trigonometric functions in the denominator of the summand is squared. For example, for non-negative integer d he evaluated

$$\sum_{j=1}^{n-1}\frac{\sin(2j+d)}{\sin^2 j\sin^2(j+d)},$$

$$\sum_{j=1}^{n-1}\frac{\sin(2j+3d)}{\sin^2 j\sin(j+d)\sin(j+2d)\sin^2(j+3d)}.$$

In this paper, we generalize the results in Melham (2016) by allowing arbitrary factors in the denominator of the summand. We use the elementary partial fraction decomposition method to prove our claims, which leads to a constructive way to obtain Melham's results in a straightforwad and easy way. (See Chu (2007) as an application of the partial fraction method for reciprocal trigonometric functions.) One can easily follow similar steps to obtain various new type of summations. We present our main results in the following section. In the last section, we give some particular results for sums which stem from Melham (2017) in which the denominator of the summand includes a square of some factors. Again we use the partial fraction decomposition method. The aim of this section is to show that this method is simpler and also less mysterious way to compute these types of sums, in comparison to Melham's approach.

2. Main results

We have the following four generalizations from the results in Melham (2016).

Theorem 1. *For even* $m \geq 2$, $m = 2t$,

$$\sum_{k=1}^{n} \frac{1}{\prod_{j=0}^{m-1} \sin(k+jd)} = \sum_{j=0}^{t-1} B_j \sum_{k=1}^{(m-2j-1)d} \big(\cot(k+jd) - \cot(n+jd+k)\big),$$

where for $0 \leq j \leq t-1$,

$$B_j = \frac{(-1)^j}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld}.$$

Proof. Noticing that

$$\sin(k+jd) = \frac{e^{i(k+jd)} - e^{-i(k+jd)}}{2i} = \frac{e^{2ijd}z^2 - 1}{2ize^{ijd}}, \quad (1)$$

with $z = e^{ik}$ and $i = \sqrt{-1}$, the summand can be rewritten as

$$\frac{1}{\prod_{j=0}^{m-1} \sin(k+jd)} = \frac{(2i)^m z^m e^{i(2t^2-t)}}{\prod_{j=0}^{m-1} (e^{2idj}z^2 - 1)}.$$

Let $z^2 = q$ and

$$h(q) = \frac{(2i)^m q^t e^{i(2t^2-t)}}{\prod_{j=0}^{m-1} (e^{2idj}q - 1)}.$$

Applying partial fraction decomposition, we have

$$h(q) = \sum_{j=0}^{m-1} \frac{A_j}{(e^{2idj}q - 1)}, \quad (2)$$

with (for $0 \leq j \leq t-1$)

$$\begin{aligned} A_j &= (e^{2idj}q - 1)h(q)\Big|_{q=e^{-2idj}} \\ &= \frac{(2i)^m e^{-idmj} e^{i(2t^2-t)}}{\prod_{l=0}^{j-1}(e^{-2idj+2idl} - 1) \prod_{l=1}^{m-j-1} (e^{2idl} - 1)} \\ &= (-1)^j \frac{(2i)^m e^{id(t^2-t+j+(j-t)^2)}}{\prod_{l=1}^{j}(e^{2idl} - 1)^2 \prod_{l=j+1}^{m-j-1} (e^{2idl} - 1)}. \end{aligned}$$

For the instance $t \leq j \leq m-1$, the coefficients can be computed in a similar way, leading to

$$A_j = -A_{m-1-j},$$

for $0 \leq j \leq t-1$. In this way, all t different coefficients have been computed. Let us rewrite the A_j's in terms of trigonometric functions by using equation (1). For $0 \leq j \leq t-1$,

$$A_j = \frac{(-1)^j (2i)}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld}.$$

So, following (2), the partial fraction decomposition has been obtained:

$$h(q) = \sum_{j=0}^{t-1} \frac{(-1)^j}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld} \left(\frac{2i}{(e^{2idj}q - 1)} - \frac{2i}{(e^{2id(m-1-j)}q - 1)} \right).$$

Since $z^2 = q$ and $z = e^{ik}$, by using the equation (1)

$$\begin{aligned} \frac{2i}{(e^{2idj}q - 1)} &= \frac{e^{-i(k+dj)}}{\sin(k+dj)} = \frac{\cos(k+dj) - i\sin(k+dj)}{\sin(k+dj)} \\ &= \cot(k+dj) - i. \end{aligned}$$

Finally,

$$h(e^{2ik}) = \sum_{j=0}^{t-1} \frac{(-1)^j}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld} \big(\cot(k+dj) - \cot(k+(m-1-j)d)\big).$$

Summing this equation over $1 \leq k \leq n$, we get

$$\begin{aligned} \sum_{k=1}^{n} h(e^{2ik}) = \sum_{j=0}^{t-1} &\frac{(-1)^j}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld} \\ &\times \sum_{k=1}^{(m-2j-1)d} \big(\cot(k+jd) - \cot(n+jd+k)\big), \end{aligned}$$

as claimed. □

As seen in the proof, the main approach is to rewrite the summand in terms of variable q. For more detail about q-calculus, we refer to Garg *et al*. (2013) and references therein.

When $d = 1$ in Theorem 1, we have the following corollary.

Corollary 2. *For even $m \geq 2$, $m = 2t$,*

$$\sum_{k=1}^{n} \frac{1}{\prod_{j=0}^{m-1} \sin(k+j)} = \sum_{j=0}^{t-1} B_j \sum_{k=1}^{m-2j-1} \big(\cot(k+j) - \cot(n+j+k)\big),$$

where for $0 \leq j \leq t-1$,

$$B_j = \frac{(-1)^j}{\prod_{l=1}^{j} \sin^2 l \prod_{l=j+1}^{m-j-1} \sin l}.$$

For the second sum we have:

Theorem 3. *For odd $m \geq 3$, $m = 2t+1$,*

$$\begin{aligned}\sum_{k=1}^{n} \frac{\cos(k+td)}{\prod_{j=0}^{m-1} \sin(k+jd)} &= \sum_{j=0}^{t-1} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(k+jd) - \cot(n+k+jd)\big) \\ &+ \sum_{j=0}^{t-1} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(n+k+(m-2-j)d) \\ &\qquad - \cot(k+(m-2-j)d)\big),\end{aligned}$$

where for $0 \leq j \leq t$,

$$B_j = \frac{1}{2\sin td} \prod_{l=1}^{j} \frac{1}{\sin^2 ld} \prod_{l=j+1}^{m-j-2} \frac{1}{\sin ld}.$$

Proof. Similar to Theorem 1, the summand could be written as

$$\frac{\cos(k+td)}{\prod_{j=0}^{m-1} \sin(k+jd)} = \frac{2^{m-1} i^m q^t e^{2idt^2} (e^{2idt} q + 1)}{\prod_{j=0}^{m-1} (e^{2idj} q - 1)} = h(q),$$

where $q = e^{2ik}$. Applying again partial fraction decomposition, we have

$$h(q) = \sum_{j=0}^{m-1} A_j \frac{(2i)}{(e^{2idj} q - 1)},$$

where

$$A_j = \frac{(-1)^j \cos(t-j)d}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld}, \text{ for } j \leq t,$$

$$A_j = A_{m-j-1}, \text{ for } t < j < m.$$

Finally, since $\sum_{j=0}^{m-1} A_j = -h(0)/2i = 0$, we obtain

$$\frac{\cos(k+td)}{\prod_{j=0}^{m-1} \sin(k+jd)} = \sum_{j=0}^{m-1} A_j (\cot(k+jd) - i) = \sum_{j=0}^{m-1} A_j \cot(k+jd).$$

Then we can write

$$\begin{aligned}|A_0| &= B_0 \\ |A_1| &= B_1 + B_0 \\ &\vdots \\ |A_{t-1}| &= B_{t-1} + B_{t-2} \\ |A_t| &= B_{t-1} + B_{t-1},\end{aligned}$$

with

$$B_j = \frac{1}{2\sin td} \prod_{l=1}^{j} \frac{1}{\sin^2 ld} \prod_{l=j+1}^{m-j-2} \frac{1}{\sin ld}.$$

Indeed, $|A_0| = B_0$, $\frac{|A_t|}{2} = B_{t-1}$ and for $1 \leq j \leq t-1$,

$$\begin{aligned}B_j + B_{j-1} &= \frac{1}{2\sin td} \frac{\sin jd + \sin(m-j-1)d}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld} \\ &= \frac{1}{2\sin td} \frac{2\sin td \cos(t-j)d}{\prod_{l=1}^{j} \sin^2 ld \prod_{l=j+1}^{m-j-1} \sin ld} = |A_j|.\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{\cos(k+td)}{\prod_{j=0}^{m-1} \sin(k+jd)} &= \sum_{j=0}^{t-1} (-1)^j B_j \big(\cot(k+jd) - \cot(k+(j+1)d)\big) \\ &+ \sum_{j=0}^{t-1} (-1)^j B_j \big(\cot(k+(m-1-j)d) - \cot(k+(m-2-j)d)\big).\end{aligned}$$

Summing over $1 \le k \le n$, we obtain

$$\sum_{k=1}^{n} \frac{\cos(k+td)}{\prod_{j=0}^{m-1} \sin(k+jd)}$$
$$= \sum_{j=0}^{t-1} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(k+jd) - \cot(n+k+jd)\big)$$
$$+ \sum_{j=0}^{t-1} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(n+k+(m-2-j)d) - \cot(k+m-2-j)d)\big),$$

as claimed □

Corollary 4. *For odd* $m \ge 3$, $m = 2t+1$,

$$\sum_{k=1}^{n} \frac{\cos(k+t)}{\prod_{j=0}^{m-1} \sin(k+j)}$$
$$= \sum_{j=0}^{t-1} (-1)^j B_j \big(\cot(1+j) - \cot(n+1+j)\big)$$
$$+ \sum_{j=0}^{t-1} (-1)^j B_j \big(\cot(n+2t-j) - \cot(2t-j)\big),$$

where for $0 \le j \le t$

$$B_j = \frac{1}{2\sin t} \prod_{l=1}^{j} \frac{1}{\sin^2 l} \prod_{l=j+1}^{m-j-2} \frac{1}{\sin l}.$$

For the third sum:

Theorem 5. *For* $m \ge 4$, we have

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{m-2} \cot(k+jd)}{\sin k \sin(k+(m-1)d)}$$
$$= \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(k+jd) - \cot(n+k+jd)\big)$$
$$+ (-1)^{m+1} \sum_{j=0}^{\lfloor \frac{m-3}{2} \rfloor} (-1)^j B_j \sum_{k=1}^{d} \big(\cot(n+k+(m-2-j)d) - \cot(k+(m-2-j)d)\big),$$

where for $0 \le j \le \lfloor \frac{m-2}{2} \rfloor$,

$$B_j = \frac{1}{\sin(m-1)d} \prod_{l=1}^{j} \cot^2 ld \prod_{l=j+1}^{m-2-j} \cot ld,$$

where $\lfloor \cdot \rfloor$ *is the floor function.*

Proof. By partial fraction decomposition, the summand can be similarly rewritten as:

$$\frac{\prod_{j=1}^{m-2} \cot(k+jd)}{\sin k \sin(k+(m-1)d)} = \sum_{j=0}^{m-1} A_j \cot(k+jd),$$

where

$$A_0 = \frac{1}{\sin(m-1)d} \prod_{l=1}^{m-2} \cot ld,$$
$$A_j = (-1)^j \frac{\prod_{l=1}^{j-1} \cot^2 ld \prod_{l=j}^{m-2-j} \cot ld}{\sin jd \sin(m-1-j)d}, \quad \text{for } 1 \le j \le \left\lfloor \frac{m-1}{2} \right\rfloor,$$
$$A_j = (-1)^{m+1} A_{m-1-j}, \quad \text{for } \left\lfloor \frac{m+1}{2} \right\rfloor \le j \le m-1.$$

It is easily checked that $A_0 = B_0$ and $A_{\lfloor \frac{m}{2} \rfloor} = 2B_{\lfloor \frac{m-2}{2} \rfloor}$ when m is odd. Furthermore, for $1 \le j \le \lfloor \frac{m-2}{2} \rfloor$,

$$B_j + B_{j-1}$$
$$= \frac{\prod_{l=1}^{j-1} \cot^2 ld \prod_{l=j}^{m-2-j} \cot ld}{\sin(m-1)d} \big(\cot jd + \cot(m-1-j)d\big)$$
$$= \frac{\prod_{l=1}^{j-1} \cot^2 ld \prod_{l=j}^{m-2-j} \cot ld}{\sin jd \sin(m-1-j)d} = |A_j|.$$

So finally we obtain

$$\frac{\prod_{j=1}^{m-2} \cot(k+jd)}{\sin k \sin(k+(m-1)d)}$$
$$= \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} (-1)^j B_j \big(\cot(k+jd) - \cot(k+(j+1)d)\big)$$
$$+ (-1)^{m+1} \sum_{j=0}^{\lfloor \frac{m-3}{2} \rfloor} (-1)^j B_j \big(\cot(k+(m-1-j)d) - \cot(k+(m-2-j)d)\big).$$

After summing over $1 \le k \le n$, we get the claim. □

As an example, we write this out for $d = 1$ and $m = 5$:

$$\sum_{k=1}^{n}\frac{\cot(k+1)\cot(k+2)\cot(k+3)}{\sin k\sin(k+4)}$$
$$=\frac{\cot 1\cot 2\sin n\sin(n+5)}{\sin 4}$$
$$\times\Big(\frac{\cos 3}{\sin 1\sin 4\sin(n+1)\sin(n+4)}$$
$$-\frac{\cos 1}{\sin 2\sin 3\sin(n+2)\sin(n+3)}\Big).$$

Now we move to the fourth sum:

Theorem 6. *For $m\geq 4$, we have*

$$\sum_{k=1}^{n}\frac{\cos\big((m-2)k+\binom{m-1}{2}d\big)}{\prod\limits_{j=0}^{m-1}\sin(k+jd)}$$
$$=\sum_{j=0}^{\lfloor\frac{m-2}{2}\rfloor}(-1)^{j}B_{j}\sum_{k=1}^{d}\big(\cot(k+jd)-\cot(n+k+jd)\big)$$
$$+(-1)^{m+1}\sum_{j=0}^{\lfloor\frac{m-3}{2}\rfloor}(-1)^{j}B_{j}\sum_{k=1}^{d}\big(\cot(n+k+(m-2-j)d)$$
$$-\cot(k+(m-2-j)d)\big),$$

where for $0\leq j\leq\lfloor\frac{m-2}{2}\rfloor$,

$$B_{j}=\frac{1}{\sin(m-1)d}\frac{\cos\big(\binom{m-1}{2}-j(m-2)-j\big)d}{\prod\limits_{l=1}^{j}\sin^{2}ld\prod\limits_{l=j+1}^{m-j-2}\sin ld}.$$

Proof. We omit some details due to the similarity with the proof of Theorem 3. By partial fraction decomposition, we write the summand as:

$$\frac{\cos\big((m-2)k+\binom{m-1}{2}d\big)}{\prod\limits_{j=0}^{m-1}\sin(k+jd)}=\sum_{j=0}^{m-1}A_{j}\cot(k+jd),$$

with

$$A_{j}=\frac{(-1)^{j}\cos(\binom{m-1}{2}-j(m-2))d}{\prod\limits_{l=1}^{j}\sin^{2}ld\prod\limits_{l=j+1}^{m-j-1}\sin ld},\quad\text{for } j\leq\Big\lfloor\frac{m-1}{2}\Big\rfloor$$
$$A_{j}=(-1)^{m+1}A_{m-1-j},\quad\text{for } j\geq\Big\lfloor\frac{m+1}{2}\Big\rfloor.$$

One can easily check that for $1\leq j\leq\lfloor\frac{m-2}{2}\rfloor$, $B_0=A_0$, $|A_j|=B_j+B_{j-1}$ and when m is odd $A_{\lfloor\frac{m}{2}\rfloor}=2B_{\lfloor\frac{m-2}{2}\rfloor}$. So, the proof is complete. □

When $d=1$ and $m=4$, we have following summation as example.

$$\sum_{k=1}^{n}\frac{\cos(2k+3)}{\prod\limits_{j=0}^{3}\sin(k+j)}$$
$$=\frac{\sin n}{\sin^{2}1\sin 3}\Big(\frac{\sin(n+4)\cos 3}{\sin 3\sin(n+1)\sin(n+3)}-\frac{1}{\sin 2\sin(n+2)}\Big).$$

Note that all the summations, where the terms 'sin' and 'cos' are exchanged with 'cos' and 'sin', respectively, can be similarly done. We leave the details to the interested reader.

We would like to note that all results can be naturally generalized by multiplying the argument by θ in every occurrence of the trigonometric functions, where θ is assumed to be any real number except for a rational multiple of π to avoid possible zeros in the denominators. For brevity, we omit the details. As an example, when $m=2$, by Theorem 1 we have

$$\sum_{k=1}^{n}\frac{1}{\sin(k\theta)\sin((k+d)\theta)}$$
$$=\frac{1}{\sin^{2}(d\theta)}\sum_{k=1}^{d}\big(\cot(k\theta)-\cot((n+k)\theta)\big).$$

3. Further results

In this section, we present some special sums in which the denominator of the summand term includes squares of some factors by using the partial fraction method. Unfortunately, we could not find generalizations of these type of sums as in previous sections because of the duplicated factors. Nevertheless we show here how one can constructively compute these sums.

Example 7. *For $n\geq 1$*

$$\sum_{k=1}^{n}\frac{\sin(2k+d)}{\sin^{2}k\sin^{2}(k+d)}=\frac{1}{\sin d}\sum_{k=1}^{d}\Big(\frac{1}{\sin^{2}k}-\frac{1}{\sin^{2}(n+k)}\Big).$$

Proof. Similarly, for $q=e^{2ik}$, the summand term can be rewritten as:

$$f(q)=\frac{(2i)^{3}e^{id}q(e^{2id}q^{2}-1)}{(q-1)^{2}(e^{2id}q-1)^{2}}$$
$$=\frac{A_0}{q-1}+\frac{A_1}{(e^{2id}q-1)}+\frac{B_0}{(q-1)^{2}}+\frac{B_1}{(e^{2id}q-1)^{2}}.$$

The coefficients can be computed as follows:

$$A_j = (e^{2ijd}q-1)^2 f(q)\Big|_{q=e^{-2ijd}},$$
$$B_j = e^{-2ijd}\frac{d}{dq}(e^{2ijd}q-1)^2 f(q)\Big|_{q=e^{-2ijd}}.$$

So $-A_0 = A_1 = -B_0 = B_1 = M$, with $M = \frac{8ie^{id}}{(e^{2id}-1)}$. Thus

$$f(q) = \frac{-Mq}{(q-1)^2} + \frac{e^{2id}Mq}{(e^{2id}q-1)^2}.$$

Finally, if we convert to trigonometric functions, we can write this as

$$\frac{\sin(2k+d)}{\sin^2 k\sin^2(k+d)} = \frac{1}{\sin d}\Big(\frac{1}{\sin^2 k} - \frac{1}{\sin^2(k+d)}\Big),$$

which completes the proof. □

Example 8. *For* $n \geq 1$,

$$\sum_{k=1}^{n}\frac{\cos(k+d)}{\sin^2 k\sin(k+d)\sin^2(k+2d)}$$
$$= \frac{-1}{2\sin^4 d}\sum_{j=0}^{1}\sum_{k=1}^{d}\big((\cot(k+jd) - \cot(n+k+jd)\big)$$
$$+ \frac{1}{4\sin^3 d\cos d}\sum_{k=1}^{2d}\Big(\frac{1}{\sin^2 k} - \frac{1}{\sin^2(n+k)}\Big).$$

Proof. Similarly, we can write

$$\frac{\cos(k+d)}{\sin^2 k\sin(k+d)\sin^2(k+2d)}$$
$$= \frac{2^4 i^5 q^2 e^{4id}(e^{2id}q+1)}{(q-1)^2(e^{2id}q-1)(e^{4id}q-1)^2},$$

where $q = e^{2ik}$. With the help of Mathematica and after some algebraic manipulations,

$$\frac{2^4 i^5 q^2 e^{4id}(e^{2id}q+1)}{(q-1)^2(e^{2id}q-1)(e^{4id}q-1)^2}$$
$$= \frac{-M}{q-1} + \frac{qM(e^{2id}-1)}{(e^{2id}+1)(q-1)^2} + \frac{2M}{(e^{2id}q-1)}$$
$$- \frac{M}{(e^{4id}q-1)} - \frac{qM(e^{2id}-1)}{(e^{4id}q-1)^2},$$

where $M = \frac{16ie^{4id}}{(e^{2id}-1)^4}$. If we write the above equation in terms of trigonometric functions, we obtain

$$\frac{\cos(k+d)}{\sin^2 k\sin(k+d)\sin^2(k+2d)}$$
$$= \frac{i}{2\sin^4 d} - \frac{1}{2\sin^4 d}\cot k + \frac{1}{4\sin^3 d\cos d\sin^2 k}$$
$$- \frac{i}{\sin^4 d} + \frac{1}{\sin^4 d}\cot(k+d)$$
$$+ \frac{i}{2\sin^4 d} - \frac{1}{2\sin^4 d}\cot(k+2d)$$
$$- \frac{1}{4\sin^3 d\cos d\sin^2(k+2d)}$$
$$= -\frac{1}{2\sin^4 d}\big(\cot k - 2\cot(k+d) + \cot(k+2d)\big)$$
$$+ \frac{1}{4\sin^3 d\cos d}\Big(\frac{1}{\sin^2 k} - \frac{1}{\sin^2(k+2d)}\Big).$$

The proof follows now after summing over $1 \leq k \leq n$. □

Finally, we have a sum including four different factors in the denominator of the summand term.

Example 9. *For* $n \geq 1$,

$$\sum_{k=1}^{n}\frac{\sin(2k+3d)}{\sin^2 k\sin^2(k+d)\sin^2(k+2d)\sin^2(k+3d)}$$
$$= \frac{1}{\sin^2 d\sin^2 2d\sin 3d}$$
$$\times\sum_{k=1}^{d}\big(\cot^2 k - \cot^2(n+k)$$
$$+ \cot^2(k+2d) - \cot^2(n+k+2d)\big)$$
$$+ \frac{2\cos d}{\sin^3 d\sin 2d\sin 3d}\sum_{k=1}^{d}\big(\cot^2(k+d) - \cot^2(n+k+d)\big)$$
$$- \frac{2+4\cos 2d}{\sin^2 d\sin^3 2d\sin 3d}\sum_{k=1}^{d}\big(\cot k - \cot(n+d)$$
$$+ \cot(n+k+2d) - \cot(k+2d)\big).$$

Proof. Similarly, we can write the summand in terms of $q = e^{2ik}$ as follows

$$h(q) = \frac{(2i)^7 q^3 e^{9id}(e^{6id}q^2-1)}{(q-1)^2(e^{2id}q-1)^2(e^{4id}q-1)^2(e^{6id}q-1)^2}$$
$$= \sum_{j=0}^{3}\frac{A_j}{(e^{2ijd}q-1)} + \sum_{j=0}^{3}\frac{B_j}{(e^{2ijd}q-1)^2},$$

where for $0 \leq j \leq 3$,

$$A_j = (e^{2ijd}q-1)^2 f(q)\Big|_{q=e^{-2ijd}},$$
$$B_j = e^{-2ijd}\frac{d}{dq}(e^{2ij}q-1)^2 f(q)\Big|_{q=e^{-2ijd}}.$$

After computing these coefficients with the help of a computer, we write them in terms of trigonometric functions:

$$\frac{B_0}{(q-1)^2} = \frac{(-i+\cot k)^2}{\sin^2 d \sin^2 2d \sin 3d},$$
$$\frac{B_1}{(e^{2id}q-1)^2} = \frac{(-i+\cot(k+d))^2}{\sin^3 d \sin^2 2d},$$
$$\frac{B_2}{(e^{4id}q-1)^2} = -\frac{(-i+\cot(k+2d))^2}{\sin^3 d \sin^2 2d},$$
$$\frac{B_3}{(e^{6id}q-1)^2} = -\frac{(-i+\cot(k+3d))^2}{\sin^3 d \sin^2 2d}$$

and

$$\frac{A_0}{(q-1)} = -\frac{2+4\cos 2d - 2i\sin 2d}{\sin^2 d \sin^3 2d \sin 3d} \cot k,$$
$$\frac{A_1}{(e^{2id}q-1)} = \frac{2+2i\sin 2d}{\sin^3 d \sin^3 2d} \cot(k+d),$$
$$\frac{A_2}{(e^{4id}q-1)} = \frac{2-2i\sin 2d}{\sin^3 d \sin^3 2d} \cot(k+2d),$$
$$\frac{A_3}{(e^{6id}q-1)} = -\frac{2+4\cos 2d + 2i\sin 2d}{\sin^2 d \sin^3 2d \sin 3d} \cot(k+3d).$$

After some simplifications and algebraic manipulations, we can rewrite the summand terms as follows:

$$\begin{aligned}
&\frac{\sin(2k+3d)}{\sin^2 k \sin^2(k+d) \sin^2(k+2d) \sin^2(k+3d)} \\
&= \frac{1}{\sin^2 d \sin^2 2d \sin 3d}\big(\cot^2 k - \cot^2(k+d) \\
&\qquad + \cot^2(k+2d) - \cot^2(k+3d)\big) \\
&+ \frac{2\cos d}{\sin^3 d \sin 2d \sin 3d}\big(\cot^2(k+d) - \cot^2(k+2d)\big) \\
&- \frac{2+4\cos 2d}{\sin^2 d \sin^3 2d \sin 3d}\big(\cot k - \cot(k+d) \\
&\qquad + \cot(k+3d) - \cot(k+2d)\big),
\end{aligned}$$

so the claim follows. □

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تقييم بعض عمليات الجمع المثلثية التبادلية من خلال تحلل الكسر الجزئي

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الملخص

في الآونة الأخيرة، قام ملهام بحساب بعض عمليات الجمع المنتهية التي يتضمن فيها المقام ناتج "جيب الزاوية" أو "جيب التمام". في هذا البحث، تم تقديم تعاميم لعمليات الجمع التي درسها في عام 2016، من خلال السماح بعوامل عشوائية في المقام. ونهجنا الحالي يستخدم التقنية الأولية لتحلل الكسر الجزئي. علاوة على ذلك، تم التعامل مع بعض العمليات التي درسها في عام 2017 بنفس الأسلوب.