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On pseudo null and null cartan darboux helices in minkowski 3-space

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Abstract

In this paper, we introduce the notion of pseudo null and null Cartan Darboux helices in Minkowski 3-space. We characterize the Darboux vectors of the pseudo null and the null Cartan curves in terms of the Darboux helices and the rectifying curves. We find an explicit parameter equation of a pseudo null curve of constant precession and prove that there are no null Cartan curves of constant precession. Finally, we give some examples.

Keywords: Darboux helix; Darboux vector; null Cartan curve; Pseudo null curve; rectifying curve.

MSC Classification: 53C50, 53C40.

1. Introduction

Izumiya & Takeuchi, 2004 introduced the notion of the *slant helix* in $\mathbb{E}^3$ as the space curve whose principal normal vector makes a constant angle with a fixed direction. In Minkowski 3-space, slant helices are studied by Ali & Lopez, 2011. Ziplar *et al.* 2012 defined special helices, called the *Darboux helices*, whose Darboux vector makes a constant angle with a fixed straight line. Senol *et al.* 2012 gave the necessary and sufficient conditions for a non-null curve in Minkowski 3-space to be a Darboux helix. The *rectifying curves* in the Euclidean 3-space are defined by Chen, 2003 as the space curves, whose position vector always lies in its rectifying plane. Some characterizations of Euclidean rectifying curves are given by Chen & Dillen, 2005. In Minkowski 3-space, rectifying curves are studied by Ilarslan *et al.* 2003. The curves of the constant precession in $\mathbb{E}^3$ are introduced by Scofield, 1995. Such curves can be considered as a special kind of the Darboux helices.

In this paper, we introduce the notion of the pseudo null and the null Cartan Darboux helices in Minkowski 3-space. We obtain the relations between the pseudo null and null Cartan Darboux helices and the slant helices. We characterize the Darboux vectors of the pseudo null and the null Cartan curves in terms of the Darboux helices and the rectifying curves. As an application, we find an explicit parameter equation of a pseudo null curve of constant precession and characterize such curve in terms of the rectifying curves. We also prove that there are no null Cartan curves of constant precession. Finally, we give some examples of the pseudo null and the null Cartan Darboux helices.

2. Preliminaries

The Minkowski 3-space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ equipped with the standard flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{R}^3_1 \setminus \{0\}$ can have one of three causal characters: it can be *spacelike*, *timelike* or *null* (*lightlike*), if $\langle x, x\rangle$ is positive, negative or zero, respectively (O'Neill, 1983). In particular, the vector $x = 0$ is a spacelike. The *norm* (length) of a vector $x \in \mathbb{R}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$. An arbitrary curve $\alpha : I \to \mathbb{R}^3_1$ can locally be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ satisfy $\langle \alpha'(s), \alpha'(s)\rangle > 0$, $\langle \alpha'(s), \alpha'(s)\rangle < 0$ or $\langle \alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, respectively (O'Neill, 1983).

A spacelike curve $\alpha : I \to \mathbb{R}^3_1$ is called a *pseudo null curve*, if its principal normal vector $N(s)$ and its binormal vector $B(s)$ are linearly independent null vectors. The Frenet formulae of a pseudo null curve α have the form (Walrave, 1995)

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.2}$$

where the first curvature $\kappa(s) = 1$. The second curvature (torsion) $\tau(s)$ is an arbitrary function of the arclength parameter s of α. The Frenet's frame vectors of α satisfy the equations

$$\begin{aligned} &\langle T, T\rangle = 1, \ \langle N, N\rangle = \langle B, B\rangle = 0, \\ &\langle T, N\rangle = \langle T, B\rangle = 0, \ \langle N, B\rangle = 1, \end{aligned} \tag{2.3}$$

and

$$T \times N = N, \ N \times B = T, \ B \times T = B. \tag{2.4}$$

In particular, a curve $\alpha : I \to \mathbb{R}^3_1$ is called a *null curve*, if its tangent vector $\alpha'(s) = T(s)$ is a null vector. A null curve α is called a *null Cartan curve*, if it is parameterized by the pseudo-arc length function $s = s(t)$ defined by Bonnor (1969)

$$s(t) = \int_0^t \sqrt{||\alpha''(u)||}\, du. \tag{2.5}$$

It is known that there exists a unique Cartan frame $\{T, N, B\}$ along a null Cartan curve α satisfying the Cartan equations (Duggal & Jin, 2007):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & -\kappa \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

where the first curvature $\kappa(s) = 1$. The second curvature (torsion) $\tau(s)$ is an arbitrary function of the pseudo-arc length parameter s. The Cartan's frame vectors of α satisfy the relations

$$\begin{aligned} \langle T, T\rangle = \langle B, B\rangle = 0,\ \langle N, N\rangle = 1, \\ \langle T, N\rangle = \langle N, B\rangle = 0,\ \langle T, B\rangle = 1, \end{aligned} \tag{2.7}$$

and

$$T \times B = N,\ N \times T = T,\ B \times N = B. \tag{2.8}$$

Definition 2.1. (Ilarslan *et al.* 2003) A non-null (null Cartan) curve $\alpha : I \to \mathbb{R}^3_1$ with the Frenet (Cartan) frame $\{T, N, B\}$ is called a *rectifying curve*, if its position vector with respect to some chosen origin always lies in its rectifying plane $N^{\perp}$.

The rectifying plane $N^{\perp}$ represents the orthogonal complement of the principal normal vector field N of α. Therefore, the position vector α of rectifying curve satisfies the relation $\langle \alpha, N\rangle = 0$.

Definition 2.2. (Ali & Lopez, 2011) A non-null (null Cartan) curve $\alpha : I \to \mathbb{R}^3_1$ with the Frenet (Cartan) frame $\{T, N, B\}$ is called a *slant helix*, if there exists a non-zero fixed direction $U \in \mathbb{R}^3_1$ such that $\langle N, U\rangle = constant$.

A fixed direction U in Definition 2.2 is called an *axis* of the helix.

3. Pseudo null darboux helices in $\mathbb{R}^3_1$

When the Frenet frame $\{T, N, B\}$ of a non-geodesic pseudo null curve α makes an instantaneous helix motion in $\mathbb{R}^3_1$, there exists an axis of the frame's rotation. The direction of such axis is given by the vector

$$D(s) = -\tau(s)T(s) + N(s), \tag{3.1}$$

which is called a *Darboux vector (centrode)*. A Darboux vector D satisfies the *Darboux equations*

$$\begin{aligned} T'(s) &= T(s) \times D(s), \\ N'(s) &= N(s) \times D(s), \\ B'(s) &= B(s) \times D(s). \end{aligned}$$

We introduce the notion of a pseudo null Darboux helix in $\mathbb{R}^3_1$ as follows.

Definition 3.1. A pseudo null curve $\alpha : I \to \mathbb{R}^3_1$ with the Darboux vector D is called a *Darboux helix*, if there exists a non-zero fixed direction $U \in \mathbb{R}^3_1$, such that holds $\langle D, U \rangle = constant$.

Theorem 3.1. *Every pseudo null curve* $\alpha : I \to \mathbb{R}^3_1$ *is a Darboux helix.*

Proof. Assume that α is a pseudo null curve in $\mathbb{R}^3_1$ parameterized by the arc-length parameter s. Denote by $\{T, N, B\}$ the Frenet frame of α. We distinguish the next three cases, depending on the torsion $\tau(s)$ of α.

(A) $\tau(s) = 0$. Then α is a pseudo null circle with parameter equation

$$\alpha(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s).$$

By using the equation (3.1) we easily obtain

$$D(s) = N(s) = (1, 1, 0).$$

Therefore, any non-zero constant vector U in $\mathbb{R}^3_1$ obviously satisfies $\langle U, D \rangle = constant$, which means that α is a Darboux helix.

(B) $\tau(s) = constant \neq 0$. Consider a non-zero vector $U \in \mathbb{R}^3_1$ given by

$$U(s) = \left(\frac{a_0}{\tau}e^{\tau s} + b_0\right)T + \left(-\frac{a_0}{2\tau^2}e^{\tau s} - \frac{b_0}{\tau} + c_0 e^{-\tau s}\right)N + a_0 e^{\tau s}B, \tag{3.2}$$

where $a_0 \in \mathbb{R}^+$, $b_0 \in \mathbb{R}$, $c_0 \in \mathbb{R}$. Differentiating the equation (3.2) with respect to s and using the equation (2.2), we obtain $U'(s) = 0$. Consequently, U is a fixed vector. By using the equations (2.3), (3.1) and (3.2) we get

$$\langle D, U \rangle = -\tau b_0 = constant.$$

According to the Definition 3.1, the curve α is a Darboux helix with an axis U.

(C) $\tau(s) \neq constant$. Consider a non-zero vector $U \in \mathbb{R}^3_1$ given by

$$U(s) = e^{-\int \tau(s)\, ds} N(s). \tag{3.3}$$

Differentiating the equation (3.3) with respect to s and using the equation (2.2), we obtain $U'(s) = 0$. Consequently, U is a fixed vector. By using the equations (2.3), (3.1) and (3.3) we get

$$\langle D, U \rangle = 0.$$

Hence α is a Darboux helix with an axis U.

Theorem 3.2. *Every pseudo null curve* $\alpha : I \to \mathbb{R}^3_1$ *is a slant helix.*

Proof. Assume that α is a pseudo null curve in $\mathbb{R}^3_1$ parameterized by the arc-length parameter s. Denote by $\{T, N, B\}$ the Frenet frame of α. Depending on the torsion $\tau(s)$ of α, we distinguish the next three cases.

(A) $\tau(s) = 0$. Then α is a pseudo null circle with parameter equation

$$\alpha(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s)$$

whose principal normal vector is given by

$$N(s) = \alpha''(s) = (1, 1, 0).$$

Consequently, any non-zero constant vector U in $\mathbb{R}^3_1$ satisfies $\langle U, N \rangle = constant$. According to the Definition 2.2, α is a slant helix.

(B) $\tau(s) = constant \neq 0$. Consider a non-zero vector $U \in \mathbb{R}^3_1$ given by

$$U(s) = a_0 T(s) + \left(-\frac{a_0}{\tau} + b_0 e^{-\tau s}\right) N(s), \tag{3.4}$$

where $a_0 \in \mathbb{R}$, $b_0 \in \mathbb{R}$. Differentiating the equation (3.4) with respect to s and using the equation (2.2), we obtain $U'(s) = 0$. Hence U is a fixed vector. The vector U is a spacelike if $a_0 \neq 0$, or null if $a_0 = 0$. The relations (2.3) and (3.4) imply

$$\langle N, U \rangle = 0,$$

which means that α is a slant helix.

(C) $\tau(s) \neq constant$. Consider a non-zero vector U in $\mathbb{R}^3_1$ given by

$$U(s) = a_0 T(s) + e^{-\int \tau(s)\, ds}(c_0 - a_0 \int e^{\int \tau(s)\, ds}\, ds) N(s), \tag{3.5}$$

where $a_0 \in \mathbb{R}$, $b_0 \in \mathbb{R}$. Differentiating the equation (3.5) with respect to s and using the equation (2.2), we obtain $U'(s) = 0$. Hence U is a fixed vector. The vector U is a spacelike if $a_0 \neq 0$, or null if $a_0 = 0$. From the relations (2.3) and (3.5) we find

$$\langle N, U \rangle = 0.$$

Therefore, α is a slant helix.

The next corollary gives a relationship between pseudo null Darboux helices and pseudo null slant helices with a non-zero constant torsion.

Corollary 3.1. *Every pseudo null slant helix with a non-zero constant torsion is a pseudo null Darboux helix.*

Remark 3.1. *The equation (3.2) and the Definition 2.2 imply that the converse is not true. Hence a class of pseudo null slant helices with non-zero constant torsion is a subclass of all pseudo null Darboux helices with non-zero constant torsion having a common axis.*

Corollary 3.2. *The pseudo null curve $\alpha : I \to \mathbb{R}^3_1$ with non-constant torsion is a Darboux helix, if and only if it is a slant helix with a null axis.*

Remark 3.2. *According to Theorem 3.2 and Corollary 3.2, it follows that a class of all pseudo null Darboux helices with non-constant torsion is a subclass of all pseudo null slant helices with non-constant torsion having a common axis.*

Theorem 3.3. *The Darboux vector of a pseudo null curve $\alpha : I \to \mathbb{R}^3_1$ with non-constant torsion is a Darboux helix.*

Proof. Assume that α is a pseudo null curve parameterized by the arc-length parameter s and with the Darboux vector D given by the equation (3.1). In order to prove the theorem, it is sufficient to prove that the Darboux vector D is a non-geodesic pseudo null curve. Denote by $s^\star$ the arc-length parameter of D and by $\{T_D, N_D, B_D\}$ the Frenet frame of D. Since $\tau'(s) \neq 0$, it follows that the unit tangent vector T_D of D is given by

$$T_D = \frac{dD}{ds} \cdot \frac{ds}{ds^\star} = -sgn(\tau')T.$$

Therefore,

$$T_D = \pm T. \tag{3.6}$$

Differentiating the last equation with respect to s and using the equation (2.2), we get

$$\frac{dT_D}{ds^\star}\frac{ds^\star}{ds} = \pm N. \tag{3.7}$$

Since the vector field $\frac{dT_D}{ds^{\star}}$ is collinear with a null vector field N, it follows that D is a pseudo null curve with the first curvature $\kappa_D = 1$ and a non-zero torsion τ_D. According to the Theorem 3.1, the Darboux vector D of α is a Darboux helix.

Theorem 3.4. *The Darboux vector of a pseudo null curve $\alpha : I \to \mathbb{R}^3_1$ with a non-constant torsion is a rectifying curve.*

Proof. Assume that α is a pseudo null curve parameterized by the arc-length parameter s and with the Darboux vector D given by

$$D(s) = -\tau(s)T(s) + N(s), \tag{3.8}$$

where $\{T, N, B\}$ is the Frenet frame of α. Denote by $\{T_D, N_D, B_D\}$ the Frenet frame of D. From the equations (2.5) and (3.7) we get

$$N_D|\tau'| = \pm N. \tag{3.9}$$

The equations (3.6) and (3.9) imply that the vector fields T and N are parallel with the vector fields T_D and N_D, respectively. This means that the position vector D of the Darboux vector given by the equation (3.8) lies in its rectifying plane $N_D^{\perp}$ spanned by T_D and N_D. According to the Definition 2.1, the Darboux vector D is a rectifying curve.

4. Null cartan darboux helices in $\mathbb{R}^3_1$

When the Cartan frame $\{T, N, B\}$ of a non-geodesic null Cartan curve α moves along α in $\mathbb{R}^3_1$, there exists an axis of the frame's rotation. The direction of such axis is given by the *Darboux vector (centrode),* which has the form

$$D(s) = -\tau(s)T(s) + B(s), \tag{4.1}$$

and satisfies the Darboux equations

$$\begin{aligned} T'(s) &= T(s) \times D(s), \\ N'(s) &= N(s) \times D(s), \\ B'(s) &= B(s) \times D(s). \end{aligned}$$

We introduce the notion of the null Cartan Darboux helix in $\mathbb{R}^3_1$ as follows.

Definition 4.1. A null Cartan curve $\alpha : I \to \mathbb{R}^3_1$ with the Darboux vector D is called a *Darboux helix, if there exists a non-zero fixed direction $U \in \mathbb{R}^3_1$, such that holds $\langle D, U \rangle = constant$.*

Theorem 4.1. *A null Cartan curve $\alpha : I \to \mathbb{R}^3_1$ is a Darboux helix, if and only if the* *torsion of α is a non-zero constant.*

Proof. Assume that α is a null Cartan curve with a non-zero constant torsion τ, parameterized by the pseudo-arc length function s. Consider a non-zero vector $U \in \mathbb{R}^3_1$ given by

$$U(s) = a(s)T(s) + b(s)N(s) + c(s)B(s), \tag{4.2}$$

where

$$\begin{aligned} a(s) &= \tau(a_0 \cos(\sqrt{2\tau}s) + b_0 \sin(\sqrt{2\tau}s)) + \frac{c_0}{2}, \\ b(s) &= \sqrt{2\tau}(b_0 \cos(\sqrt{2\tau}s) - a_0 \sin(\sqrt{2\tau}s)), \\ c(s) &= a_0 \cos(\sqrt{2\tau}s) + b_0 \sin(\sqrt{2\tau}s) - \frac{c_0}{2\tau}, \end{aligned}$$

and a_0, b_0, c_0 are real constants not all equal to zero. Differentiating the equation (4.2) with respect to s and using the equation (2.6) we obtain $U'(s) = 0$. Therefore, U is a fixed vector. Moreover, the equations (4.1) and (4.2) imply

$$\langle D, U \rangle = c_0 = constant.$$

According to the Definition 4.1, α is a Darboux helix with an axis U.

Conversely, assume that α is a null Cartan Darboux helix with an axis U. Then

$$\langle D, U \rangle = constant. \tag{4.3}$$

Differentiating the equation (4.3) with respect to s and using the equations (2.6) and (4.1), we obtain

$$\tau' \langle T, U \rangle = 0.$$

It follows that $\tau' = 0$ or $\langle T, U \rangle = 0$. If

$$\langle T, U \rangle = 0, \tag{4.4}$$

differentiating the equation (4.4) with respect to s and using the equation (2.6), we find

$$\langle N, U \rangle = 0. \tag{4.5}$$

Differentiating the last equation with respect to s and using the equations (2.6) and (4.4), we get

$$\langle B, U \rangle = 0. \tag{4.6}$$

With respect to the pseudo-orthonormal frame $\{T, N, B\}$, an axis U of α can be decomposed as

$$U = \langle B, U \rangle T + \langle N, U \rangle N + \langle T, U \rangle B.$$

Substituting the equations (4.4), (4.5) and (4.6) in the last relation, we get $U = 0$, which is a contradiction. Therefore, $\tau' = 0$ and hence $\tau(s) = constant \neq 0$.

Theorem 4.2. *Every null Cartan curve $\alpha : I \to \mathbb{R}^3_1$ with non-zero constant torsion is a slant helix with non-null axis.*

Proof. Assume that α is a null Cartan curve parameterized by the pseudo-arc s and with a non-zero constant torsion τ. Consider a non-zero vector $U \in \mathbb{R}^3_1$ given by

$$U(s) = c_0 T(s) - \frac{c_0}{\tau} B(s), \quad c_0 \in \mathbb{R}, \quad c_0 \neq 0. \tag{4.7}$$

Differentiating the equation (4.7) with respect to s and using the Cartan equations (2.6), we find $U'(s) = 0$. This means that U is a fixed vector. From the equations (2.7) and (4.7) we get $\langle N, U \rangle = 0$. Therefore, the curve α is a slant helix with an axis U.

By using the Theorem 4.2, we obtain the following relationship between null Cartan slant helices and null Cartan Darboux helices.

Corollary 4.1. *Every null Cartan slant helix $\alpha : I \to \mathbb{R}^3_1$ with non-zero constant torsion is a Darboux helix.*

Proof. Assume that α is a null Cartan slant helix with a non-zero constant torsion. According to the proof of the Theorem 4.2, its axis is given by the equation (4.7). By using the equations (2.7), (4.1) and (4.7), we get

$$\langle U, D \rangle = \langle c_0 T - \frac{c_0}{\tau} B, -\tau T + B \rangle = 2c_0 = constant \neq 0.$$

According to the Definition 2.2, α is a Darboux helix.

By using the equation (4.2), it can be verified that the converse is not true. Namely, there holds the next corollary.

Corollary 4.2. *A null Cartan Darboux helix $\alpha : I \to \mathbb{R}^3_1$ with non-zero constant torsion is a slant helix, if an axis U is orthogonal to the principal normal vector N of α.*

Remark 4.1. *According to the Corollary 4.1 and the Corollary 4.2, it follows that a class of the null Cartan slant helices with a non-zero constant torsion is the subclass of all null Cartan Darboux helices having a common axis.*

Now we can ask the following question: "*Can the Darboux vector of a null Cartan curve with non-constant torsion be a Darboux helix?*"

Theorem 4.3. *The Darboux vector of a null Cartan curve* $\alpha : I \to \mathbb{R}^3_1$ *with a non-constant torsion* τ *is a Darboux helix, if the torsion* τ *of* α *satisfies the differential equation*

$$\frac{\tau''^2}{4\tau'^3} - \left(\frac{\tau''}{\tau'}\right)' + \frac{2\tau}{\tau'} = constant \neq 0. \tag{4.8}$$

Proof. Assume that the torsion $\tau(s)$ of α satisfies the equation (4.8), where s is the pseudo-arc length parameter of α. Denote by D a Darboux vector of α given by the equation (4.1). Since $D'(s) = -\tau'(s)T(s)$, it follows that $\langle D', D'\rangle = 0$, so D is a null curve. In particular, $D''(s) = -\tau''(s)T(s) - \tau'(s)N(s)$ and thus $\langle D'', D''\rangle = \tau'^2$. We can reparameterize a Darboux vector D by using its pseudo-arc length function $s^\star$ defined by

$$s^\star(s) = \int_0^s \sqrt{||D''(t)||}\, dt.$$

Then $D(s^\star)$ is a null Cartan curve, such that $\langle D''(s^\star), D''(s^\star)\rangle = 1$. Next we will calculate the torsion τ_D of D. Denote by $\{T_D, N_D, B_D\}$ the Cartan frame along D. A null tangent vector T_D of D is given by

$$T_D = \frac{dD}{ds} \cdot \frac{ds}{ds^\star} = -sgn(\tau')\sqrt{|\tau'|}T. \tag{4.9}$$

Differentiating the previous equation with respect to $s^\star$ and using the equation (2.6), we find

$$N_D = \frac{dT_D}{ds} \cdot \frac{ds}{ds^\star} = -\frac{\tau''}{2|\tau'|}T - sgn(\tau')N. \tag{4.10}$$

Differentiating the last relation with respect to $s^\star$ and using the equation (2.6), we get

$$\begin{aligned} N_D' &= \frac{dN_D}{ds} \cdot \frac{ds}{ds^\star} \\ &= \left[-\frac{\tau''}{2|\tau'|\sqrt{|\tau'|}} + sgn(\tau')\frac{\tau}{\sqrt{|\tau'|}}\right] T - \frac{\tau''}{2|\tau'|\sqrt{|\tau'|}} N \\ &+ sgn(\tau')\frac{1}{\sqrt{|\tau'|}} B. \end{aligned}$$

By using the equation (2.7) and the last equation, we obtain

$$\langle N_D', N_D'\rangle = sgn(\tau')\left(\frac{\tau''^2}{4\tau'^3} - \left(\frac{\tau''}{\tau'}\right)' + \frac{2\tau}{\tau'}\right). \tag{4.11}$$

On the other hand, by using the Cartan equations (2.6) we have

$$\langle N_D', N_D' \rangle = 2\tau_D. \tag{4.12}$$

The equations (4.11) and (4.12) yield

$$\tau_D = \frac{sgn(\tau')}{2}\left(\frac{\tau''^2}{4\tau'^3} - \left(\frac{\tau''}{\tau'}\right)' + \frac{2\tau}{\tau'}\right). \tag{4.13}$$

By assumption, the torsion τ of α satisfies the equation (4.8). By using the equations (4.8) and (4.13), we conclude that the torsion τ_D of D is a non-zero constant. According to the Theorem 4.1, it follows that a Darboux vector D of α is a Darboux helix, which completes the proof.

Next we characterize the Darboux vectors of the null Cartan curves in terms of the rectifying curves.

Theorem 4.4. *The Darboux vector of a null Cartan curve $\alpha : I \to \mathbb{R}^3_1$ parameterized by the pseudo-arc length function s is a rectifying curve if and only if the torsion τ of α is a linear function given by $\tau(s) = a_0 s + b_0$, $a_0 \neq 0$, $a_0, b_0 \in \mathbb{R}$.*

Proof. Assume that α is a null Cartan curve parameterized by the pseudo-arc length function s and with the torsion

$$\tau(s) = a_0 s + b_0, \quad a_0 \neq 0, \quad a_0, b_0 \in \mathbb{R}. \tag{4.14}$$

Denote by D Darboux vector of α with Cartan frame $\{T_D, N_D, B_D\}$ along D. By using the equations (4.9), (4.10) and (4.14), it follows that the vector fields T and N are parallel with the vector fields T_D and N_D, respectively. This means that B and B_D are also parallel. Consequently, a Darboux vector D of α given by equation (4.1) lies in its rectifying plane $N_D^{\perp} = span\{T_D, B_D\}$. According to the Definition 2.1, the Darboux vector D is a rectifying curve.

Conversely, assume that the Darboux vector D of α is a rectifying curve. Then D satisfies the equation

$$\langle D, N_D \rangle = 0,$$

where N_D is the principal normal vector of D. Substituting (4.1) and (4.10) in the last relation, we get

$$\langle -\tau T + B, -\frac{\tau''}{2|\tau'|}T - sgn(\tau')N \rangle = 0. \tag{4.15}$$

The relations (2.7) and (4.15) imply $\tau'' = 0$. Hence $\tau(s) = a_0 s + b_0$, $a_0 \neq 0$, $a_0, b_0 \in \mathbb{R}$, which completes the proof of the theorem.

5. Pseudo null and null cartan curves of constant precession

According to Scofield, 1995 the Euclidean *curves of constant precession* are defined as the curves, whose Darboux vectors make a constant angle with a fixed direction and rotate about it with a constant speed. Therefore, the curves of constant precession are the special kind of Darboux helices. In Minkowski 3-space, the spacelike curves of constant precession with the spacelike and the timelike principal normal vector are studied by Nešović *et al.* 2005.

Consider the pseudo null Darboux helix α with the Darboux vector D. Assume that D has a constant speed, i.e. $||D'(s)|| = |\tau'(s)| = constant \neq 0$. The last condition gives $\tau(s) = c_1 s + c_2$, $c_1 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$, where $\mathbb{R}_0$ denotes $\mathbb{R}\backslash\{0\}$. Hence the following theorem is proved.

Theorem 5.1. *The pseudo null Darboux helix $\alpha : I \to \mathbb{R}^3_1$ parameterized by the arc-length function s and with the torsion $\tau(s) \neq constant$ is the curve of constant precession, if and only if it has the natural equations*

$$\kappa(s) = 1, \quad \tau(s) = c_1 s + c_2, \quad c_1 \in \mathbb{R}_0, \ c_2 \in \mathbb{R}. \tag{5.1}$$

Remark 5.1. *The curvature functions of the pseudo null curve α of constant precession are the eigenfunctions of the Laplacian Δ of α corresponding to the zero eigenvalue.*

By using the natural equations (5.1) and the Frenet equations (2.2), we obtain the explicit parameter equation of the pseudo null curve α of constant precession as follows (Figure 1):

$$\alpha(s) = (\int (\int e^{c_1 \frac{s^2}{2} + c_2 s}\, ds)\, ds, \int (\int e^{c_1 \frac{s^2}{2} + c_2 s}\, ds)\, ds, s), \tag{5.2}$$

where $c_1 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$.

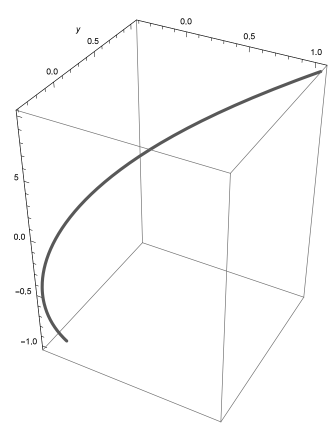

Fig. 1. The pseudo null curve of constant precession

Theorem 5.2. *Let $\alpha : I \to \mathbb{R}^3_1$ be a pseudo null curve parameterized by the arc-length function s. Up to isometries of $\mathbb{R}^3_1$, the curve α is a rectifying curve if and only if its torsion is given by*

$$\tau(s) = -\frac{s + b'(s) + c}{b(s)}, \quad c \in \mathbb{R}, \tag{5.3}$$

where $b(s)$ is some non-zero differentiable function.

Proof. Assume that α is a pseudo null rectifying curve. Then its position vector satisfies the relation

$$\alpha(s) = a(s)T(s) + b(s)N(s),$$

for some arbitrary differentiable functions $a(s)$ and $b(s)$. Differentiating the last relation with respect to s and using the equation (2.2), we get

$$a'(s) = 1, \quad a(s) + b'(s) + b(s)\tau(s) = 0.$$

It follows that

$$a(s) = s + c, \quad \tau(s) = -\frac{s + b'(s) + c}{b(s)}, \quad c \in \mathbb{R}.$$

Conversely, assume that the torsion of α is given by the equation (5.3). Applying the Frenet equations (2.2), we easily find

$$\frac{d}{ds}[\alpha(s) - (s + c)T(s) - b(s)N(s)] = 0.$$

Up to isometries of $\mathbb{R}^3_1$, α is a rectifying curve, which proves the theorem.

Corollary 5.1. *Every pseudo null curve of constant precession in $\mathbb{R}^3_1$ is a rectifying curve.*

In general case, the converse is not true. By using the Theorem 4.1, we get the next corollary.

Theorem 5.3. *There are no null Cartan curves of constant precession in $\mathbb{R}^3_1$.*

6. Some examples of pseudo null and null cartan darboux helices in $\mathbb{R}^3_1$

Example 6.1. Consider a pseudo null curve α in $\mathbb{R}^3_1$ with the parameter equation (Figure 2)

$$\alpha(s) = \left(\frac{s^3}{12}, \frac{s^3 + 12s}{12\sqrt{2}}, \frac{s^3 - 12s}{12\sqrt{2}}\right).$$

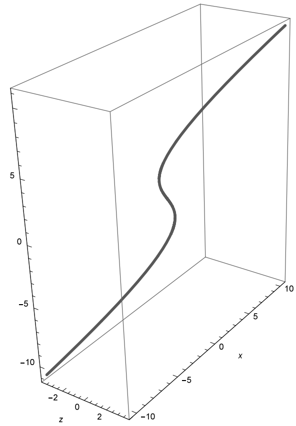

Fig. 2. The pseudo null Darboux helix α

According to the Theorem 3.1, the curve α is a Darboux helix whose Frenet frame reads

$$\begin{aligned} T(s) &= \left(\tfrac{s^2}{4}, \tfrac{s^2+4}{4\sqrt{2}}, \tfrac{s^2-4}{4\sqrt{2}}\right), \\ N(s) &= \tfrac{s}{2}\left(1, \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right), \\ B(s) &= \left(-\tfrac{s^3}{16} - \tfrac{1}{s}, \tfrac{s}{2\sqrt{2}} + \tfrac{1}{\sqrt{2}s} - \tfrac{s^3}{16\sqrt{2}}, -\tfrac{s}{2\sqrt{2}} + \tfrac{1}{\sqrt{2}s} - \tfrac{s^3}{16\sqrt{2}}\right). \end{aligned} \tag{6.1}$$

The curvature functions of α are given by

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{s}. \tag{6.2}$$

From the equations (3.1), (6.1) and (6.2), it follows that the Darboux vector of α is given by

$$D(s) = \left(\frac{s}{4}, \frac{\sqrt{2}}{8}\left(\frac{s^2-4}{s}\right), \frac{\sqrt{2}}{8}\left(\frac{s^2+4}{s}\right)\right).$$

Since $\tau(s) \neq constant$, by using the relations (3.3), (6.1) and (6.2) we find that an axis of α has the form

$$U = (\frac{1}{2}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}).$$

Therefore, U is a null axis. According to the Corollary 3.2, the curve α is a slant helix.

Example 6.2. Let us consider a pseudo null curve α in $\mathbb{R}^3_1$ given by (Figure 3)

$$\alpha(s) = \left(\frac{s^3+3s^2}{9}, \frac{\sqrt{2}(s^3+3s^2+9s)}{18}, \frac{\sqrt{2}(s^3+3s^2-9s)}{18} \right).$$

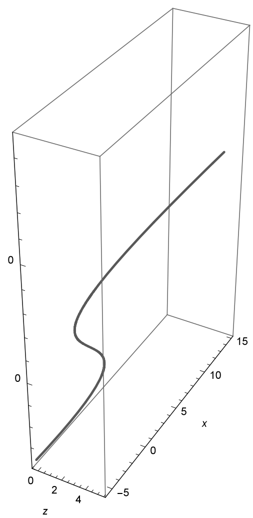

Fig. 3. The pseudo null Darboux helix α

By Theorem 3.1, the curve α is a Darboux helix whose Frenet frame reads

$$\begin{aligned} T(s) &= \left(\tfrac{s^2+2s}{3}, \tfrac{\sqrt{2}(s^2+2s+3)}{6}, \tfrac{\sqrt{2}(s^2+2s-3)}{6} \right), \\ N(s) &= \left(\tfrac{s+1}{3} \right) \left(2, \sqrt{2}, \sqrt{2}\right), \\ B(s) &= \left(-\tfrac{(s^2+2s)^2+9}{12(s+1)}, \tfrac{9-(s^2+2s)^2-6(s^2+2s)}{12\sqrt{2}(s+1)}, \tfrac{9-(s^2+2s)^2+6(s^2+2s)}{12\sqrt{2}(s+1)} \right). \end{aligned} \tag{6.3}$$

The first and the second curvature of α have the form

$$\kappa(s) = 1, \quad \tau(s) = \frac{1}{1+s}. \tag{6.4}$$

From the equations (3.1), (6.3) and (6.4), we obtain that the Darboux vector of α reads

$$D(s) = \left(\frac{2(s^2+2s+2)}{6(s+1)}, \frac{\sqrt{2}(s^2+2s-1)}{6(s+1)}, \frac{\sqrt{2}(s^2+2s+5)}{6(s+1)} \right).$$

Relations (3.3), (6.3) and (6.4) imply that an axis of α is given by

$$U = (\frac{2}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}).$$

Since U is a null axis, the Corollary 3.2 implies that α is a slant helix.

Example 6.3. Let α be a pseudo null helix in $\mathbb{R}^3_1$ with the parameter equation (Figure 4)

$$\alpha(s) = (e^s, e^s, s).$$

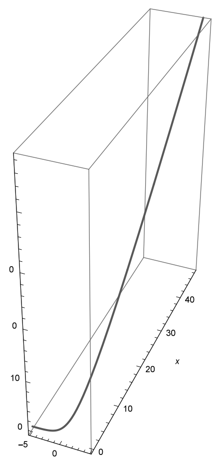

Fig. 4. The pseudo null Darboux helix α

According to the Theorem 3.1, the curve α is a pseudo null Darboux helix. The Frenet frame of α reads

$$\begin{aligned} T(s) &= (e^s, e^s, 1), \\ N(s) &= (e^s, e^s, 0), \\ B(s) &= \left(-\tfrac{e^{2s}+1}{2e^s}, \tfrac{1-e^{2s}}{2e^s}, -1\right). \end{aligned} \tag{6.5}$$

The curvature functions of α are given by

$$\kappa(s) = 1, \quad \tau(s) = 1. \tag{6.6}$$

From the equations (3.1), (6.5) and (6.6), it follows that the Darboux vector of α is given by

$$D(s) = (0, 0, -1).$$

Then obviously $\langle D, U\rangle = constant$ for every constant vector $U \in \mathbb{R}^3_1$.

(i) If U is a spacelike constant vector given by $U = (1, 1, 2)$, relation (6.5) implies $\langle U, N\rangle = 0$. Consequently, the curve α is Darboux and slant helix having a common spacelike axis U.

(ii) If U is a null vector given by $U = (2, 2, 0)$, by using (6.5) we get $\langle U, N\rangle = 0$. Therefore, the curve α is Darboux and slant helix having a common null axis U.

Example 6.4. Let us consider a null Cartan curve α in $\mathbb{R}^3_1$ with the parameter equation (Figure5)

$$\alpha(s) = \left(-\frac{s}{2}, \frac{1}{4}\sin(2s), \frac{1}{4}\cos(2s)\right).$$

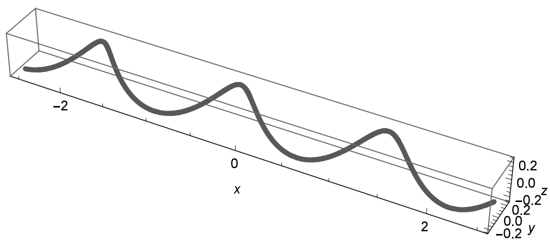

Fig. 5. The null Cartan Darboux helix α

The Cartan frame of the curve α reads

$$\begin{aligned} T(s) &= \left(-\tfrac{1}{2}, \tfrac{1}{2}\cos(2s), -\tfrac{1}{2}\sin(2s)\right), \\ N(s) &= (0, -\sin(2s), -\cos(2s)), \\ B(s) &= (1, \cos(2s), -\sin(2s)). \end{aligned} \tag{6.7}$$

The curvatures κ and τ of α have the form

$$\kappa(s) = 1, \quad \tau(s) = 2. \tag{6.8}$$

According to the Theorem 4.1, the curve α is a Darboux helix. By using the equations (4.1), (6.7) and (6.8), we find that the Darboux vector of α is given by

$$D(s) = (2, 0, 0).$$

Hence $\langle D, U\rangle = constant$ for every constant vector $U \in \mathbb{R}^3_1$. Taking $U = (1, 0, 0)$, relation (6.7) implies $\langle U, N\rangle = 0$. Therefore, α is Darboux and slant helix having a common timelike axis U.

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خلاصة

نقدم في هذا البحث لوالب كارتان داربو الصفرية و شبة الصفرية في فضاء مينكوفسكي الثلاثي. و نقوم بإيجاد خصائص متجهات داربو لمنحنيات كارتان الصفرية و شبة الصفرية و ذلك باستخدام لوالب داربو و المنحنيات المقومة. و نوجد معادلة وسيطيه صريحة لمنحنى شبة صفري ثابت الدقة. وأخيراً نقوم بإعطاء بعض الأمثلة.