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Generalized pseudo Ricci symmetric manifolds with semi-symmetric metric connection

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ABSTRACT

In this paper, firstly an example of a manifold with almost constant curvature and nearly quasi constant curvature which is neither quasi Einstein nor nearly quasi Einstein is given. Later, the existence of a generalized pseudo Ricci symmetric manifold is proven by a nontrivial concrete example by using Exp-function Method and Differential Transform Method.

Keywords: Generalized pseudo Ricci symmetric manifold; sectional curvature; semisymmetric manifold; semi-symmetric metric connection.

MSC Classification: 53B15, 53B20, 34A25.

INTRODUCTION

Semi-symmetric metric connection plays an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one defined point, i.e. the north pole, then this displacement is semi-symmetric and metric (Schouten, 1954). Mathematicians invented the "Moscow displacement" during a mathematical congress in Moscow in 1934. The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walks in the streets always facing the Kremlin, then this displacement is semi-symmetric and metric (Schouten, 1954; Shaikh *et al.*, 2010).

Freidmann & Schouten (1924) introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Then, Hayden introduced the idea of metric connection with torsion on a Riemannian manifold (Hayden, 1932). Yano (1970) considered a semi-symmetric metric connection on a Riemannian manifold. Imai (1972) found some properties of a Riemannien manifold with a semi-symmetric metric-connection. Nakao (1976) studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection. Binh (1990) considered a semi-symmetric connection and investigated some of its properties. Manifolds with a semi-symmetric metric connection satisfying some special conditions were studied by some authors (Altay & Özen, 2007; De, 1990; Shaikh *et al.*, 2008; Shaikh & Hui, 2010). The concept of a semi-symmetric metric connection has been applied to Kenmotsu manifold (Pathak & De, 2002), almost contact manifold (De & Sengupta, 2000) and Sasakian manifold (Pujar & De, 2000).

This paper is organized as follows: In the introduction section, the necessary notions and results which will be used in the next sections are given. In the next section, some theorems are proven related to $G(PRS)_n$ manifold admitting semi-symmetric metric connection. Later, in the following two sections, solution methods of a special differential equation are introduced. Finally, solution of a special differential equation belonging to $G(PRS)_n$ manifold is obtained in the last section.

By a triple (M, g, T), we mean (M, g) is a Riemannian manifold with a torsion tensor T. More generally, on a differentiable manifold equipped with an affine connection (that is a connection in the tangent bundle TM), torsion and curvature form the two fundamental invariants of the connection.

Let (M_n, g) be an n-dimensional differentiable Riemannian manifold of class C^∞ with the metric tensor g and ∇^* be a linear connection. A smooth linear connection ∇^* on (M_n, g) is said to be a semi-symmetric if the torsion tensor T of ∇^* satisfies the relation

$$T(X, Y) = \omega(Y)X - \omega(X)Y \tag{1}$$

for any vector fields X and Y on M_n and w is a 1-form associated with the torsion tensor T of the connection ∇^* given by $g(X, \rho) = \omega(X)$.

If ∇^* further satisfies the condition $\nabla^* g = 0$, then ∇^* is called semi-symmetric metric connection (Yano, 1970).

The relation between the semi-symmetric metric connection ∇^* and the Riemannian connection ∇ of (M_n, g) is given by Yano (1970)

$$\nabla^*_X Y = \nabla_X Y + w(Y)X - g(X, Y)\rho \tag{2}$$

for any vector field X, Y on M. In particular, if the 1-form ω vanishes identically then a semi-symmetric metric connection reduces to the Riemannian connection.

In this section, firstly a brief review will be given for the curvature tensor of Riemannian manifold with semi-symmetric metric connection. If $R^*(X, Y)Z$ and $R(X, Y)Z$ denote the curvature tensors with respect to the connections ∇^* and ∇, respectively, then we have (Yano, 1970)

$$\begin{aligned} R^*(X, Y)Z &= R(X, Y)Z - \alpha(Y, Z)X \\ &+\alpha(X, Z)Y - g(Y, Z)LX + g(X, Z)LY \end{aligned} \tag{3}$$

where α is a tensor field of type $(0, 2)$ defined by

$$\begin{aligned} \alpha(X, Y) &= g(LX, Y) \\ &= (\nabla_X w)(Y) - w(X)w(Y) + \frac{1}{2}w(\rho)g(X, Y) \end{aligned} \tag{4}$$

for any vector fields X and Y. Using (3), we get

$$S^*(Y, Z) = S(Y, Z) - (n - 2)\alpha(Y, Z) - \theta g(Y, Z) \tag{5}$$

where S^* and S denote respectively the Ricci tensor with respect to ∇^* and ∇, $\theta = g^{ih}\alpha_{ih} = trace\alpha$. The tensor α of type (0,2) given in equation (4) is not symmetric in general and hence it follows from (5) that the Ricci tensor S^* is not symmetric. But if we consider that the 1-form ω associated with the torsion tensor T is closed then it can be easily shown that the relation

$$(\nabla_X\omega)(Y) = (\nabla_Y\omega)(X) \tag{6}$$

holds for all vector fields X, Y. r^* and r denote the scalar curvatures with respect to the linear connection ∇^* and Levi-Civita connection ∇, respectively; then, they are related by the following form:

$$r^* = r - 2(n - 1)\theta \tag{7}$$

Chaki & Koley (1993) introduced the notion of a generalized pseudo Ricci symmetric manifold, if its Ricci tensor as of type (0, 2) is not identically zero and satisfies the condition

$$\begin{aligned} \nabla_X S(Y, Z) &= 2A(X)S(Y, Z) \\ &+B(Y)S(X, Z) + D(Z)S(X, Y) \end{aligned} \tag{8}$$

for all vector fields X, Y, Z where A, B, D are distinct non-zero 1-forms (not simultaneously zero). Such a manifold is called a generalized pseudo Ricci symmetric and n-dimensional manifold of this kind which is denoted by $G(PRS)_n$. In the study of a $G(PRS)_n$, an important role is played by the 1-form δ defined by $\delta(X) = B(X) - D(X)$ for all X such that $\delta(X) \neq 0$.

Now, we can state the following lemma which will be used in our subsequent work:

Lemma 1. *In a $G(PRS)_n$ with the defined metric, if $\delta(x) \neq 0$, then the scalar curvature r is none-zero and the Ricci tensor is of the form* (De & De, 1997)

$$S(X, Y) = rA(X)A(Y), A(X) = \frac{\delta(X)}{\| \delta(X) \|} \tag{9}$$

We will give some definitions and introduce some properties which will be used in the next section.

Einstein manifolds play an important role in Riemannian geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensors.

The notion of quasi-constant curvature tensor was introduced by Chen & Yano (1972) as follows:

$$\begin{aligned} R(X, Y, Z, W) &= p(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ &+q(g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W) \\ &+g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)) \end{aligned} \tag{10}$$

where p and q are scalar functions and A is non-zero 1-form.

The notion of quasi-Einstein manifold was introduced by Chaki & Maity (2000). A non-flat Riemannian manifold is called a quasi-Einstein manifold if its Ricci tensor $S(Y, Z)$ satisfies the following condition

$$S(Y, Z) = \alpha g(Y, Z) + \beta U(X)U(Y) \tag{11}$$

where α and β ($\beta \neq 0$) are scalar functions and U is non-zero 1-form such that $g(X, V) = U(X)$ for all vector fields X; V being a unit vector field. If $\beta = 0$, then the manifold reduces to an Einstein manifold.

In De & Gazi (2008), the authors introduced a type of non-flat Riemannian manifold (M_n, g) $(n \geq 2)$ whose Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(Y, Z) = ag(Y, Z) + bE(X, Y) \tag{12}$$

where a and b are non-zero scalar functions and E is non-zero symmetric tensor type (0,2). A non-flat Riemannian manifold which satisfies equation (12) is called a nearly quasi-Einstein manifold.

It is known from De *et al.* (2008) that the outer product of two covariant vectors is a covariant tensor of type (0,2), but the converse is not true in general. Hence, the manifolds which are quasi-Einstein are also nearly quasi-Einstein, but the converse is not true in general. For this reason, the name nearly quasi-Einstein was chosen.

A Riemannian manifold is said to be a manifold of nearly quasi-constant curvature, if the curvature tensor $R(X, Y, Z, W)$ of type (0,4) satisfies the condition (Gazi & De, 2009)

$$\begin{aligned} R(X, Y, Z, W) &= p(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ &+q(g(X, W)B(Y, Z) - g(X, Z)B(Y, W) \\ &+g(Y, Z)B(X, W) - g(Y, W)B(X, Z)) \end{aligned} \tag{13}$$

where p and q are scalar functions and B is non-zero symmetric tensor of type (0,2). An n-dimensional Riemannian manifold of nearly quasi-constant curvature will be denoted by $N(QC)_n$.

Chern (1956) studied a type of Riemannian manifold whose curvature tensor $R(X, Y, Z, W)$ of type (0,4) satisfies the condition

$$R(X, Y, Z, W) = F(X, Z)F(Y, W) - F(Y, Z)F(X, W) \tag{14}$$

where F is non-zero symmetric tensor of type (0,2). Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor F and denoted by $\psi(F_n)$. This manifold is important for the following reasons:

A Riemannian manifold (M_n, g) $(n \geq 3)$ is called semisymmetric if

$$R.R = 0 \tag{15}$$

holds on M. It is well known that the class of semisymmetric manifolds includes the set of locally symmetric manifolds $(\nabla R = 0)$ as a proper subset. A fundamental study on Riemannian semisymmetric manifolds was made by Szabo (1982) and Szabo (1985).

A Riemannian manifold (M_n, g) $(n \geq 3)$ is called Ricci semisymmetric if

$$R.S = 0 \tag{16}$$

holds on M (Mikes, 1980).

The class of Ricci semisymmetric manifolds includes the set of Ricci symmetric manifolds $(\nabla S = 0)$ as a proper subset. Every semisymmetric manifold is Ricci semisymmetric. The converse statement is not true in general.

ON A RIEMANNIAN MANIFOLD HAVING SEMI-SYMMETRIC METRIC CONNECTION

Let M be an n-dimensional Riemannian manifold. For the vector fields X, Y, Z and Levi-Civita connection ∇ of M, the curvature tensor R and Ricci operator S of M are defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \tag{17}$$

and

$$S(X, Y) = g(QX, Y) \tag{18}$$

respectively. Furthermore, for the vector field W, the Riemann Christoffel curvature tensor R of M is defined by $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ (Tanno, 1969).

Let Π be tangent plane to M at $P \in M$ given by X, Y. Then, the sectional curvature $K(\Pi)$ of Π defined by

$$K(X, Y)(g(X, X)g(Y, Y) - g(X, Y)^2) = g(R(X, Y)Y, X) \tag{19}$$

which is independent of choice of the basis X, Y for π. A tensor field R of type (1,2) on M is called algebraic curvature tensor field, if it has symmetric properties of the curvature tensor field of Riemannian manifolds. The curvature tensor R satisfies the second Bianchi identity if

$$(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(X, Z, W) + (\nabla_Z R)(X, Y, W) = 0 \tag{20}$$

The Weyl conformal curvature tensor C of a Riemannian manifold is defined by

$$\begin{aligned} C(X, Y, Z, W) = {} & R(X, Y, Z, W) - \tfrac{1}{(n-2)}(g(X, W)S(Y, Z)) \\ & -g(X, Z)S(Y, W) + g(Y, Z)S(X, W) \\ & -g(Y, W)S(X, Z)) + \tfrac{R}{(n-1)(n-2)}(g(X, W)g(Y, Z) \\ & -g(X, Z)g(Y, W) \end{aligned} \tag{21}$$

where $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ and r denote the scalar curvature of Riemannian manifold. If $C = 0$, then the manifold of dimension > 3 is called conformally flat.

Let $R^*(\Pi)$ be curvature tensor of (0,4) type by $R^*(X, Y, Z, W) = g(R^*(X, Y, Z, W)$. Let Π be tangent plane to Riemannian manifold with semi-symmetric metric connection M_n at $P \in M_n$ given by X, Y. Then, sectional curvature $K^*(\Pi)$ of Π defined by

$$K^*(X, Y)(g(X, X)g(Y, Y) - g^2(X, Y)) = g(R^*(X, Y)Y, X) \tag{22}$$

which is independent of the choice of the basis X, Y form. The case of a 2-dimensional Riemannian manifold having semi-symmetric metric connection need not to be considered, since it has only one plane at each point. Equation (22) can be rewritten as follows:

$$R^*(X, Y)Z = K^*(\pi)(g(Y, Z)X - g(X, Z)Y) \tag{23}$$

From the first Bianchi identity, we have

$$R^*(X, Y)Z + R^*(Y, Z)X + R^*(Z, X)Y = 0 \tag{24}$$

From equation (24), it is clear that the first Bianchi identity with respect to linear connection is satisfied.

A necessary and sufficient condition that 1-form ω of the semi-symmetric metric connection to be closed is that the first Bianchi identity with respect to linear connection ∇^* is satisfied (De & De, 1995).

In Yano (1970), it has been shown that "If a Riemannian manifold admitting a semi-symmetric metric connection is constant sectional curvature and 1-form ω is closed, then the Riemannian manifold is conformally flat".

If a Riemannian manifold admits a semi-symmetric metric connection with constant sectional curvature, then 1-form ω is closed. Thus we have

$$C_{\alpha\beta\lambda\mu} = 0 \tag{25}$$

According to Yano (1970), in order that a Riemannian manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian metric be conformally flat.

Since this manifold is conformally flat, we then have

$$R^*_{\alpha\beta\lambda\mu} = 0 \tag{26}$$

Remembering that a Riemannian manifold admitting a semi-symmetric metric connection with constant sectional curvature is conformally flat, then we get

$$\begin{aligned} R_{ijkh} = {} & \frac{1}{(n-2)}(g_{jk}R_{ih} - g_{ik}R_{jh} + g_{ih}R_{jk} - g_{jh}R_{ik}) \\ & - \frac{r}{(n-1)(n-2)}(g_{jk}g_{ih} - g_{jh}g_{ik}) \end{aligned} \tag{27}$$

Thus from (9) and (27), we have

$$R_{ijkl} = b(-g_{jl}A_iA_k + g_{jk}A_iA_l - g_{ik}A_jA_l + g_{il}A_jA_k) \\ +a(g_{il}g_{jk} - g_{jl}g_{ik}) \tag{28}$$

where

$$a = \frac{-r}{(n-1)(n-2)}, \qquad b = \frac{r}{(n-2)} \tag{29}$$

Smaranda (1989) calls a Riemannian manifold, whose curvature tensor satisfies equation (28), a space of almost constant curvature.

The notion of "almost constant curvature" is the same notion as "quasi constant curvature" introduced by Chen & Yano (1972). Later, Mocanu (1987) pointed out that both the notions are the same. In addition, Vranceanu (1968) defined the notion of almost constant curvature by the same equation (28).

We state the first theorem of this section:

Theorem 1. If a $G(PRS)_n$ admits a semi-symmetric metric connection with constant sectional curvature, then it is a manifold with quasi-constant curvature.

From equations (5), (7) and (26), we get

$$R_{\beta\mu} = \frac{r}{2(n-1)} g_{\beta\mu} + (n-2)\alpha_{\mu\beta} \tag{30}$$

Applying equation (9) in equation (30) we obtain

$$\alpha_{\beta\mu} = \frac{r}{(n-2)} \left(\frac{-g_{\beta\mu}}{2(n-1)} + A_\beta A_\mu \right) \tag{31}$$

From equation (28), it is clear that the manifold under consideration is also nearly quasi-constant curvature; but, from Lemma 1., this manifold is neither quasi-Einstein manifold nor nearly quasi-Einstein manifold.

Thus we have the following theorem:

Theorem 2. A $G(PRS)_n$ of definite metric admitting semi-symmetric metric connection with constant sectional curvature can be neither nearly quasi-Einstein manifold nor quasi-Einstein manifold.

Now, we need the following theorem:

Theorem A. (De & De, 1997): Let (M, g) be conformally flat $G(PRS)_n$

$(n > 3)$, then (M, g) is a subprojective manifold in the sense of Kagan and it can be expressed as a warped product $1 \times e^q M^*$ where M^* is an Einstien manifold and (M, g) has a proper concircular vector field.

Theorem A leads to the following theorem:

Theorem 3. If a $G(PRS)_n$ with definite metric manifold $(n > 3)$ admits a semi-symmetric metric connection with constant sectional curvature, then

(i) this manifold is a subprojective manifold in the sense of Kagan.

(ii) this manifold can be expressed as a warped product $1 \times e^q M^*$ where M^* is an Einstein manifold.

(iii) this manifold has a proper concircular vector field.

Let us introduce Lemma 2 which will be used in the next theorem:

Lemma 2. It is well known that every simply connected conformally flat $G(PRS)_n$ $(n > 3)$ can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface, (De & De, 1997).

This gives the assertion of the following theorem:

Theorem 4. Every simply connected $G(PRS)_n$ with definite metric manifold $(n > 3, r > 0)$, a semi-symmetric connection with constant sectional curvature can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

On the other hand, by equation (28), we have the following relations:

$$R_{ijkh} = F_{ik}F_{jh} - F_{jk}F_{ih} \tag{32}$$

where $F_{ij} = \sqrt{a}(g_{ij} - (n-1)A_iA_j)$ and $a = r/(n-2)(n-1), r > 0$. Now, from the equation (14), we can state easily the following:

Theorem 5. If a $G(PRS)_n$ with definite metric $(r > 0)$ admits a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23), then the manifold is special manifold with associated symmetric tensor F.

Considering a nearly quasi-umbilical hypersurface of a manifold of special curvature $\psi(F)_n$ Gazi & De (2009) obtained the following theorem:

Theorem B. (Gazi & De, 2009). A nearly quasi-umbilical hypersurface of a manifold of special curvature $\psi(F)_n$ is a manifold nearly quasi-constant curvature.

So, by virtue of Theorem 5 and Theorem B, we obtain the following theorem:

Theorem 6. A nearly quasi-umbilical hypersurface of a $G(PRS)_n$ of definite metric with $(r > 0)$ admitting a semi-symmetric metric connection whose constant sectional curvature $K^*(\pi)$ is manifold of nearly quasi-constant curvature.

Now by virtue of equations (9) and (27) yields,

$$A_h R^h_{ijk} = \frac{r}{(n-1)}(g_{jk}A_i - g_{ik}A_j) \tag{33}$$

We suppose that a $G(PRS)_n$ admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23) is semisymmetric. From equation (16), we get

$$\nabla_m \nabla_l R_{ij} - \nabla_l \nabla_m R_{ij} = 0 \tag{34}$$

By using equations (9) and (34), we find that

$$\begin{aligned} &r[A_j(\nabla_m \nabla_l A_i - \nabla_l \nabla_m Ai) \\ &+A_i(\nabla_m \nabla_l A_j - \nabla_l \nabla_m Aj)] = 0 \end{aligned} \tag{35}$$

In $G(PRS)_n$, since the scalar curvature is not zero, by using (35) we get

$$A_j(\nabla_m \nabla_l A_i - \nabla_l \nabla_m Ai) + A_i(\nabla_m \nabla_l A_j - \nabla_l \nabla_m A_j) = 0 \tag{36}$$

Transvecting equation (36) with A^j and using Ricci identity property and equation (33), we get

$$r(g_{ij}A_k - g_{ik}A_j) = 0 \tag{37}$$

In $G(PRS)_n$, because of both the scalar curvature r and A_k are not zero, this relation does not occur.

Conversly, we suppose that a $G(PRS)_n$ (definite metric) admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23) is satisfied in the following condition

$$\nabla_m \nabla_l A_i - \nabla_l \nabla_m Ai = 0 \tag{38}$$

From equations (33) and (38), we get

$$r(g_{ij}A_k - g_{ik}A_j) = 0 \tag{39}$$

From equation (39), since $A_k \neq 0$, we have $r = 0$. However, this is not possible in $G(PRS)_n$. This leads to the following theorem:

Theorem 7. A $G(PRS)_n$ with definite metric admitting a semi-symmetric metric connection with constant sectional curvature is not semisymmetric.

We obtain Corollary 1 and 2 as follows:

Corollary 1. A $G(PRS)_n$ with definite metric admitting a semi-symmetric metric connection with constant sectional curvature is not Ricci semisymmetric.

Corollary 2. In $G(PRS)_n$ with definite metric admitting a semi-symmetric metric connection with constant sectional curvature, $\nabla_m \nabla_l T_i - \nabla_l \nabla_m Ti \neq 0$.

Considering parallel vector field A in $G(PRS)_n$ results in:

Case 1. We consider a $G(PRS)_n$ of definite metric with parallel vector field A admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23). From equation (33), we get the scalar curvature is zero or $A_k = 0$.

Thus, the vector field A is not parallel.

Considering concurrent vector field A in $G(PRS)_n$ leads to result:

Case 2. Similarly, we now assume that $G(PRS)_n$ of definite metric with concurrent vector field A admitting a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23). After some manipulation, from equation (33) we get $r(g_{ij}A_k - g_{ik}A_j) = 0$. For similar reasons, A vector field can not be concurrent.

Considering recurrent vector field A in $G(PRS)_n$ results in:

Case 3. We assume that a $G(PRS)_n$ of definite metric with recurrent vector field A admits a semi-symmetric metric connection whose sectional curvature $K^*(\pi)$ is given by equation (23). Since the vector field is recurrent (i.e. $\nabla_j A_i = \phi_j A_i$) we have

$$\nabla_l \nabla_m A_k - \nabla_m \nabla_l A_k = (\nabla_l \phi_m - \nabla_m \phi_l) A_k \tag{40}$$

Using equation (40) and Ricci identity, we get

$$\nabla_l \phi_m - \nabla_m \phi_l = 0 \tag{41}$$

By using equation equation (40), we have $\nabla_l \nabla_m A_k - \nabla_m \nabla_l A_k = 0$.

Considering parallel, concurrent and recurrent vector field A in $G(PRS)_n$ with definite metric and by using Corollary 2, we obtain the following theorem:

Theorem 8. A $G(PRS)_n$ with definite metric admitting a semi-symmetric metric connection with constant sectional curvature does not exist if 1-form A satisfies one of the following conditions:

(i) A is parallel vector field

(ii) A is concurrent

(iii) A is recurrent

THE EXP-FUNCTION METHOD

We consider a general nonlinear ODE in the form

$$Q(u, u', u'', u''', ...) = 0 \tag{42}$$

where the prime denotes the derivation with respect to x. According to Exp-function method, we assume that the solution can be expressed in the form

$$u(x) = \frac{\sum_{n=-c}^{d} a_n exp(nx)}{\sum_{m=-p}^{q} b_m exp(mx)} \tag{43}$$

where c, d, p and q are positive integer which could be freely chosen, a_n and b_m are unknown constants to be determined. To determine the values of c and p, we balance the linear term of highest order in equation (42) with the highest order nonlinear term. Similarly, to determine the values of d and q, we balance the linear term of lowest order in equation (42) with the lowest order nonlinear term.

This method was used to solve the travelling wave equations (in various scientific and engineering fields) to deal with nonlinear physical situations. The solution procedure of this method, by the help of Maple of Mathematica, is of utter simplicity and this method can be easily extended to all kinds of nonlinear evolution equations (Zhang, 2007).

ANALYSIS OF THE DIFFERENTIAL TRANSFORM METHOD

The differential transform method is an analytical method for solving differential equations. The concept of the differential transform was first introduced by Zhou (1986). Its main application is to solve both linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. This transform method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. The method is well addressed in (Liu & Sang 2007; Jang *et al.*, 2001).

Now, we will discuss a brief outline of the differential transformation method (DTM). The basic theory of the method depends on the following definitions.

If $y(x)$ is analytic in the X domain, then it can be defined as

$$\theta(x,k) = \frac{d^k y(x)}{dx^k} \qquad \forall x \in X \tag{44}$$

where k belongs to the set of nonnegative integers denoted by the K domain and $\frac{d^k}{dx^k}$ means the k-th derivative with respect to x. Then, we can give the following definition.

The inverse differential transformation of a sequence $\{Y(k)\}|_{k=0}^{\infty}$ is defined as

$$y(x) = \infty \sum_{k=0} Y(k)(x-x_0)^k \tag{45}$$

Table1: Operations of the DTM Method

Function	Transformation
$y(x) = u(x) \mp v(x)$	$Y(k) = U(k) \mp V(k)$
$y(x) = au(x)$	$Y(k) = aU(k)$ where a is a constant
$y(x) = u(x)v(x)$	$Y(k) = U(k) \otimes V(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$y(x) = u(x)v(x)w(x)$	$Y(k) = \sum_{s=0}^{k} \sum_{m=0}^{k-s} u(s)v(m)w(k-s-m)$
$y(x) = p(x)\frac{d^s u(\alpha x+\beta)}{dx^s}$	$Y(k) = P(k) \otimes \frac{(k+s)!}{k!} \alpha^{k+s} \sum_{r=0}^{N-k-s} \frac{(k+s+r)!}{(k+s)!r!} (\frac{\beta^r}{\alpha}) U(k+s+r)$

EXAMPLE OF $G(PRS)_n$

This section deals with an example of $G(RPS)_n$.

Let each Latin index run over 1,2,3,...,n and each Greek index over 2,3,...,n-1. We define the metric g in R^n $(n \geq 4)$ by the formula (Roter 1974),

$$ds^2 = \phi(dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n \tag{46}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constant entries and ϕ is a function independent of x^n.

In the metric considered, the only non-vanishing components of the Christoffel symbols and the curvature tensor R_{hijk} are given by

$$\Gamma^{\beta}_{11} = -\frac{1}{2} k^{\alpha\beta} \phi_{.\alpha}, \quad \Gamma^{n}_{11} = \frac{1}{2}\phi_{.1}, \quad \Gamma^{n}_{1\alpha} = \frac{1}{2}\phi_{.\alpha} \tag{47}$$

and

$$R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta}, \quad R_{1\alpha\beta 1} = \frac{1}{2}k^{\alpha\beta}\phi_{.\alpha\beta} \tag{48}$$

where (.) denotes the partial differentiation and $[k^{\alpha\beta}]$ is the inverse matrix of $[k_{\alpha\beta}]$. Here, we assume $k_{\alpha\beta} = \delta_{\alpha\beta}$ and f is a continuously differentiable non constant function of x^1 and $f''(x^1) \neq f(x^1)$ where the prime denotes the differentiation with respect to x^1.

$$\phi = (f(x^1) - f''(x^1))\delta_{\alpha\beta}x^\alpha x^\beta, \tilde{}\tilde{}\tilde{} f''(x^1) \neq f(x^1) \tag{49}$$

In this case, ϕ reduces to

$$\phi = n - 1\sum_{\alpha=2}(f(x^1) - f''(x^1))(x^\alpha)^2 \tag{50}$$

Then, it follows from (48) that the only non-zero components of R_{ij} is R_{11}, where

$$R_{11} = (n-2)(f(x^1) - f''(x^1)) \tag{51}$$

Therefore, our space with the considered metric is neither Ricci symmetric nor Ricci recurrent. Now, we shall show that this space is $G(RPS)_n$. Let us consider the 1 forms

$$A_i = \begin{Bmatrix} A_1 & i = 1 \\ 0 & otherwise \end{Bmatrix} \tag{52}$$

$$B_i = \begin{Bmatrix} B_1 & i = 1 \\ 0 & otherwise \end{Bmatrix} \tag{53}$$

$$D_i = \begin{Bmatrix} D_1 & i = 1 \\ 0 & otherwise \end{Bmatrix} \tag{54}$$

where

$$2A_1 + B_1 + D_1 = \frac{3f'f^2}{f - f''} \tag{55}$$

On the other hand, equation (8) reduces to the following equation in our space

$$f''(x^1) - f(x^1) + f^3(x^1) = 0 \tag{56}$$

By the same procedure as illustrated in Section 3, we can determine values of c and p by balancing for the linear term of highest order f'' and the highest order nonlinear term f^3 in equation (43). We have

$$f'' = \frac{c_1 exp((3p + c)x^1) + ...}{c_2 exp(4px^1) + ...} \tag{57}$$

and

$$f^3 = \frac{c_3 exp((p + 3c)x^1) + ...}{c_4 exp(4px^1) + ...} \tag{58}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in equations (57) and (58), we have

$$p + 3c = 3p + c \tag{59}$$

which leads to the result

$$p = c \tag{60}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in equation (56)

$$f'' = \frac{... + d_1 exp(-(3q + d)x^1)}{... + d_2 exp(-4qx^1)} \tag{61}$$

and

$$f^3 = \frac{... + d_3 exp(-(3d + q)x^1)}{... + d_4 exp(-4qx^1)} \tag{62}$$

where d_i are determined coefficients only for simplicity. By a similar calculation as illustrated in Section 3, we obtain

$$q = d \tag{63}$$

For simplicity, we choose $p = c = 1$ and $q = d = 1$, equation (43) becomes

$$f(x^1) = \frac{a_1 exp(x^1) + a_0 + a_{-1} exp(-x^1)}{exp(x^1) + b_0 + b_{-1} exp(-x^1)} \tag{64}$$

Substituting equation (64) into equation (56) and equating to zero, the coefficients of all powers of $exp(nx)$ yields a set of algebraic equations for a_0, b_0, a_1, a_{-1} and b_{-1}. Solving the system of algebraic equations with the aid of Maple, we obtain

$$a_0 = 2\sqrt{2}, \quad a_1 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_{-1} = 1 \tag{65}$$

Then, equation (64) becomes

$$f(x^1) = \frac{\sqrt{2}}{chx^1} \tag{66}$$

In Bekir & Boz (2008), similar solution method was used to find the exact solution of the Klein-Gordon equation. Equation (64) satisfies equation (66). Thus, it reduces to $\phi = n - 1 \sum_{\alpha=2} \frac{2\sqrt{2}}{ch^3 x^1} (x^\alpha)^2$ and in this case, the metric is determined. By using DTM, we get a series solution of equation (56). Assuming the solution of Eq. (55) satisfies following the initial conditions

$$f(0) = \sqrt{2}, \qquad f'(0) = 0 \tag{67}$$

From equations (56) and (65) and Table.1, we have

$$\begin{aligned} &(k+1)(k+2)Y(k+2) \\ &= Y(k) - k \sum_{k_2=0} k_2 \sum_{k_1=0} Y(k_1)Y(k_2 - k_1)Y(k - k_2) \end{aligned} \tag{68}$$

where $Y(k)$ is the differential transform of $f(x^1)$ and the transform of the initial conditions are $Y(0) = \sqrt{2}$ and $Y(1) = 0$. By using equation (68) and the transformed initial condition, the following solution is evaluated by using Mathematica up to $(x^1)^{10}$

$$\begin{aligned} f(x^1) &= \sqrt{2} - \frac{\sqrt{2}}{2}(x^1)^2 + \frac{5\sqrt{2}}{24}(x^1)^4 - \frac{61\sqrt{2}}{720}(x^1)^6 \\ &+ \frac{277\sqrt{2}}{8064}(x^1)^8 - \frac{50521\sqrt{2}}{3628800}(x^1)^{10} + O((x^1)^{12}) \end{aligned} \tag{69}$$

We can also write the solution in the form

$$f(x^1) = \sqrt{2}(1 - \frac{1}{2}(x^1)^2 + \frac{5}{24}(x^1)^4 - \frac{61}{720}(x^1)^6 + \frac{277}{8064}(x^1)^8 - \frac{50521}{3628800}(x^1)^{10} + O((x^1)^{12})) \tag{70}$$

The series equation (70) is the Maclaurin series of $f(x^1) = \sqrt{2}sech(x^1)$. Thus, we can easily see that this solution is the same as that in equation (66).

Let us introduce a new approximation for the solution of equation (56). Multiplying both sides of equation (56) by f', then integrating, yields

$$f' = \sqrt{f^2 - \frac{1}{2}f^4 + c_1} \tag{71}$$

where c_1 is the integration constant.

Consider the case $c_1 = 0$. Thus, we have

$$(x^1 + c_2) = \int \frac{df}{f\sqrt{1 - (\frac{f}{\sqrt{2}})^2}} \tag{72}$$

where c_2 is the integration constant. If this integral is calculated, we finally obtain

$$-(x^1 + c_2) = sech^{-1}(\frac{f}{\sqrt{2}}) \tag{73}$$

From (72), we have

$$f = \sqrt{2}sech(-(x^1 + c_2)) \tag{74}$$

If we take c_2 as zero and remembering that $sech(-x^1)$ is an even function, we find

$$f = \sqrt{2}sech(x^1) \tag{75}$$

It is clear that this solution is the same as the solution that we have found before. Hence, an example of the existence of a generalized pseudo Ricci symmetric manifold is given.

Thus, we can state the following theorem:

Theorem 9. Let (M^n, g) be a manifold endowed with the metric (46), then (M^n, g) is a generalized pseudo Ricci symmetric manifold with vanishing scalar curvature which is neither Ricci symmetric nor Ricci recurrent.

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منطويات شبيهة ريتشي معممة متناظرة لها صلات مقاسية مثيل متناظرة

سيزجن التاي ديمربا – ج
قسم الرياضيات – كلية الآداب والعلوم – جامعة اسطنبول التقنية – تركيا

خلاصة

في هذا البحث نقوم أولاً بإعطاء مثال لمنطوي له تقوس قرب الثابت وتقوس شبيه الثابت تقريباً بحيث لا يكون شبيه أينشتاين ولا شبيه أينشتاين تقريباً. نقوم بعد ذلك بإثبات وجود منطويات شبيهة ريتشي معممة ومتناظرة وذلك بإعطاء مثال محدد غير تافه.

مجلة دراسات الخليج والجزيرة العربية

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رئيس التحرير

أ. د. بـدر عمـر العمـر

ترحـب المجلة بنشر البحوث والدراسات العلمية المتعلقة بشؤون منطقة الخليج والجزيرة العربية في مختلف علوم البحث والدراسة .

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