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Bäcklund transformation for spacelike curves in the Minkowski space-time

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ABSTRACT

The main purpose of this paper is to construct a Bäcklund transformation between two spacelike curves with the same constant curvature in Minkowski space-time by considering some assumptions. Moreover, we give the relations between curvatures of these two spacelike Bäcklund curves.

Keywords: Bäcklund transformation; constant torsion curve; spacelike curve.

INTRODUCTION

In 1883, Bäcklund proved that for pseudospherical congruences, which satisfy two additional conditions- namely the distance ρ between corresponding points, is constant and the normal of the focal surfaces at these points form a constant angle θ- have the same constant curvature $\frac{-\sin^2\theta}{\rho^2}$ (Eisenhart, 1909; Tenenblat & Terng, 1980; Rogers & Schief, 2002). This allows an iterative construction of pseudospherical surfaces with the same constant curvature. Therefore, Bäcklund transformations have different physical applications (Rogers & Shadwick, 1982; Deng, 2005; Zuo, *et al.,* 2002; Nemeth, 2000). Moreover, Bäcklund transformation was discussed in Minkowski 3 space by Abdel-baky (2005) and the curvatures are found as $\frac{-\sinh^2\theta}{\rho^2}$, $\frac{\sin^2\theta}{\rho^2}$ and $\frac{\sinh^2\theta}{\rho^2}$ for spacelike surfaces with spacelike congruence, timelike surfaces with spacelike congruence and timelike surfaces with timelike congruence, respectively (Abdel-baky, 2005). Also, the Bäcklund transformation for construction of timelike surfaces with positive Gaussian curvature and imaginary principal curvatures was established in Gu *et al.* (2002). Furthermore, Bäcklund transformation on surfaces with Gaussian curvature $K = -1$ in Minkowski 3-space was given by Tian (1997).

Bäcklund transformation maps asymptotic curves to asymptotic curves and asymptotic curves have constant torsion. So, Bäcklund transformation for pseudospherical surfaces, which is equivalent to that of sine-Gordon equation, can be restricted to give a transformation on space curves that preserves constant torsion. The curve with constant torsion, which is associated with sine-Gordon equation, constructs a pseudospherical surface, as it moves and at each instant, forms an asymptotic line on the surface (Rogers & Schief, 2002; Schief & Rogers, 1999). The binormal motion of curves with constant torsion is shown to lead to integrable extensions of the classical sine-Gordon equations. Motion of curves and its applications were given by several authors (e.g.Rogers & Schief, 2002; Schief & Rogers, 1999).

The curve with constant torsion, which is constructed by a given curve, was investigated by Calini & Ivey (1996). Nemeth (1998) proved that if there is a correspondence between points of two unit speed curves $\gamma,\widetilde{\gamma}$ having the property, where line joining the corresponding points $\gamma(s)$ and $\widetilde{\gamma}(s)$ is the intersection of the osculating planes of these curves, then the angle between $\widetilde{\gamma}(s)-\gamma(s)$ and tangent vectors of the curves $\widetilde{\gamma}(s)$ and $\gamma(s)$ are the same. Under same assumptions, Nemeth proved that these two curves $\gamma(s)$and $\widetilde{\gamma}(s)$ have the same constant torsion $\mathbf{k}_{n-1}=\widetilde{\mathbf{k}}_{n-1}=\frac{\sin\theta}{\rho}$ in n dimensional Euclidean space. Here

$$\mathbf{k}_{n-1}=\left\langle E_n', E_{n-1}\right\rangle \text{ and } \widetilde{\mathbf{k}}_{n-1}=\left\langle \widetilde{\mathbf{E}}_n', \widetilde{\mathbf{E}}_{n-1}\right\rangle.$$

In three dimensional Euclidean space, Bäcklund transformation was given as follows:

$$\widetilde{\gamma}=\gamma+\frac{2C}{C^2+\tau^2}(\cos\beta\mathbf{T}+\sin\beta\mathbf{B})$$

where β is the angle between $\widetilde{\gamma}(s)-\gamma(s)$ and tangent vectors of the curves $\widetilde{\gamma}(s)$ and $\gamma(s)$, $\frac{d\beta}{ds}=C\sin\beta-\kappa$ and κ, τ, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ are the Frenet apparatus (Nemeth, 1998). In four dimensional Euclidean space, Bäcklund transformation of two dimensional surface was given by Aminov & Sym (2000). Furthermore, Bäcklund transformation in Minkowski 3-space was discussed in (Özdemir & Çöken, 2009).

In this paper, we construct a Bäcklund transformation between two spacelike curves with the same constant curvature in Minkowski space-time by considering some assumptions. Moreover, we give the relations between curvatures of these two spacelike curves in five different cases.

PRELIMINARIES

Minkowski space-time is four dimensional Euclidean space which is provided with Lorentzian inner product

$$\left\langle \overrightarrow{u}, \overrightarrow{v} \right\rangle_L = -u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

for vectors $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and denoted by E_1^4. We say that a vector $\overrightarrow{u} \in E_1^4$ is called spacelike, lightlike (null) or timelike if $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle_L > 0$, $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle_L = 0$ or $\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle < 0$, respectively. Norm of a vector $\overrightarrow{u} \in E_1^4$ is defined by

$$\|\overrightarrow{u}\| = \sqrt{\left|\left\langle \overrightarrow{u}, \overrightarrow{u} \right\rangle_L\right|}.$$

An arbitrary curve $\gamma = \gamma(s) : I \to E_1^4$ is called spacelike, timelike or lightlike, if its velocity vectors are spacelike, timelike or lightlike, respectively. If $\|\gamma'(s)\| = 1$, then γ is called a unit speed curve (O'Neill, 1983).

Let γ be a unit speed non-lightlike curve and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$ be the Frenet vector fields of γ then the Frenet formulas are given as follows:

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \\ \mathbf{E}_4' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\mathbf{k}_1 & 0 & 0 \\ -\varepsilon_1\mathbf{k}_1 & 0 & \varepsilon_3\mathbf{k}_2 & 0 \\ 0 & -\varepsilon_2\mathbf{k}_2 & 0 & \varepsilon_4\mathbf{k}_3 \\ 0 & 0 & -\varepsilon_3\mathbf{k}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} \tag{1}$$

where $\varepsilon_i = \langle \mathbf{E}_i, \mathbf{E}_i \rangle_L$ and $\mathbf{k}_{n-1} = \left\langle \mathbf{E}_n', \mathbf{E}_{n-1} \right\rangle_L$. Here, we call $\mathbf{k}_3$ as torsion of the curve.

BACKLUND TRANSFORMATION FOR SPACELIKE CURVES

First, let us suppose that φ is a transformation between two spacelike curves γ and $\widetilde{\gamma}$ in Minkowski space-time such that $\widetilde{\gamma} = \varphi(\gamma(s))$. Here s is the arc length parameter of the curve γ. Now, let us assume that following properties for corresponding points of these two spacelike curves are satisfied:

i. $\left\|\overrightarrow{\gamma(s)\widetilde{\gamma}(s)}\right\| = \rho$ such that ρ is constant.

ii. The Frenet frame $\left\{\widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{E}}_4\right\}$ of $\tilde{\gamma}$ can be obtained from the Frenet frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$ of γ by a rotation with constant angle θ around a plane which contains $\mathbf{E}_4$.

iii. $\mathbf{F}_\rho$ is not perpendicular to tangent vectors of the curves and is contained in the intersection of osculating hyperplanes such that $\mathbf{F}_1$ is unit vector of the vector $\overrightarrow{\gamma(s)\widetilde{\gamma}(s)}$.

iv. $\langle \mathbf{E}_1, \mathbf{F}_\tau \rangle_L = -\langle \mathbf{E}_3, \mathbf{F}_\rho \rangle_L$ where $\mathbf{E}_1$ is tangent vector of the curve γ, $\mathbf{F}_\tau$ is perpendicular to the intersection space of the osculating hyperplanes, but is not a Frenet vector.

Since the distance between corresponding points of the curves γ and $\widetilde{\gamma}$ is constant ρ, then we can define the transformation φ as $\varphi(\gamma) = \widetilde{\gamma} = \gamma + \rho \mathbf{F}_\rho$. If the above properties i, ii, iii and iv are satisfied, then the curves γ and $\widetilde{\gamma}$ are called Bäcklund curves.

Let γ and $\widetilde{\gamma}$ be two spacelike Bäcklund curves in Minkowski space-time, then according to property ii, we can write $\widetilde{\mathbf{E}} = A^T \Omega A \mathbf{E}$ where $\widetilde{\mathbf{E}}^T = \left\{ \widetilde{\mathbf{E}}_1, \widetilde{\mathbf{E}}_2, \widetilde{\mathbf{E}}_3, \widetilde{\mathbf{E}}_4 \right\}$ and $\mathbf{E}^T = \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4\}$ positively oriented Frenet frames of $\widetilde{\gamma}$ and γ, respectively. Here Ω, $A = \left[a_{ij}\right]_{4\times4} \in SO(1,3)$ with the property $a_{i4} = a_{4i} = \varepsilon_4 \delta_{i4}$. We examine the matrix Ω with respect to kind of rotation and casual characters of the Frenet vectors $\mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4$. We know that $\mathbf{E}_1$ is spacelike for spacelike curves. Thus, there are five different cases for spacelike Bäcklund curves.

Case 1 . $\mathbf{E}_2$ is timelike and the rotation is around a spacelike plane which contains $\mathbf{E}_4$

In this case, the rotation matrix with respect to index $(+,-,+,+)$ is of the form:

$$\Omega_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\cos\theta & \sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

So, according to the transformation $\widetilde{\mathbf{E}} = A^T \Omega_1 A \mathbf{E}$, the relations between the Frenet vectors of γ and $\widetilde{\gamma}$ can be given as follows:

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 - \left(a_{31}^2 \mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\right)(1+\cos\theta) + \mathbf{E}_4 a_{31}\sin\theta \\ -\mathbf{E}_2 - \left(a_{31}a_{32}\mathbf{E}_1 + a_{32}^2\mathbf{E}_2 + a_{32}a_{33}\mathbf{E}_3\right)(1+\cos\theta) + \mathbf{E}_4 a_{32}\sin\theta \\ \mathbf{E}_3 - \left(a_{31}a_{33}\mathbf{E}_1 + a_{32}a_{33}\mathbf{E}_2 + a_{33}^2\mathbf{E}_3\right)(1+\cos\theta) + \mathbf{E}_4 a_{33}\sin\theta \\ \sin\theta(\mathbf{E}_1 a_{31} + \mathbf{E}_2 a_{32} + \mathbf{E}_3 a_{33}) + \cos\theta \mathbf{E}_4 \end{bmatrix}. \quad (2)$$

Moreover, the frame $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{E}_4\}$ can be obtained by $\mathbf{F}{=}A\mathbf{E}$ or $\widetilde{\mathbf{F}} = \Omega_1$

where $\mathbf{F}^T = \{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{E}_4\}$. So, the vectors $\{\mathbf{F}_1, \mathbf{F}_2\}$ form a frame for the intersection space of the osculating hyperplanes of the curves. We know that the Frenet formulas for a unit speed spacelike curve with timelike $\mathbf{E}_2$ can be stated as follows:

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \\ \mathbf{E}_4' \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{k}_1 & 0 & 0 \\ -\mathbf{k}_1 & 0 & \mathbf{k}_2 & 0 \\ 0 & \mathbf{k}_2 & 0 & \mathbf{k}_3 \\ 0 & 0 & -\mathbf{k}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix}. \tag{3}$$

Using these formulas, we can show that $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ is a unit speed curve for the unit speed curve γ. Here, $\mathbf{F}_1$ is unit vector of $\overrightarrow{\gamma(s)\widetilde{\gamma}(s)}$. Since $\mathbf{F}_1 = \mathbf{E}_1 a_{11} + \mathbf{E}_2 a_{12} + \mathbf{E}_3 a_{13}$, we can write $\gamma = \gamma + \rho(\mathbf{E}_1 a_{11} + \mathbf{E}_2 a_{12} + \mathbf{E}_3 a_{13})$. So $\left\langle \mathbf{F}_1, \widetilde{\mathbf{E}}_1 \right\rangle_L = \langle \mathbf{F}_1, \mathbf{E}_1 \rangle_L = a_{11}$. On the other hand, differentiating the equation $\langle \widetilde{\gamma} - \gamma, \widetilde{\gamma} - \gamma \rangle_L = \rho^2$ with respect to arc length parameter of γ, we get

$$\left\langle \rho\mathbf{F}_1, \tilde{\gamma}' - \mathbf{E}_1 \right\rangle_L = 0 \rightarrow \left\langle \mathbf{F}_1, \tilde{\gamma}' \right\rangle_L = a = \left\langle \mathbf{F}_1, \widetilde{\mathbf{E}}_1 \right\rangle_L.$$

Considering $\tilde{\gamma}' = \left\| \tilde{\gamma}' \right\| \widetilde{\mathbf{E}}_1$, we obtain $\tilde{\gamma}' = \widetilde{\mathbf{E}}_1$. It means that, $\tilde{\gamma}$ is also a unit speed curve, γ and $\tilde{\gamma}$ have the same arc length parameter. Moreover, $\widetilde{\mathbf{E}}_1$ is also spacelike and that means $\tilde{\gamma}$ is a spacelike curve. Now, let's see the relations between the curvatures of γ and $\widetilde{\gamma}$.

Lemma 1. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_2$ is timelike and θ is the rotation angle in spacelike plane between the Frenet frames of curves, then relations between the curvatures of γ and $\widetilde{\gamma}$ are obtained as follows:

$$\begin{aligned} a_{32}\widetilde{\mathbf{k}}_1 &= -2a_{31}' + \mathbf{k}_1 a_{32}, \\ a_{32}\widetilde{\mathbf{k}}_2 &= a_{32}\mathbf{k}_2 + 2a_{33}', \\ \widetilde{\mathbf{k}}_3 &= -\mathbf{k}_3, \end{aligned}$$

and

$$\begin{aligned} &a_{31}{}' - a_{32}\mathbf{k}_1 = -\mathbf{k}_3 a_{31} a_{33} \cot\frac{\theta}{2}, \\ &-\mathbf{k}_1 a_{31} + a_{32}{}' + a_{33}\mathbf{k}_2 = -\mathbf{k}_3 a_{32} a_{33} \cot\frac{\theta}{2}, \\ &a_{32}\mathbf{k}_2 + a_{33}{}' = \mathbf{k_3}\left(\mathbf{1} - \mathbf{a_{33}^2}\right) \cot\frac{\theta}{\mathbf{2}}. \end{aligned}$$

Proof. If we differentiate the Frenet vector $\widetilde{\mathbf{E}}_4$ of curve $\tilde{\gamma}$ in (2) and use the Frenet formulas of γ in (3), then we get

$$\begin{aligned}\widetilde{\mathbf{E}}_4' &= \sin\theta\big(a_{31}'\mathbf{E}_1 + a_{32}'\mathbf{E}_2 + a_{33}'\mathbf{E}_3 + a_{31}\mathbf{E}_1' + a_{32}\mathbf{E}_2' + a_{33}\mathbf{E}_3'\big) + \cos\theta\mathbf{E}_4' \\ &= \sin\theta\big[a_{31}'\mathbf{E}_1 + a_{32}'\mathbf{E}_2 + a_{33}'\mathbf{E}_3 + a_{31}(-\mathbf{k}_1\mathbf{E}_2) + a_{32}(-\mathbf{k}_1\mathbf{E}_1 + \mathbf{k}_2\mathbf{E}_3) + a_{33}(\mathbf{k}_2\mathbf{E}_2 + \mathbf{k}_3\mathbf{E}_4)\big] \\ &+ \cos\theta(-\mathbf{k}_3\mathbf{E}_3).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\widetilde{\mathbf{E}}_4' &= -\widetilde{\mathbf{k}}_3\widetilde{\mathbf{E}}_3 \\ &= -\widetilde{\mathbf{k}}_3\Big[\widetilde{\mathbf{E}}_3 - (a_{32}a_{33}\mathbf{E}_2 + a_{33}^2\mathbf{E}_3)(1-\cos\theta)\Big].\end{aligned}$$

Equality of above obtained vectors and using the identity $\frac{\cos\theta+1}{\sin\theta} = \cot\frac{\theta}{2}$, we get

$$\begin{aligned}\widetilde{\mathbf{k}}_3 &= -\mathbf{k}_3, \\ a_{31}' - a_{32}'\mathbf{k}_1 &= -\mathbf{k}_3 a_{31}a_{33}\cot\frac{\theta}{2} \\ -\mathbf{k}_1 a_{31} + a_{32}' + a_{33}\mathbf{k}_2 &= -\mathbf{k}_3 a_{32}a_{33}\cot\frac{\theta}{2} \\ a_{32}\mathbf{k}_2 + a_{33}\,' &= \mathbf{k}_3(1-a_{33}^2)\cot\frac{\theta}{2}.\end{aligned}$$

Similarly, by differentiating the Frenet vectors $\widetilde{E}_1$ and $\widetilde{E}_2$ of the curve $\tilde{\gamma}$ in γ, using the Frenet formulas of γ and above obtained relations, we get

$$\begin{aligned}a_{32}\widetilde{\mathbf{k}}_1 &= -2a_{31}' + \mathbf{k}_1 a_{32}, \\ a_{32}\widetilde{\mathbf{k}}_2 &= a_{32}\mathbf{k}_2 + 2a_{33}'.\end{aligned}$$

Theorem 1. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_2$ is timelike and θ is the rotation angle in spacelike plane between the Frenet frames of curves, then the curvatures of γ are obtained as follows:

$$\begin{aligned}\mathbf{k}_1 &= \frac{a_{31}^2(1+\cos\theta) + \rho a_{11}'}{\rho a_{12}}, \\ \mathbf{k}_2 &= -\frac{\rho a_{13}' + (1+\cos\theta)a_{31}a_{33}}{\rho a_{12}}, \\ \mathbf{k}_3 &= -\frac{\sin\theta}{\rho}.\end{aligned}$$

Proof. If we differentiate $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ and use the Frenet formulas of γ in (3), then we obtain

$$\begin{aligned}\widetilde{\gamma}' &= \gamma' + \rho \mathbf{F}_1' \\ &= \mathbf{E}_1 + \rho\big[(-\mathbf{E}_2\mathbf{k}_1)a_{11} + (-\mathbf{E}_1\mathbf{k}_1 + \mathbf{E}_3\mathbf{k}_2)a_{12} + (\mathbf{E}_2\mathbf{k}_2 + \mathbf{E}_4\mathbf{k}_3)a_{13} + \mathbf{E}_1 a_{11}' + \mathbf{E}_2 a_{12}' + \mathbf{E}_3 a_{13}'\big].\end{aligned}$$

Comparing this equality to

$$\widetilde{\mathbf{E}}_1 = \mathbf{E}_1 - \left(a_{31}^2\mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\right)(1 + cos\theta) + sin\theta\mathbf{E}_4 a_{31},$$

we get

$$\begin{aligned} 1 - a_{31}^2(1+\cos\theta) &= 1 - \rho\mathbf{k}_1 a_{12} + \rho a_{11}', \\ -(1+\cos\theta)a_{31}a_{32} &= -\rho\mathbf{k}_1 a_{11} + \rho\mathbf{k}_2 a_{13} + \rho a_{12}', \\ -(1+\cos\theta)a_{13}a_{33} &= \rho\mathbf{k}_2 a_{12} + \rho a_{13}', \\ \sin\theta a_{31} &= \rho\mathbf{k}_3 a_{13}. \end{aligned}$$

By considering to iv'th property of Bäcklund curves, we see that the relations $\langle \mathbf{E}_1, \mathbf{F}_3\rangle_L = -\langle \mathbf{E}_3, \mathbf{F}_1\rangle_L$ and $a_{31} = -a_{13}$ are satisfied. Using these relations and above equalities, we get

$$\begin{aligned} \mathbf{k}_1 &= \frac{a_{31}^2(1+\cos\theta) + \rho a_{11}'}{\rho a_{12}}, \\ \mathbf{k}_2 &= -\frac{\rho a_{13}' + (1+\cos\theta)a_{31}a_{33}}{\rho a_{12}}, \\ \mathbf{k}_3 &= -\frac{\sin\theta}{\rho}. \end{aligned}$$

Case 2. $\mathbf{E}_2$ is timelike and the rotation is around a Lorentzian plane which contains $\mathbf{E}_4$

In this case, the rotation matrix with respect to index $(+,-,+,+)$ is of the form:

$$\Omega_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & 0 & \sinh\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\theta & 0 & \cosh\theta \end{bmatrix}.$$

Also, relations between the Frenet vectors of γ and $\widetilde{\gamma}$ are given as follows:

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 + \left(a_{21}^2\mathbf{E}_1 + a_{21}a_{22}\mathbf{E}_2 + a_{21}a_{23}\mathbf{E}_3\right)(1+\cosh\theta) + a_{21}\mathbf{E}_4\sinh\theta \\ -\mathbf{E}_2 + \left(a_{21}a_{22}\mathbf{E}_1 + a_{22}^2\mathbf{E}_2 + a_{22}a_{23}\mathbf{E}_3\right)(1+\cosh\theta) + a_{22}\mathbf{E}_4\sinh\theta \\ \mathbf{E}_3 + \left(a_{21}a_{23}\mathbf{E}_1 + a_{22}a_{23}\mathbf{E}_2 + a_{23}^2\mathbf{E}_3\right)(1+\cosh\theta) + a_{23}\mathbf{E}_4\sinh\theta \\ \sinh\theta(\mathbf{E}_1 a_{21} + \mathbf{E}_2 a_{22} + \mathbf{E}_3 a_{23}) + \cosh\theta\mathbf{E}_4 \end{bmatrix} \tag{4}$$

Moreover, the frame $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{E}_4\}$ can be obtained by $\mathbf{F} = A\mathbf{E}$ where $\mathbf{F}^T = \{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{E}_4\}$. So, the vectors $\{\mathbf{F}_1, \mathbf{F}_3\}$ form a frame for the intersection space of the osculating hyperplanes of the curves. Considering property iv of Bäcklund curves, we have $\langle \mathbf{E}_1, \mathbf{F}_2\rangle_L = -\langle \mathbf{E}_3, \mathbf{F}_1\rangle_L$ or$a_{21} = -a_{13}$. It can be shown that $\widetilde{\gamma}$ is also a unit speed curve.

Lemma 2. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_2$ is timelike and θ is the rotation angle in the Lorentzian plane between the Frenet frames of curves, then the relations between the curvatures of γ and $\widetilde{\gamma}$ are obtained as follows:

$$\begin{aligned} \widetilde{\mathbf{k}}_1 a_{22} &= -2a'_{21} + a_{22}\mathbf{k}_1, \\ \widetilde{\mathbf{k}}_2 a_{22} &= \mathbf{k}_2 a_{22} + 2a'_{23}, \\ \widetilde{\mathbf{k}}_3 &= -\mathbf{k}_3, \\ \mathbf{k}_3(a_{21}a_{23})\coth\frac{\theta}{2} &= -a_{22}\mathbf{k}_1 + a'_{21}, \\ \mathbf{k}_3(a_{22}a_{23})\coth\frac{\theta}{2} &= -a_{21}\mathbf{k}_1 + a_{23}\mathbf{k}_2 + a'_{22}, \\ \mathbf{k}_3\left(1 + a_{23}^2\right)\coth\frac{\theta}{2} &= a_{22}\mathbf{k}_2 + a'_{23}. \end{aligned}$$

Proof. The proof can be done similarly to the proof of Lemma 1 by differentiating the Frenet vectors of $\widetilde{\gamma}$ in (4) and using the Frenet formulas of γ, $\widetilde{\gamma}$ and the identity $\frac{1+\cosh\theta}{\sinh\theta} = \coth\frac{\theta}{2}$.

Theorem 2. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_2$ is timelike and θ is the rotation angle in the Lorentzian plane between the Frenet frames of curves, then the curvatures of γ are obtained as follows:

$$\begin{aligned} \mathbf{k}_1 &= \frac{\rho a'_{11} - a_{12}^2(1+\cosh\theta)}{\rho a_{12}}, \\ \mathbf{k}_2 &= \frac{-\rho a'_{13} + a_{21}a_{23}(1+\cosh\theta)}{\rho a_{12}}, \\ \mathbf{k}_3 &= -\frac{\sinh\theta}{\rho}. \end{aligned}$$

Proof. If we differentiate $\widetilde{\gamma} = \gamma + \rho \mathbf{F}_1$ and use the Frenet formulas of the curve γ in (3), then we get

$$\begin{aligned} \widetilde{\gamma}' &= \gamma' + \rho \mathbf{F}_1' \\ &= \mathbf{E}_1 + \rho\big[(-\mathbf{E}_2\mathbf{k}_1)a_{11} + (-\mathbf{E}_1\mathbf{k}_1 + \mathbf{E}_3\mathbf{k}_2)a_{12} + (\mathbf{E}_2\mathbf{k}_2 + \mathbf{E}_4\mathbf{k}_3)a_{13} + \mathbf{E}_1 a_{11}' + \mathbf{E}_2 a_{12}' + \mathbf{E}_3 a_{13}'\big]. \end{aligned}$$

Comparing this equality to

$$\widetilde{\mathbf{E}}_1 = \mathbf{E}_1 + \left(a_{21}^2 \mathbf{E}_1 + a_{21}a_{22}\mathbf{E}_2 + a_{21}a_{23}\mathbf{E}_3\right)(1 + \cosh\theta) + a_{21}\mathbf{E}_4 \sinh\theta,$$

we obtain

$$\begin{aligned} 1 + a_{21}^2(1+\cosh\theta) &= 1 - \rho \mathbf{k}_1 a_{12} + \rho a_{11}', \\ a_{21}a_{22}(1+\cosh\theta) &= -\rho \mathbf{k}_1 a_{11} + \rho \mathbf{k}_2 a_{13} + \rho a_{12}', \\ a_{21}a_{23}(1+\cosh\theta) &= \rho \mathbf{k}_2 a_{12} + \rho a_{13}', \\ \sinh\theta a_{21} &= \rho \mathbf{k}_3 a_{13}. \end{aligned}$$

By considering to iv'th property of Bäcklund curves, we can see that the relations$\langle \mathbf{E}_1, \mathbf{F}_2\rangle_L = -\langle \mathbf{E}_3, \mathbf{F}_1\rangle_L$and $a_{21} = -a_{13}$ are satisfied. Using these relations and above equalities, we get

$$\begin{aligned} \mathbf{k}_1 &= \frac{\rho a_{11}' - a_{12}^2(1+\cosh\theta)}{\rho a_{12}}, \\ \mathbf{k}_2 &= \frac{-\rho a_{13}' + a_{21}a_{23}(1+\cosh\theta)}{\rho a_{12}}, \\ \mathbf{k}_3 &= -\frac{\sinh\theta}{\rho}. \end{aligned}$$

Case 3. E$_4$ is timelike and the rotation is around a Lorentzian plane which contains E$_4$

In this case, the rotation matrix with respect to index $(+,+,+,-)$ is of the form:

$$\Omega_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix}.$$

So, the relations between the Frenet frames of γ and $\widetilde{\gamma}$ can be given as follows:

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 - \left(a_{31}^2\mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\right)(1-\cosh\theta) - a_{31}\mathbf{E}_4\sinh\theta \\ \mathbf{E}_2 - \left(a_{31}a_{32}\mathbf{E}_1 + a_{32}^2\mathbf{E}_2 + a_{32}a_{33}\mathbf{E}_3\right)(1-\cosh\theta) - a_{32}\mathbf{E}_4\sinh\theta \\ \mathbf{E}_3 - \left(a_{31}a_{33}\mathbf{E}_1 + a_{32}a_{33}\mathbf{E}_2 + a_{33}^2\mathbf{E}_3\right)(1-\cosh\theta) - a_{33}\mathbf{E}_4\sinh\theta \\ -\sinh\theta(\mathbf{E}_1a_{31} + \mathbf{E}_2a_{32} + \mathbf{E}_3a_{33}) + \cosh\theta\mathbf{E}_4 \end{bmatrix}. \tag{5}$$

In this case, the vectors $\{\mathbf{F}_1, \mathbf{F}_2\}$ form a frame for the intersection space of the osculating hyperplanes of the curves and we have $\langle\mathbf{E}_1, \mathbf{F}_3\rangle_L = -\langle\mathbf{E}_3, \mathbf{F}_1\rangle_L$ or$a_{31} = -a_{13}$. Here, we can also show that $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ is a unit speed curve by using the Frenet formulas

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \\ \mathbf{E}_4' \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{k}_1 & 0 & 0 \\ -\mathbf{k}_1 & 0 & \mathbf{k}_2 & 0 \\ 0 & -\mathbf{k}_2 & 0 & -\mathbf{k}_3 \\ 0 & 0 & -\mathbf{k}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} \tag{6}$$

of a unit speed spacelike curve with timelike $\mathbf{E}_4$.

Lemma 3. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_4$ is timelike and θ is the rotation angle in the Lorentzian plane between the Frenet frames of curves, then the relations between the curvatures of γ and $\widetilde{\gamma}$ are obtained as follows:

$$\begin{aligned} \widetilde{\mathbf{k}}_1 a_{32} &= -2a_{31}' - a_{32}\mathbf{k}_1, \\ \widetilde{\mathbf{k}}_2 a_{32} &= -a_{32}\mathbf{k}_2 - 2a_{33}', \\ \widetilde{\mathbf{k}}_3 &= \mathbf{k}_3, \\ a_{32}\mathbf{k}_1 - a_{31}' &= -\mathbf{k}_3 a_{31}a_{33}\tanh\frac{\theta}{2}, \\ a_{33}\mathbf{k}_2 - a_{31}\mathbf{k}_1 - \mathbf{a}_{32}' &= -\mathbf{k}_3 a_{32}a_{33}\tanh\frac{\theta}{2}, \\ a_{32}\mathbf{k}_2 + a_{33}' &= \mathbf{k}_3\left(a_{33}^2 + 1\right)\tanh\frac{\theta}{2}. \end{aligned}$$

Proof. The proof can be done similar to proof of Lemma 1 by differentiating the Frenet vectors of $\widetilde{\gamma}$ in (5), using the Frenet formulas of γ, $\widetilde{\gamma}$ and the identity $\frac{1+\cosh\theta}{\sinh\theta} = \coth\frac{\theta}{2}$.

Theorem 3. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_4$ is timelike and θ is the rotation angle in the Lorentzian plane between the Frenet frames of curves, then the curvatures of γ are obtained as follows:

$$\mathbf{k}_1 = \frac{\rho a'_{11} - a_{31}^2(-1+\cosh\theta)}{\rho a_{12}},$$
$$\mathbf{k}_2 = -\frac{\rho a'_{13} - a_{33}a_{31}(-1+\cosh\theta)}{\rho a_{12}},$$
$$\mathbf{k}_3 = \frac{\sinh\theta}{\rho}.$$

Proof. If we differentiate $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ and use the Frenet formulas of γ in (6), then we get

$$\begin{aligned}\widetilde{\gamma}' &= \gamma' + \rho\mathbf{F}'_1 \\ &= \mathbf{E}_1 + \rho\big[(\mathbf{E}_2\mathbf{k}_1)a_{11} + (-\mathbf{E}_1\mathbf{k}_1 + \mathbf{E}_3\mathbf{k}_2)a_{12} + (-\mathbf{E}_2\mathbf{k}_2 - \mathbf{E}_4\mathbf{k}_3)a_{13} + \mathbf{E}_1a'_{11} + \mathbf{E}_2a'_{12} + \mathbf{E}_3a'_{13}\big].\end{aligned}$$

Comparing this equality to

$$\widetilde{\mathbf{E}}_1 = \mathbf{E}_1 - \big(a_{31}^2\mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\big)(1-\cosh\theta) - a_{31}\mathbf{E}_4\sinh\theta,$$

we obtain

$$\begin{aligned}1 - a_{31}^2(1-\cosh\theta) &= 1 - \rho\mathbf{k}_1a_{12} + \rho a'_{11},\\ -a_{31}a_{32}(1-\cosh\theta) &= \rho\mathbf{k}_1a_{11} - \rho\mathbf{k}_2a_{13} + \rho a'_{12},\\ -a_{31}a_{33}(1-\cosh\theta) &= \rho\mathbf{k}_2a_{12} + \rho a'_{13},\\ -\sinh\theta a_{31} &= \rho\mathbf{k}_3a_{13}.\end{aligned}$$

By iv'th property of Bäcklund curves, we have $\langle\mathbf{E}_1,\mathbf{F}_3\rangle_L = -\langle\mathbf{E}_3,\mathbf{F}_1\rangle_L$ and$a_{31} = -a_{13}$. We get

$$\mathbf{k}_1 = \frac{\rho a'_{11} - a_{31}^2(-1+\cosh\theta)}{\rho a_{12}},$$
$$\mathbf{k}_2 = -\frac{\rho a'_{13} - a_{33}a_{31}(-1+\cosh\theta)}{\rho a_{12}},$$
$$\mathbf{k}_3 = \frac{\sinh\theta}{\rho}.$$

Case 4. $\mathbf{E}_3$ is timelike and the rotation is around a Lorentzian plane which contains $\mathbf{E}_4$

In this case, the rotation matrix with respect to index $(+,+,-,+)$ is of the form:

$$\Omega_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix}.$$

So, the relations between the Frenet frames of γ and $\widetilde{\gamma}$ are given as;

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 + \left(a_{31}^2\mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\right)(1+\cosh\theta) + a_{31}\mathbf{E}_4\sinh\theta \\ \mathbf{E}_2 + \left(a_{31}a_{32}\mathbf{E}_1 + a_{32}^2\mathbf{E}_2 + a_{32}a_{33}\mathbf{E}_3\right)(1+\cosh\theta) + a_{32}\mathbf{E}_4\sinh\theta \\ -\mathbf{E}_3 + \left(a_{31}a_{33}\mathbf{E}_1 + a_{32}a_{33}\mathbf{E}_2 + a_{33}^2\mathbf{E}_3\right)(1+\cosh\theta) + a_{33}\mathbf{E}_4\sinh\theta \\ \sinh\theta(\mathbf{E}_1a_{31} + \mathbf{E}_2a_{32} + \mathbf{E}_3a_{33}) + \cosh\theta\mathbf{E}_4 \end{bmatrix}. \quad (7)$$

In this case, the vectors $\{\mathbf{F}_1, \mathbf{F}_2\}$ form a frame for the intersection space of the osculating hyperplanes of curves. We have $\langle\mathbf{E}_1, \mathbf{F}_3\rangle_L = -\langle\mathbf{E}_3, \mathbf{F}_1\rangle_L$ or$a_{31} = -a_{13}$. By the Frenet formulas

$$\begin{bmatrix} \mathbf{E}_1' \\ \mathbf{E}_2' \\ \mathbf{E}_3' \\ \mathbf{E}_4' \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{k}_1 & 0 & 0 \\ -\mathbf{k}_1 & 0 & -\mathbf{k}_2 & 0 \\ 0 & -\mathbf{k}_2 & 0 & \mathbf{k}_3 \\ 0 & 0 & \mathbf{k}_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \\ \mathbf{E}_4 \end{bmatrix} \quad (8)$$

of a unit speed spacelike curve with timelike $\mathbf{E}_3$, we can show that $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ is unit speed curve.

Lemma 4. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_3$ is timelike and θ is the rotation angle in the Lorentzian plane between the Frenet frames of curves, then the relations between curvatures of $\gamma$and $\widetilde{\gamma}$ are obtained as follows:

$$\begin{aligned} \widetilde{\mathbf{k}}_1 a_{32} &= 2a_{31}' - a_{32}\mathbf{k}_1, \\ \widetilde{\mathbf{k}}_2 a_{32} &= a_{32}\mathbf{k}_2 - 2a_{33}', \\ \widetilde{\mathbf{k}}_3 &= \mathbf{k}_3, \\ \mathbf{k}_3(a_{31}a_{33})\coth\frac{\theta}{2} &= -a_{32}\mathbf{k}_1 + a_{31}', \\ \mathbf{k}_3(a_{32}a_{33})\coth\frac{\theta}{2} &= a_{31}\mathbf{k}_1 - a_{33}\mathbf{k}_2 + a_{32}', \\ \mathbf{k}_3\left(-1 + a_{33}^2\right)\coth\frac{\theta}{2} &= -a_{32}\mathbf{k}_2 + a_{33}'. \end{aligned}$$

Proof. The proof can be done similarly to proof of Lemma 1 by differentiating the Frenet vectors of $\widetilde{\gamma}$ in (7) and using the Frenet formulas of γ, $\widetilde{\gamma}$ and the identity $\frac{1+\cosh\theta}{\sinh\theta} = \coth\frac{\theta}{2}$.

Theorem 4. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_3$ is timelike and θ is the rotation angle in the Lorentzian plane between Frenet frames of curves, then the curvatures of γ are obtained as follows:

$$\begin{aligned}\mathbf{k}_1 &= \frac{\rho a'_{11} - a_{31}^2(1+\cosh\theta)}{\rho a_{12}},\\ \mathbf{k}_2 &= \frac{\rho a'_{13} - a_{33}a_{31}(1+\cosh\theta)}{\rho a_{12}},\\ \mathbf{k}_3 &= -\frac{\sinh\theta}{\rho}.\end{aligned}$$

Proof. If we differentiate $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ and use the Frenet formulas of γ in (8), then we get

$$\begin{aligned}\widetilde{\gamma}' &= \gamma' + \rho\mathbf{F}_1'\\ &= \mathbf{E}_1 + \rho\big[(\mathbf{E}_2\mathbf{k}_1)a_{11} + (-\mathbf{E}_1\mathbf{k}_1 - \mathbf{E}_3\mathbf{k}_2)a_{12} + (-\mathbf{E}_2\mathbf{k}_2 + \mathbf{E}_4\mathbf{k}_3)a_{13} + \mathbf{E}_1 a'_{11} + \mathbf{E}_2 a'_{12} + \mathbf{E}_3 a'_{13}\big].\end{aligned}$$

Comparing this equality to

$$\widetilde{\mathbf{E}}_1 = \mathbf{E}_1 + \left(a_{31}^2\mathbf{E}_1 + a_{31}a_{32}\mathbf{E}_2 + a_{31}a_{33}\mathbf{E}_3\right)(1+\cosh\theta) + a_{31}\mathbf{E}_4\sinh\theta,$$

we obtain

$$\begin{aligned}1 + a_{31}^2(1+\cosh\theta) &= 1 - \rho\mathbf{k}_1 a_{12} + \rho a'_{11}\\ a_{31}a_{32}(1+\cosh\theta) &= \rho\mathbf{k}_1 a_{11} - \rho\mathbf{k}_2 a_{13} + \rho a'_{12}\\ a_{31}a_{33}(1+\cosh\theta) &= -\rho\mathbf{k}_2 a_{12} + \rho a'_{13}\\ \sinh\theta a_{31} &= \rho\mathbf{k}_3 a_{13}.\end{aligned}$$

By iv'th property of Bäcklund curves, we have $\langle\mathbf{E}_1, \mathbf{F}_3\rangle_L = -\langle\mathbf{E}_3, \mathbf{F}_1\rangle_L$ and$a_{31} = -a_{13}$. We get

$$\begin{aligned}\mathbf{k}_1 &= \frac{\rho a'_{11} - a_{31}^2(1+\cosh\theta)}{\rho a_{12}},\\ \mathbf{k}_2 &= \frac{\rho a'_{13} - a_{33}a_{31}(1+\cosh\theta)}{\rho a_{12}},\\ \mathbf{k}_3 &= -\frac{\sinh\theta}{\rho}.\end{aligned}$$

Case 5. $\mathbf{E}_3$ is timelike and the rotation is around a spacelike plane which contains $\mathbf{E}_4$

In this case, the rotation matrix with respect to index $(+,+,-,+)$ is of the form:

$$\Omega_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\theta & 0 & \sin\theta \\ 0 & 0 & -1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

So, the relations between the Frenet frames of γ and $\widetilde{\gamma}$ are given as follows:

$$\begin{bmatrix} \widetilde{\mathbf{E}}_1 \\ \widetilde{\mathbf{E}}_2 \\ \widetilde{\mathbf{E}}_3 \\ \widetilde{\mathbf{E}}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{E}_1 - \left(a_{21}^2\mathbf{E}_1 + a_{21}a_{22}\mathbf{E}_2 + a_{21}a_{23}\mathbf{E}_3\right)(1+\cos\theta) + a_{21}\mathbf{E}_4\sin\theta \\ \mathbf{E}_2 - \left(a_{21}a_{22}\mathbf{E}_1 + a_{22}^2\mathbf{E}_2 + a_{22}a_{23}\mathbf{E}_3\right)(1+\cos\theta) + a_{22}\mathbf{E}_4\sin\theta \\ -\mathbf{E}_3 - \left(a_{21}a_{23}\mathbf{E}_1 + a_{22}a_{23}\mathbf{E}_2 + a_{23}^2\mathbf{E}_3\right)(1+\cos\theta) + a_{23}\mathbf{E}_4\sin\theta \\ \sin\theta(\mathbf{E}_1a_{21} + \mathbf{E}_2a_{22} + \mathbf{E}_3a_{23}) + \cos\theta\mathbf{E}_4 \end{bmatrix}. \quad (9)$$

In this case, the vectors $\{\mathbf{F}_1, \mathbf{F}_3\}$ form a frame for the intersection space of the osculating hyperplanes of the curves and we have $\langle \mathbf{E}_1, \mathbf{F}_2\rangle_L = -\langle \mathbf{E}_3, \mathbf{F}_1\rangle_L$ or$a_{21} = -a_{13}$. Using the Frenet formulas in (8), we can also show that $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ is a unit speed curve.

Lemma 5. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_3$ is timelike and θ is the rotation angle in spacelike plane between the Frenet frames of curves, then the relations between curvatures of γ and $\widetilde{\gamma}$ are obtained as follows:

$$\begin{aligned} \widetilde{\mathbf{k}}_1 a_{22} &= 2a_{21}' - a_{22}\mathbf{k}_1, \\ \widetilde{\mathbf{k}}_2 a_{22} &= a_{22}\mathbf{k}_2 - 2a_{23}', \\ \widetilde{\mathbf{k}}_3 &= \mathbf{k}_3, \end{aligned}$$

and

$$\begin{aligned} &\mathbf{k}_3(a_{21}a_{23})\cot\frac{\theta}{2} = a_{22}\mathbf{k}_1 + a_{31}', \\ &\mathbf{k}_3(a_{22}a_{23})\cot\frac{\theta}{2} = -a_{21}\mathbf{k}_1 + a_{23}\mathbf{k}_2 - a_{22}', \\ &\mathbf{k}_3\left(1 + a_{23}^2\right)\cot\frac{\theta}{2} = a_{22}\mathbf{k}_2 - a_{23}'. \end{aligned}$$

Proof. The proof can be done similarly to proof of Lemma 1 by differentiating Frenet vectors of $\widetilde{\gamma}$ in (9) and using the Frenet formulas of γ, $\widetilde{\gamma}$ and the identity $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$.

Theorem 5. Let γ and $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ be two spacelike Bäcklund curves in E_1^4. If the Frenet vector $\mathbf{E}_3$ is timelike and θ is the rotation angle in spacelike plane between the Frenet frames of curves, then the curvatures of γ are obtained as follows:

$$\begin{aligned}
\mathbf{k}_1 &= \frac{\rho a_{11}' + a_{21}^2(1+\cos\theta)}{\rho a_{12}}, \\
\mathbf{k}_2 &= \frac{\rho a_{13}' + a_{23}a_{21}(1+\cos\theta)}{\rho a_{12}}, \\
\mathbf{k}_3 &= -\frac{\sin\theta}{\rho}.
\end{aligned}$$

Proof. If we differentiate $\widetilde{\gamma} = \gamma + \rho\mathbf{F}_1$ and use the Frenet formulas in (8), then we get

$$\begin{aligned}
\widetilde{\gamma}' &= \gamma' + \rho\mathbf{F}_1' \\
&= \mathbf{E}_1 + \rho\big[(\mathbf{E}_2\mathbf{k}_1)a_{11} + (-\mathbf{E}_1\mathbf{k}_1 - \mathbf{E}_3\mathbf{k}_2)a_{12} + (-\mathbf{E}_2\mathbf{k}_2 + \mathbf{E}_4\mathbf{k}_3)a_{13} + \mathbf{E}_1 a_{11}' + \mathbf{E}_2 a_{12}' + \mathbf{E}_3 a_{13}'\big]
\end{aligned}$$

Comparing this equality to

$$\widetilde{\mathbf{E}}_1 = \mathbf{E}_1 - \big(a_{21}^2\mathbf{E}_1 + a_{21}a_{22}\mathbf{E}_2 + a_{21}a_{23}\mathbf{E}_3\big)(1+\cos\theta) + a_{21}\mathbf{E}_4\sin\theta,$$

we obtain

$$\begin{aligned}
1 - a_{21}^2(1+\cos\theta) &= 1 - \rho\mathbf{k}_1 a_{12} + \rho a_{11}', \\
-a_{21}a_{22}(1+\cos\theta) &= \rho\mathbf{k}_1 a_{11} - \rho\mathbf{k}_2 a_{13} + \rho a_{12}', \\
-a_{21}a_{23}(1+\cos\theta) &= -\rho\mathbf{k}_2 a_{12} + \rho a_{13}', \\
\sin\theta a_{21} &= \rho\mathbf{k}_3 a_{13}.
\end{aligned}$$

By considering to iv'th property of Bäcklund curves, the relations $\langle\mathbf{E}_1, \mathbf{F}_2\rangle_L = -\langle\mathbf{E}_3, \mathbf{F}_1\rangle_L$ and $a_{21} = -a_{13}$ are satisfied. Thus, we get

$$\mathbf{k}_1 = \frac{\rho a'_{11} + a_{21}^2(1+\cos\theta)}{\rho a_{12}},$$
$$\mathbf{k}_2 = \frac{\rho a'_{13} + a_{23}a_{21}(1+\cos\theta)}{\rho a_{12}},$$
$$\mathbf{k}_3 = -\frac{\sin\theta}{\rho}.$$

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تحويل بوكلند على منحنيات كفضائية في زمكان مينكوفسكي

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خلاصة

الهدف الأساسي لهذا البحث هو إنشاء تحويل بوكلند بين منحنيين كفضائيين لهما نفس التقوس الثابت في زمكان مينكوفسكي، وذلك باستخدام بعض الافتراضات. نقوم كذلك بإيجاد العلاقات بين تقوسات هذين المنحنيين.