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On pointwise statistical convergence of order α of sequences of fuzzy mappings

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ABSTRACT

In this paper, we introduce the concept of pointwise statistical convergence of order α of sequences of fuzzy mappings. Furthermore we give the concept of α-statistically Cauchy sequence for sequences of fuzzy mappings and prove that it is equivalent to pointwise statistical convergence of order α of sequences of fuzzy mappings. Also some relations between $S^{\beta}(F)$-statistical convergence and strong $w_p^{\alpha}(F)$-summability are given.

Keywords: Cesàro summability; pointwise statistical convergence; sequences of fuzzy mappings.

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INTRODUCTION

The idea of statistical convergence was given by Zygmund (1979) in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951) and later reintroduced by Schoenberg (1959) independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with Summability theory by Connor (1988); Et & Nuray (2001); Et (2003); Et *et al.* (2006); Fridy (1985); Gökhan & Güngör (2002); Güngör *et al.* (2004); Işık (2011); Rath & Tripathy (1994); Salat (1980); Tripathy (1997) and many others.

The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Altın *et al.* (2006,2007); Altın *et al.* (2007); Altınok *et al.* (2009); Burgin (2000); Colak *et al.* (2009); Gökhan *et al.*

(2009); Mursaleen & Başarır (2003); Talo & Başar (2009); Tripathy & Sarma (2011) and Tripathy & Dutta (2006,2007) extended the idea to apply to sequences of fuzzy numbers.

In the present paper, we introduce the concept of pointwise statistical convergence of order α of sequences of fuzzy mappings. In section 2 we give a brief overview about statistical convergence, $p-$Cesàro summability and fuzzy numbers. In section 3 we give the concept of pointwise statistical convergence of order α and the concept $\alpha-$statistically Cauchy sequence of sequences of fuzzy mappings and prove that it is equivalent to pointwise statistical convergence of order α of sequences of fuzzy mappings. We also establish some inclusion relations between $w_p^{\alpha}(F)$ and $S^{\beta}(F)$ and between $S^{\alpha}(F)$ and $S(F)$.

DEFINITIONS AND PRELIMINARIES

The definitions of statistical convergence and strong $p-$Cesàro convergence of a sequence of real numbers were introduced in the literature independently of one another and followed different lines of development since their first appearance. It turns out, however, that the two definitions can be simply related to one another in general and are equivalent for bounded sequences. The idea of statistical convergence depends on the density of subsets of the set N of natural numbers. The density of a subset E of N is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of N has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

The $\alpha - density$ of a subset E of N was defined by Çolak (2010). Let α be real number such that $0 < \alpha \leq 1$. The $\alpha - density$ of a subset E of N is defined by

$$\delta_{\alpha}(E) = \lim_{n} \frac{1}{n^{\alpha}} |\{k \leq n \ : \ k \in E\}|, \text{ provided the limit exists,}$$

where $|\{k \leq n \ : \ k \in E\}|$ denotes the number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that x_k satisfies property $P(k)$ for almost all k except a set of $\alpha - density$ zero, then we say that x_k satisfies property $P(k)$ for "*almost all k according to* α " and we abbreviate this by " $a.a.k(\alpha)$ ".

It is clear that any finite subset of N has zero $\alpha - density$ and $\delta_{\alpha}(E^c) = 1 - \delta_{\alpha}(E)$ does not hold for $0 < \alpha < 1$ in general, the equality holds only if $\alpha = 1$. Note that the $\alpha - density$ of any set reduces to the natural density of the set in case $\alpha = 1$.

The order of statistical convergence of a sequence of numbers was given by Gadjiev & Orhan (2002) and after then statistical convergence of order α and strong $p-$Cesàro summability of order α studied by Çolak (2010).

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh (1965) a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \to [0, 1]$. The function u itself is often used for the fuzzy set.

Let $C(\mathrm{R}^n)$ denote the family of all nonempty, compact, convex subsets of R^n. If $\alpha, \beta \in \mathrm{R}$ and $A, B \in C(\mathrm{R}^n)$, then

$$\alpha(A + B) = \alpha A + \alpha B, \qquad (\alpha\beta)A = \alpha(\beta A), \qquad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$\delta_\infty(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\},$$

where $\| . \|$ denotes the usual Euclidean norm in R^n. It is well known that $(C(\mathrm{R}^n), \delta_\infty)$ is a complete metric space.

Denote

$$L(\mathrm{R}^n) = \{u : \mathrm{R}^n \to [0, 1] : u \text{ satisfies } (i) - (iv) \text{ below}\},$$

where

i) u is normal, that is, there exists an $x_0 \in \mathrm{R}^n$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is, for $x, y \in \mathrm{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]$;

iii) u is upper semicontinuous;

iv) the closure of $\{x \in \mathrm{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathrm{R}^n)$, then u is called a fuzzy number and $L(\mathrm{R}^n)$ is said to be a fuzzy number space.

For $0 < \alpha \leq 1$, the α-level set $[u]^\alpha$ is defined by

$$[u]^\alpha = \{x \in \mathrm{R}^n : u(x) \geq \alpha\}$$

Then from $(i) - (iv)$, it follows that the α-level sets $[u]^\alpha \in C(\mathrm{R}^n)$.

Some arithmetic operations for $\alpha-$level sets are defined as follows:

Let $u, v \in L(\mathrm{R})$ and the $\alpha-$level sets be $[u]^{\alpha}= \left[u_1^{\alpha}, u_2^{\alpha}\right]$, $[v]^{\alpha}= \left[v_1^{\alpha}, v_2^{\alpha}\right]$, $\alpha \in [0, 1]$. Then we have

$$[u+v]^{\alpha}= \left[u_1^{\alpha}+v_1^{\alpha}, u_2^{\alpha}+v_2^{\alpha}\right]$$

$$[u-v]^{\alpha}= \left[u_1^{\alpha}-v_2^{\alpha}, u_2^{\alpha}-v_1^{\alpha}\right]$$

$$[ku]^{\alpha} = \begin{cases} \left[ku_1^{\alpha}, ku_2^{\alpha}\right], & \text{if } k \geq 0 \\ \left[ku_2^{\alpha}, ku_1^{\alpha}\right], & \text{otherwise.} \end{cases}$$

Define, for each $1 \leq q < \infty$,

$$d_q(u, v) = \left(\int_0^1 [\delta_{\infty}([u]^{\alpha}, [v]^{\alpha})]^q d\alpha \right)^{1/q}$$

and $d_{\infty}(u, v) = \sup_{0 \leq \alpha \leq 1} \delta_{\infty}([u]^{\alpha}, [v]^{\alpha})$, where δ_{∞} is the Hausdorff metric. Clearly $d_{\infty}(u, v) = \lim_{q \to \infty} d_q(u, v)$ with $d_q \leq d_s$ if $q \leq s$, Diamond & Kloeden (1990); Lakshmikantham & Mohapatra (2003). For simplicity in notation, throughout the paper d will denote the notation d_q with $1 \leq q \leq \infty$.

A fuzzy mapping X is a mapping from a set $T(\subset \mathrm{R}^n)$ to the set of all fuzzy numbers. A sequence of fuzzy mappings is a function whose domain is the set of positive integers and whose range is a set of fuzzy mappings. We denote a sequence of fuzzy mappings by (X_k). If (X_k) is a sequence of fuzzy mappings then $(X_k(t))$ is a sequence of fuzzy numbers for every $t \in T$. Corresponding to a number t in the domain of each of terms of the sequences of fuzzy mappings (X_k), there is a sequence of fuzzy numbers $(X_k(t))$. If $(X_k(t))$ converges for each number t in a set T and we get $\lim_k X_k(t) = X(t)$, then we say that (X_k) converges pointwise to X on T Matloka (1987).

MAIN RESULTS

In this section we introduce the notion of pointwise statistical convergence of order α and the concept of $\alpha-$statistically Cauchy sequence for sequences of fuzzy mappings. We give the relations between the statistical convergence of order α and the statistical convergence of order β $(\alpha \leq \beta)$ of sequences of fuzzy mappings, the relations between strong $p-$Cesàro summability of order α and strong $p-$Cesàro summability of order β $(\alpha \leq \beta)$ and the relations between

strong $p-$Cesàro summability of order α and the statistical convergence of order β $(\alpha \leq \beta)$ sequences of fuzzy mappings.

Before giving the inclusion relations we will give two new definitions.

Definition 3.1 Let $0 < \alpha \leq 1$ be given. A sequence of fuzzy mappings (X_k) is said to be pointwise statistically convergent of order α or pointwise α - statistically convergent to fuzzy number X on a set T if, for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} |\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}| = 0,$$

i.e. for every $t \in T$,

$$d(X_k(t), X(t)) < \varepsilon \ \text{ for } a.a.k \ (\alpha) \ . \tag{1}$$

In this case we write $S^\alpha - \lim X_k(t) = X(t)$ on T. This means that for every $\delta > 0$, there is an integer N such that

$$\frac{1}{n^\alpha} |\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}| < \delta,$$

for all $n > N(= N(\varepsilon, \delta, t))$ and for each $\varepsilon > 0$. It is clear that if the inequality (1) holds for all but finitely many k, then $\lim X_k(t) = X(t)$ on T. It can be shown that $\lim X_k(t) = X(t)$ implies $S^\alpha - \lim X_k(t) = X(t)$ on T. The set of all pointwise statistically convergent sequences of fuzzy mappings of order α will be denoted by $S^\alpha(F)$. For $\alpha = 1$, we shall write $S(F)$ instead of $S^\alpha(F)$ and in the special case $X = \bar{0}$, we shall write $S_0^\alpha(F)$ instead of $S^\alpha(F)$.

Definition 3.2 Let α be any real number such that $0 < \alpha \leq 1$ and let (X_k) be a sequence of fuzzy mappings on a set T. The sequence (X_k) is a statistically Cauchy sequence of order α, or α - statistically Cauchy sequence provided that for every $\varepsilon > 0$ there exists a number $N(= N(\varepsilon, x))$ such that

$$d(X_k(t), X_N(t)) < \varepsilon \text{ for } a.a.k \ (\alpha) \text{ and every } t \in T$$

i.e

$$\lim_{n \to \infty} \frac{1}{n^\alpha} |\{k \leq n \ : \ d(X_k(t), X_N(t)) \geq \varepsilon \text{ for every } t \in T\}| = 0.$$

Definition 3.3 A fuzzy sequence space $E(F)$ with metric d is said to be normal (or solid) if $(X_k) \in E(F)$ and (Y_k) is such that $d(Y_k, \bar{0}) \leq d(X_k, \bar{0})$ implies $(Y_k) \in E(F)$.

Definition 3.4 A fuzzy sequence space $E(F)$ is said to be monotone if $X = (X_k) \in E(F)$ implies $\chi.X \subset E(F)$ where χ is the class of all sequences of zeros and ones. The product considered is the term product.

Remark 1. From the above definitions it follows that if a fuzzy sequence space $E(F)$ is solid, then it is monotone.

Definition 3.5 A fuzzy sequence space $E(F)$ is said to be symmetric if $(X_k) \in E(F)$ implies $(X_{\pi(k)}) \in E(F)$, where π is a permutation of N.

Definition 3.6 A fuzzy sequence space $E(F)$ is said to be convergence free if $(X_k) \in E(F)$ implies $(Y_k) \in E(F)$, where $Y_k = \bar{0}$ whenever $X_k = \bar{0}$, where

$$\bar{0}(t) = \begin{cases} 1, & \text{for } t = (0, 0, 0, ..., 0) \\ 0, & \text{otherwise} \end{cases} .$$

Theorem 3.7 Let $0 < \alpha \leq 1$ and (X_k) and (Y_k) be sequences of fuzzy mappings.

(*i*) If $S^\alpha - \lim X_k(t) = X_0(t)$ and $c \in \mathrm{R}$, then $S^\alpha - \lim cX_k(t) = cX_0(t)$,

(*ii*) If $S^\alpha - \lim X_k(t) = X_0(t)$ and $S^\alpha - \lim Y_k(t) = Y_0(t)$, then $S^\alpha - \lim(X_k(t) + Y_k(t)) = X_0(t) + Y_0(t)$.

Proof. The proof is clear in case $c = 0$. Suppose that $c \neq 0$ then the proof of (*i*) follows from the following inequality

$$\frac{1}{n^\alpha} |\{k \leq n \ : \ d(cX_k(t), cX_0(t)) \geq \varepsilon\}|$$
$$\leq \frac{1}{n^\alpha} \left| \left\{ k \leq n \ : \ d(X_k(t), X_0(t)) \geq \frac{\varepsilon}{|c|} \right\} \right| .$$

ii) Suppose that $S^\alpha - \lim X_k(t) = X_0(t)$ and $S^\alpha - \lim Y_k(t) = Y_0(t)$. By triangular inequality we get

$$d(X_k(t) + Y_k(t), X_0(t) + Y_0(t)) \leq d(X_k(t), X_0(t)) + d(Y_k(t), Y_0(t)) .$$

Therefore given $\varepsilon > 0$ we have

$$\frac{1}{n^\alpha} | \{k \leq n \ : \ d(X_k(t) + Y_k(t), X_0(t) + Y_0(t)) \geq \varepsilon\} |$$
$$\leq \frac{1}{n^\alpha} \left| \left\{ k \leq n \ : \ d(X_k(t), X_0(t)) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{1}{n^\alpha} \left| \left\{ k \leq n \ : \ d(Y_k(t), Y_0(t)) \geq \frac{\varepsilon}{2} \right\} \right| .$$

Hence $S^\alpha - \lim (X_k(t) + Y_k(t)) = X_0(t) + Y_0(t)$.

Theorem 3.8 Let (X_k) be a sequence of fuzzy mappings defined on a set T. The

following statements are equivalent:

(*i*) (X_k) is a pointwise $\alpha-$statistically convergent sequence on T;

(*ii*) (X_k) is a $\alpha-$statistically Cauchy sequence on T;

(*iii*) (X_k) is a sequence of fuzzy mappings for which there is a pointwise convergent sequence of fuzzy mappings (Y_k) such that $X_k(t) = Y_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$.

Proof. $(i \Rightarrow ii)$ Suppose that $S^{\alpha}-\lim X_k(t) = X(t)$ on T and let $\varepsilon > 0$. Then $d(X_k(t), X(t)) < \frac{\varepsilon}{2}$ for $a.a.k(\alpha)$ and if N is chosen so that $d(X_N(t), X(t)) < \frac{\varepsilon}{2}$, then we have

$$d(X_k(t), X_N(t)) \leq d(X_k(t), X(t)) + d(X_N(t), X(t)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ for } a.a.k(\alpha)$$

for every $t \in T$. Hence $(X_k(t))$ is a $\alpha-$statistically Cauchy sequence.

$(ii \Rightarrow iii)$ Next, assume that $(X_k(t))$ be $\alpha-$statistically Cauchy so that the closed ball $B = \bar{B}\big(X_{N(1)}(t), 1\big)$ contains $X_k(t)$ for $a.a.k(\alpha)$ for some positive number $N(1)$ and for every $t \in T$. Also apply hypothesis to choose M so that $B' = \bar{B}\Big(X_M(t), \frac{1}{2}\Big)$ contains $X_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$. It is clear that $B_1 = B \bigcap B'$ contains $X_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$. Therefore B_1 is a closed set of diameter less than or equal to 1 that contains $X_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$. Now we proceed by choosing $N(2)$ so that $B'' = \bar{B}\Big(X_{N(2)}(t), \frac{1}{4}\Big)$ contains $X_k(t)$ for $a.a.k(\alpha)$ and by the preceding argument $B_2 = B_1 \bigcap B''$ contains $X_k(t)$ for $a.a.k\ (\alpha)$ and for every $t \in T$, and B_2 has diameter less than or equal to $\frac{1}{2}$. Continuing this process, we construct a sequence $\{B_m\}_{m=1}^{\infty}$ of closed balls such that for each m, $B_m \supset B_{m+1}$, the diameter of B_m is not greater than $\frac{1}{2^{m-1}}$ and $X_k(t) \in B_m$ for $a.a.k(\alpha)$ and for every $t \in T$. By the nest of closed set theorem, we have $\bigcap_{m=1}^{\infty} B_m \neq \phi$ and contains exactly one element. So there is a fuzzy mapping $X(t)$ which is $X(t) \in \bigcap_{m=1}^{\infty} B_m$ for every $t \in T$. Using the fact that $X_k(t) \in B_m$ for $a.a.k(\alpha)$ and for every $t \in T$, we choose an increasing positive integer sequence $\{H_m\}_{m=1}^{\infty}$ such that

$$\lim_n \frac{1}{n^{\alpha}} |\{k \leq n \ : \ X_k(t) \notin B_m \text{ for every } t \in T\}| < \frac{1}{m} \text{ if } n > H_m. \qquad (2)$$

Now we construct a subsequence $(Z_k(t))$ of $(X_k(t))$ consisting of terms of $X_k(t)$ such that if $H_m < k \leq H_{m+1}$ and $X_k(t) \notin B_m$ for every $t \in T$, then $X_k(t)$ is a term of $Z_k(t)$.

Next define the sequence $(Y_k(t))$ by

$$Y_k(t) = \begin{cases} X(t), & \text{if } X_k(t) \text{ is a term of } Z_k(t) \\ X_k(t), & \text{otherwise,} \end{cases}$$

for every $t \in T$. Then $\lim_{k\to\infty} Y_k(t) = X(t)$ on T; for if $\varepsilon > \frac{1}{m} > 0$ and $k > H_m$ then either $X_k(t)$ is a term of $(Z_k(t))$, which means $Y_k(t) = X(t)$ on T or $Y_k(t) = X_k(t) \in B_m$ on T and $d(Y_k(t), X(t)) \leq$diameter of $B_m \leq \frac{1}{2^{m-1}}$ for every $t \in T$. We also assert that $X_k(t) = Y_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$. To verify this we observe that if $H_m < n \leq H_{m+1}$ then

$$\{k \leq n \ : \ Y_k(t) \neq X_k(t) \text{ for every } t \in T\} \subseteq \{k \leq n \ : \ X_k(t) \notin B_m \text{ for every } t \in T\}$$

so by (2)

$$\begin{aligned} &\frac{1}{n^\alpha} |\{k \leq n \ : \ Y_k(t) \neq X_k(t) \text{ for every } t \in T\}| \\ &\leq \frac{1}{n^\alpha} |\{k \leq n \ : \ X_k(t) \notin B_m \text{ for every } t \in T\}| < \frac{1}{m} \end{aligned}$$

Hence, the limit is 0, as $n \to \infty$ and $X_k(t) = Y_k(t)$ for $a.a.k(\alpha)$ and for every $t \in T$. Therefore (ii) implies (iii).

Finally, assume that (iii) holds, say $X_k(t) = Y_k(t)$ for $a.a.k(\alpha)$ and for every$t \in T$, and $\lim_{k\to\infty} Y_k(t) = X(t)$ on T. Let $\varepsilon > 0$ then for each n,

$$\begin{aligned} &\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\} \\ &\subseteq \{k \leq n \ : \ X_k(t) \neq Y_k(t) \text{ for every } t \in T\} \bigcup \{k \leq n \ : \ d(Y_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\} \end{aligned}$$

Since $\lim_{k\to\infty} Y_k(t) = X(t)$ on T, the latter set contains a fixed number of integers, say $L = L(\varepsilon, t)$. Therefore

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}| \\ &\leq \lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n \ : \ X_k(t) \neq Y_k(t) \text{ for every } t \in T\}| + \lim_{n\to\infty} \frac{L}{n^\alpha} = 0 \end{aligned}$$

because $X_k(t) = Y_k(t)$ $a.a.k(\alpha)$ for every $t \in T$. Hence $d(X_k(t), X(t)) < \varepsilon$ for $a.a.k(\alpha)$ and for every $t \in T$, so (i) holds and proof is complete.

As an immediate consequence of Theorem 3.8, we have the following result.

Corollary 3.9 If (X_k) is a sequence of fuzzy mappings such that

$S^{\alpha} - \lim X_k(t) = X_k$ on T, then (X_k) has a subsequence $\left(X_{k(i)}(t)\right)$ such that $\lim X_{k(i)}(t) = X(t)$ on T.

Theorem 3.10 Let $0 < \alpha \leq \beta \leq 1$, then $S^{\alpha}(F) \subseteq S^{\beta}(F)$.

Proof. If $0 < \alpha \leq \beta \leq 1$, then

$$\frac{1}{n^{\beta}} |\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}|$$
$$\leq \frac{1}{n^{\alpha}} |\{k \leq n \ : \ d(X_k(t), X(t)) \geq \varepsilon \text{ for every } t \in T\}|$$

for every $\varepsilon > 0$ and this gives that $S^{\alpha}(F) \subseteq S^{\beta}(F)$.

The following result is a consequence of the above theorem.

Corollary 3.11 If a sequence of fuzzy mappings (X_k) is statistically convergent of order α, to X, for some $0 < \alpha \leq 1$, then it is statistically convergent to X.

Definition 3.12 Let α be any real number such that $0 < \alpha \leq 1$ and let p be a positive real number. A sequence of fuzzy mappings (X_k) is said to be strongly $p-$Cesàro summable of order α if there is a fuzzy-valued function X such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} (d(X_k(t), X(t)))^p = 0.$$

In this case we write $w_p^{\alpha} - \lim X_k(t) = X(t)$ on T. The set of all strongly $p-$Cesàro summable sequences of fuzzy mappings of order α will be denoted by $w_p^{\alpha}(F)$. In the special case $X = \bar{0}$, we shall write $w_{p_0}^{\alpha}(F)$ instead of $w_p^{\alpha}(F)$.

Theorem 3.13 Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number, then $w_p^{\alpha}(F) \subseteq w_p^{\beta}(F)$.

Proof. Let the sequence (X_k) be strongly $p-$Cesàro summable of order α. Then, given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number p we may write

$$\frac{1}{n^{\beta}} \sum_{k=1}^{n} (d(X_k(t), X(t)))^p \leq \frac{1}{n^{\alpha}} \sum_{k=1}^{n} (d(X_k(t), X(t)))^p$$

and this gives that $w_p^{\alpha}(F) \subseteq w_p^{\beta}(F)$.

Corollary 3.14 Let $0 < \alpha \leq \beta \leq 1$ and p be a positive real number. Then

(*i*) If $\alpha = \beta$, then $w_p^{\alpha}(F) = w_p^{\beta}(F)$,

(*ii*) $w_p^{\alpha}(F) \subseteq w_p(F)$ for each $\alpha \in (0, 1]$ and $0 < p < \infty$.

Theorem 3.15 Let $0 < \alpha \leq 1$ and $0 < p < q < \infty$. Then $w_q^\alpha(F) \subseteq w_p^\alpha(F)$.

Proof. Omitted.

Theorem 3.16 Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < p < \infty$. If a sequence of fuzzy mappings (X_k) is strongly $p-$Cesàro summable of order α, to X, then it is statistically convergent of order β, to X.

Proof. For any sequence of functions (X_k) defined on T, we can write

$$\sum_{k=1}^{n}(d(X_k(t), X(t)))^p \geq |\{k \leq n \ : \ d(X_k(t), X(t))^p \geq \varepsilon \text{ for every } t \in T\}| \varepsilon^p$$

and so that

$$\frac{1}{n^\alpha}\sum_{k=1}^{n}(d(X_k(t), X(t)))^p \geq \frac{1}{n^\alpha}|\{k \leq n \ : \ d(X_k(t), X(t))^p \geq \varepsilon \text{ for every } t \in T\}| \varepsilon^p$$
$$\geq \frac{1}{n^\beta}|\{k \leq n \ : \ d(X_k(t), X(t))^p \geq \varepsilon \text{ for every } t \in T\}| \varepsilon^p$$

Corollary 3.17 Let α be a fixed real number such that $0 < \alpha \leq 1$ and $0 < p < \infty$. If a sequence of fuzzy mappings (X_k) is strongly $p-$Cesàro summable of order α, to X, then it is statistically convergent of order α, to X.

Theorem 3.18 (*i*) The spaces $S_0^\alpha(F)$ and $w_{p_0}^\alpha(F)$ are solid and such as are monotone.

(*ii*) The spaces $S^\alpha(F)$ and $w_p^\alpha(F)$ are neither solid nor monotone.

Proof. (*i*) We shall prove only for $S_0^\alpha(F)$ and the other can be treated similarly. Let (X_k) be a sequence of fuzzy mappings in $S_0^\alpha(F)$. Let $d(Y_k(t), \bar{0}) \leq d(X_k(t), \bar{0})$ for all $k \in \mathrm{N}$. The proof follows from the following inclusion:

$$\{k \in \mathrm{N} \ : \ d(X_k(t), \bar{0}) \geq \varepsilon\} \supseteq \{k \in \mathrm{N} \ : \ d(Y_k(t), \bar{0}) \geq \varepsilon\}.$$

The rest of the proof follows from the Remark 1.

(*ii*) The proof follows from the following examples.

Example 1 Let $\alpha = 1$ and consider the sequences (X_k) and (Y_k) defined as follows:

$$X_k(t) = \begin{cases} 1, & \text{for } t = (1, 1, 1, ..., 1) \\ 0, & \text{otherwise} \end{cases}$$

$$Y_k(t) = \begin{cases} 1, & \text{for all } k \text{ odd and } t = (-1,-1,-1,...,-1) \\ 0, & \text{for all } k \text{ odd and } t \neq (-1,-1,-1,...,-1) \\ 1, & \text{for all } k \text{ even and } t = (1,\ 1,\ 1,...,\ 1) \\ 0, & \text{for all } k \text{ even and } t \neq (1,\ 1,\ 1,...,\ 1) \end{cases}$$

Then (X_k) belongs to $S^{\alpha}(F)$, but (Y_k) does not belong. Hence the space $S^{\alpha}(F)$ is not solid.

Example 2 Let $\alpha = 1$ and consider the sequence $\{X_k\}$ defined by

$$X_k = \bar{1}, \text{ for all } k \in \mathrm{N}.$$

Consider its J^{th} step space Z_J defined as $(Y_k) \in Z_J \Rightarrow Y_k = X_k$ for all $k = 2i + 1, i \in \mathrm{N}$ and $Y_k = \bar{0}$, otherwise. Then $\{X_k\} \in S^{\alpha}(F)$, but (Y_k) does not belong to $S^{\alpha}(F)$.

Theorem 3.19 The spaces $S_0^{\alpha}(F)$, $w_{p_0}^{\alpha}(F)$, $S^{\alpha}(F)$ and $w_p^{\alpha}(F)$ are not symmetric.

Proof . The proof follows from the following example.

Example 3 Let $\alpha = 1$ and consider the sequence (X_k) defined by

$$X_k(t) = \begin{cases} 1, & k = i^3,\ i \in \mathrm{N} \text{ and } t = (1,\ 1,\ 1,...,\ 1) \\ 0, & k = i^3,\ i \in \mathrm{N} \text{ and } t \neq (1,\ 1,\ 1,...,\ 1) \\ 1, & k \neq i^3, \text{ for any and } t = (0,\ 0,\ 0,...,\ 0) \\ 0, & k \neq i^3, \text{ for any and } t \neq (0,\ 0,\ 0,...,\ 0) \end{cases}$$

Consider the rearranged sequence (Y_k) of (X_k), defined as follows :

$$(Y_k) = (X_1,\ X_2,\ X_8,\ X_3,\ X_{27},\ X_4,\ X_{64},\ X_5,\ X_{125},\ X_6,\ X_{216},\ X_7,\ X_{343},\ X_9, ...).$$

The (X_k) belongs to the space $S^{\alpha}(F)$, but (Y_k) does not belong to $S^{\alpha}(F)$, hence the space $S^{\alpha}(F)$ is not symmetric. The others can be treated similarly.

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خلاصة

نقدم في هذا البحث، ولأول مرة مفهوم التقارب الإحصائي من المرتبة α وذلك لمتتاليات التطبيقات المشوشة. وعلاوة على ذلك، نعطي مفهوم متتالية كوشي الإحصائية ونثبت أنها مكافئة للمتتالية المتقاربة إحصائياً ونقطياً. كما نناقش كذلك بعض العلاقات بين التقارب الإحصائي والتجميعية القوية.