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Quaternions: Quantum calculus approach with applications

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Abstract

In this paper we introduce two types of quaternion sequences with components including quantum integers. We also introduce quantum quaternion polynomials. Moreover, we give some properties and identities for these quantum quaternions and polynomials. Finally, we give time evolution and rotation applications for some specific quaternion sequences. The applications can be converted into quantum integer forms under suitable conditions with similar considerations.

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1. Introduction

Let $\mathcal{F}$ be an arbitrary field with characteristic not 2. The quaternion algebra $\mathbf{H}$ is a four-dimensional central simple algebra over $\mathcal{F}$. The addition and multiplication operations together give $\mathbf{H}$ the structure of a ring, or more explicitly, a non-commutative division ring. It is well known that every quaternion algebra over the field $\mathcal{F}$ is isomorphic to the following form

$$\mathbf{H} := \left\{\mathcal{F} + \mathcal{F}\mathbf{i} + \mathcal{F}\mathbf{j} + \mathcal{F}\mathbf{ij} \mid \mathbf{i}^2 = a,\ \mathbf{j}^2 = b,\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji}\right\},$$

where a and b are nonzero invertible elements of $\mathcal{F}$. Here the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is called a standard basis for $\mathbf{H}$, and we simply write $\mathbf{H} = \left(\frac{a,b}{\mathcal{F}}\right)$. All of quaternion algebra proceeds from the equations in $\mathbf{H}$. The classical example of a quaternion algebra is the 2×2 real matrix algebra $M_2(\mathbb{R}) \simeq \left(\frac{1,1}{\mathbb{R}}\right)$. Other familiar examples are Hamilton's quaternions $\left(\frac{-1,-1}{\mathbb{R}}\right)$ and split quaternions $\left(\frac{-1,1}{\mathbb{R}}\right)$.

The *product* of two quaternions p and q in $\left(\frac{a,b}{\mathcal{F}}\right)$, where

$$p = a_0 + \mathbf{p} = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and

$$q = b_0 + \mathbf{q} = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

can be reduced to

$$pq = a_0b_0 - \mathbf{p}\cdot\mathbf{q} + a_0\mathbf{q} + b_0\mathbf{p} + \mathbf{p}\times\mathbf{q}.$$

Here, " $\cdot$ " is the dot product, and " $\times$ " is the cross product all in accordance with Hamilton's foregoing original equations.

The *conjugate* of the quaternion

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

in quaternion algebra $\mathbf{H}$ denoted by q^* is defined by $q^* = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. The *trace* and *norm* of a quaternion q is defined as usual

$$Tr(q) = q + q^* \text{ and } N(q) = qq^*.$$

In addition, the *inverse* of a quaternion q is $q^{-1} = N(q)^{-1}q^*$, and if q is a unit quaternion, then $q^{-1} = q^*$.

Quaternions whose components are from special number sequences have been studied by several authors for many years. These types of quaternions are referred to as *quaternion sequences*. There are various types of quaternion sequences which are determined by their components taken from different types of integer sequences. Horadam (1963) defined the n^{th} Fibonacci quaternion and n^{th} Lucas quaternion as

$$Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k} \tag{1.1}$$

and

$$K_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k}, \tag{1.2}$$

that is, the quaternions whose components are the well-known Fibonacci and Lucas numbers, respectively. In that paper, he essentially considered the elements of the algebra of real quaternions whose components are special integer sequences. Through natural logic, Horadam (1967) investigated the generalized Fibonacci sequence $W_n(a, b; p, q)$, where a, b, p, q are integers

$$W_0 = a, \ W_1 = b$$

and

$$W_n = pW_{n-1} + qW_{n-2}, \quad n \geq 2.$$

For $a = 0$ and $b = 1$, we write the recurrence $W_n(a, b; p, q)$ in the form

$$U_n = pU_{n-1} + qU_{n-2}, \quad n \geq 2. \tag{1.3}$$

For $a = 2$, $b = p$, we use the form

$$V_n = pV_{n-1} + qV_{n-2}, \quad n \geq 2. \tag{1.4}$$

Swamy (1973) discussed some relations for generalized Fibonacci quaternions. Horadam (1993) considered a few recurrence relations of some special quaternions. Pell, Pell-Lucas and Jacobsthal are also fundamental q sequences (see Çimen and İpek (2016); Szynal-Liana and Włoch (2016)). Other authors have investigated similar structures and obtained meaningful results (see Catarino (2016); Catarino (2019); Flaut and Savin (2015); Flaut and Shpakivskyi (2013); Flaut and Shpakivskyi (2015); Halici (2012); Iyer (1969); İpek (2017); Ramirez (2015); Savin (2017)).

In accordance with our purpose, we define the second order linear sequences $\{F_{k,n}\}$ and $\{L_{k,n}\}$ for $n > 1$ as

$$F_{k,n} = kF_{n-1} + F_{n-2}, \ F_{k,0} = 0, \ F_{k,1} = 1,$$
$$L_{k,n} = kL_{n-1} + L_{n-2}, \ \ L_{k,0} = 2, \ L_{k,1} = k.$$

Falcon and Plaza named the elements of the sequence $\{F_{k,n}\}$ as $k-$Fibonacci (see Falcon and Plaza (2007)). In Falcon (2011), the sequence $\{L_{k,n}\}$ is called as $k-$Lucas numbers. The Binet formulæ of these numbers are

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_{k,n} = \alpha^n + \beta^n , \tag{1.5}$$

where $\alpha, \beta = \left(k \pm \sqrt{k^2 + 4}\right)/2$.

Let $h(x)$ be a polynomial with real coefficients. $h(x)-$Fibonacci polynomials $\{F_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$F_{h,n}(x) = h(x)F_{h,n-1}(x) + F_{h,n-2}(x), \quad n \geq 2$$

with the initial conditions

$$F_{h,0}(x) = 0, \ F_{h,1}(x) = 1.$$

We can see that $h(x)-$Fibonacci polynomials are the generalizations of Catalan's Fibonacci polynomials, Byrd's Fibonacci polynomials and also the $k-$Fibonacci numbers. Using these polynomials Catarino studied $h(x)-$Fibonacci quaternion polynomials

$$\begin{aligned} Q_{h,n}(x) \;\; &= \;\; F_{h,n}(x) + F_{h,n+1}(x)\mathbf{i} + \\ & \qquad F_{h,n+2}(x)\mathbf{j} + F_{h,n+3}(x)\mathbf{k} \end{aligned}$$

and obtained Binet formulæ, the generating function and some identities of the $h(x)-$Fibonacci quaternion polynomials (see

Catarino (2015)). For $h(x) = k$, k being any real number, we get the $k-$Fibonacci numbers $F_{k,n}$ from the definition of $h(x)-$Fibonacci polynomials $F_{h,n}(x)$, and hence we obtain $k-$Fibonacci quaternions $Q_{k,n}$

$$Q_{k,n} = F_{k,n} + F_{k,n+1}\mathbf{i} + F_{k,n+2}\mathbf{j} + F_{k,n+3}\mathbf{k}$$

from the definition of $h(x)-$Fibonacci quaternion polynomials $Q_{h,n}(x)$. Thus, $h(x)-$Fibonacci quaternion polynomials generalize the $k-$Fibonacci quaternions, and in so doing, they also generalize the Fibonacci quaternions.

We consider two types of quaternion sequences with components including quantum integers. We begin with a brief introduction to some properties of the quantum integers. Let $\mathcal{R}$ be an associative ring with unit and q is an element of $\mathcal{R}$. If $n \in \mathbb{N}$, the *quantum integer n* or simply the *$q-$integer n* is defined by

$$[n]_q = \sum_{i=0}^{n-1} q^i.$$

The $q-$integer $(-n)$ is defined as

$$[-n]_q = -\sum_{i=1}^{n} q^{-i},$$

when q is invertible in $\mathcal{R}$. Thus for $n \in \mathbb{Z}^-$, we have $[n]_q = -q^n[-n]_q$. In particular, if $1-q$ is invertible in $\mathcal{R}$, we have

$$[n]_q = \frac{1-q^n}{1-q}.$$

For all $m, n \in \mathbb{Z}$, we have

$$[m+n]_q = [m]_q + q^m[n]_q$$

and

$$[mn]_q = [m]_q[n]_{q^m}$$

for an invertible element q in $\mathcal{R}$. For $\mathcal{R} = \mathbb{Z}$ and $q = 1$, it can be easily seen that the quantum integer $[n]_q$ will be the usual integer n (see Le Stum and Quirós (2015)).

The elements of the second order integer sequences and $q-$integers can be transformed to each other. That is, for $q = \beta/\alpha$ the Binet formulæ in (1.5) are reduced to the following $q-$integer forms:

$$F_{k,n} = \alpha^{n-1}[n]_q \quad \text{and} \quad L_{k,n} = \alpha^n \frac{[2n]_q}{[n]_q},$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$.

As before stated, we will introduce two types of quaternion sequences with components including quantum integers in Section 2. We give some properties and identities of defined quaternions. Because quantum integers are extensively used in physics, quantum quaternion types may also be applied to many applications. In Section 3, we introduce $q-$quaternion polynomials which generalize the $h(x)-$Fibonacci quaternion polynomials. We obtain the Binet formulæ and generating functions for $q-$quaternion polynomials. Furthermore, we give some properties and identities for these quantum quaternion polynomials. Finally, in Section 4, we will give time evolution and rotation applications for some specific quaternion sequences. The applications can be converted into quantum integer forms under suitable conditions with similar considerations.

2. $q-$Quaternions

The quaternion sequences up to now were the quaternions whose components are real sequences. Inspired by these studies, we consider a more general quaternion sequence by receiving components from complex sequences.

Throughout this section, we take $n \in \mathbb{N}$ and $1-q$ as a nonzero complex number.

Definition 1 *Quaternions of the form*

$$\mathbf{Q}_n = \alpha^{n-1}[n]_q + \alpha^n[n+1]_q\mathbf{i} + \alpha^{n+1}[n+2]_q\mathbf{j} + \alpha^{n+2}[n+3]_q\mathbf{k}$$

are the n^{th} $q-$Fibonacci quaternion. Quaternions of the form

$$\mathbf{V}_n = \alpha^n\frac{[2n]_q}{[n]_q} + \alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i} + \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{j} + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{k}$$

are referred to as the n^{th} $q-$Lucas quaternion.

Theorem 1 *The Binet formula of the q-Fibonacci quaternion $\mathbf{Q}_n$ is*

$$\alpha^{n-1}[n]_q\underline{\alpha}+(\alpha q)^n\underline{\beta}\ .$$

The Binet formulæ of the q-Lucas quaternion $\mathbf{V}_n$ is

$$\alpha^n\frac{[2n]_q}{[n]_q}\underline{\gamma}+\alpha^{n+1}(1-q)\underline{\beta},$$

where

$$\begin{aligned}
\underline{\alpha} &= 1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k},\\
\underline{\beta} &= \mathbf{i}+\alpha[2]_q\mathbf{j}+\alpha^2[3]_q\mathbf{k},\\
\underline{\gamma} &= 1+(\alpha q)\mathbf{i}+(\alpha q)^2\mathbf{j}+(\alpha q)^3\mathbf{k}.
\end{aligned}$$

Proof. By Definition 1**,** we have

$$\begin{aligned}
\mathbf{Q}_n &= \alpha^{n-1}[n]_q+\alpha^n[n+1]_q\mathbf{i}+\\
&\quad \alpha^{n+1}[n+2]_q\mathbf{j}+\alpha^{n+2}[n+3]_q\mathbf{k}\\
&= \alpha^{n-1}[n]_q+\alpha^n([n]_q+q^n)\mathbf{i}+\\
&\quad \alpha^{n+1}([n]_q+q^n[2]_q)\mathbf{j}+\\
&\quad \alpha^{n+2}([n]_q+q^n[3]_q)\mathbf{k}\\
&= \alpha^{n-1}[n]_q(1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k})+\\
&\quad \alpha^nq^n\left(\mathbf{i}+\alpha[2]_q\mathbf{j}+\alpha^2[3]_q\mathbf{k}\right).
\end{aligned}$$

That is,

$$\mathbf{Q}_n=\alpha^{n-1}[n]_q\underline{\alpha}+(\alpha q)^n\underline{\beta}.$$

The Binet form of $\mathbf{V}_n$ can be similarly proven. ■

Remark 1 *We can write the Binet formulæ of the $q-$quaternions $\mathbf{Q}_n$ and $\mathbf{V}_n$ in other forms. We have*

$$\begin{aligned}
\mathbf{Q}_n &= \alpha^{n-1}[n]_q+\alpha^n[n+1]_q\mathbf{i}+\\
&\quad \alpha^{n+1}[n+2]_q\mathbf{j}+\alpha^{n+2}[n+3]_q\mathbf{k}\\
\mathbf{Q}_n &= \alpha^{n-1}\frac{1-q^n}{1-q}+\alpha^n\frac{1-q^{n+1}}{1-q}\mathbf{i}+\\
&\quad \alpha^{n+1}\frac{1-q^{n+2}}{1-q}\mathbf{j}+\alpha^{n+2}\frac{1-q^{n+3}}{1-q}\mathbf{k}\\
\mathbf{Q}_n &= \frac{\alpha^n-(\alpha q)^n}{\alpha-\alpha q}+\frac{\alpha^{n+1}-(\alpha q)^{n+1}}{\alpha-\alpha q}\mathbf{i}+\\
&\quad \frac{\alpha^{n+2}-(\alpha q)^{n+2}}{\alpha-\alpha q}\mathbf{j}+\frac{\alpha^{n+3}-(\alpha q)^{n+3}}{\alpha-\alpha q}\mathbf{k}\\
\mathbf{Q}_n &= \frac{\alpha^n}{\alpha-\alpha q}\left(1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}\right)-\\
&\quad \frac{(\alpha q)^n}{\alpha-\alpha q}\left(1+(\alpha q)\mathbf{i}+(\alpha q)^2\mathbf{j}+(\alpha q)^3\mathbf{k}\right).
\end{aligned}$$

A similar consideration shows that

$$\begin{aligned}
\mathbf{V}_n &= \alpha^n\frac{[2n]_q}{[n]_q}+\alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i}+\\
&\quad \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{j}+\alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{k}\\
\mathbf{V}_n &= \alpha^n\frac{1-q^{2n}}{1-q}+\alpha^{n+1}\frac{1-q^{2n+2}}{1-q}\mathbf{i}+\\
&\quad \alpha^{n+2}\frac{1-q^{2n+4}}{1-q}\mathbf{j}+\alpha^{n+3}\frac{1-q^{2n+6}}{1-q}\mathbf{k}\\
\mathbf{V}_n &= \alpha^n(1+q^n)+\alpha^{n+1}(1+q^{n+1})\mathbf{i}+\\
&\quad \alpha^{n+2}(1+q^{n+2})\mathbf{j}+\alpha^{n+3}(1+q^{n+3})\mathbf{k}\\
\mathbf{V}_n &= \alpha^n\left(1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}\right)+\\
&\quad (\alpha q)^n\left(1+(\alpha q)\mathbf{i}+(\alpha q)^2\mathbf{j}+(\alpha q)^3\mathbf{k}\right).
\end{aligned}$$

Hence we obtain that

$$\mathbf{Q}_n=\frac{\alpha^n\underline{\alpha}-(\alpha q)^n\underline{\gamma}}{\alpha(1-q)},\qquad \mathbf{V}_n=\alpha^n\underline{\alpha}+(\alpha q)^n\underline{\gamma}, \tag{2.1}$$

respectively.

The following theorem gives the exponential generating functions of the $q-$Fibonacci quaternion $\mathbf{Q}_n$ and $q-$Lucas quaternion $\mathbf{V}_n$.

Theorem 2 *The exponential generating function for the $q-$Fibonacci quaternion $\mathbf{Q}_n$ is*

$$F(x)=\frac{e^{\alpha x}\underline{\alpha}-e^{(\alpha q)x}\underline{\gamma}}{\alpha(1-q)},$$

and the exponential generating function for the $q-$Lucas quaternion $\mathbf{V}_n$ is

$$G(x)=e^{\alpha x}\underline{\alpha}+e^{(\alpha q)x}\underline{\gamma}.$$

Proof. By using the Binet formula of $\mathbf{Q}_n$ given in (2.1) and the well-known equality

$$e^{\alpha x} = \sum \alpha^n \frac{x^n}{n!}$$

we get the result. Exponential generating function for the $q-$Lucas quaternion $\mathbf{V}_n$ can be similarly proven. ■

Example 1 *Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = -\frac{1}{\alpha^2}$. Then we have*

$$\begin{aligned}
\mathbf{Q}_n &= \alpha^{n-1}[n]_q + \alpha^n[n+1]_q\mathbf{i} + \\
&\quad \alpha^{n+1}[n+2]_q\mathbf{j} + \alpha^{n+2}[n+3]_q\mathbf{k} \\
\mathbf{Q}_n &= \frac{\alpha^n - (\alpha q)^n}{\alpha - \alpha q} + \frac{\alpha^{n+1} - (\alpha q)^{n+1}}{\alpha - \alpha q}\mathbf{i} + \\
&\quad \frac{\alpha^{n+2} - (\alpha q)^{n+2}}{\alpha - \alpha q}\mathbf{j} + \frac{\alpha^{n+3} - (\alpha q)^{n+3}}{\alpha - \alpha q}\mathbf{k}.
\end{aligned}$$

Since $F_n = \frac{\alpha^n - (\alpha q)^n}{\alpha - (\alpha q)}$ for $\alpha = \frac{1+\sqrt{5}}{2}$ and $q = -\frac{1}{\alpha^2}$, we obtain that

$$\begin{aligned}
\mathbf{Q}_n &= \frac{\alpha^n - (\alpha q)^n}{\alpha - (\alpha q)} + \frac{\alpha^{n+1} - (\alpha q)^{n+1}}{\alpha - (\alpha q)}\mathbf{i} + \\
&\quad \frac{\alpha^{n+2} - (\alpha q)^{n+2}}{\alpha - (\alpha q)}\mathbf{j} + \frac{\alpha^{n+3} - (\alpha q)^{n+3}}{\alpha - (\alpha q)}\mathbf{k} \\
\mathbf{Q}_n &= F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k},
\end{aligned}$$

which gives the Fibonacci quaternions Q_n (1.1). *On the other hand, we have*

$$\begin{aligned}
\mathbf{V}_n &= \alpha^n \frac{[2n]_q}{[n]_q} + \alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i} + \\
&\quad \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{j} + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{k} \\
\mathbf{V}_n &= \alpha^n(1+q^n) + \alpha^{n+1}(1+q^{n+1})\mathbf{i} + \\
&\quad \alpha^{n+2}(1+q^{n+2})\mathbf{j} + \alpha^{n+3}(1+q^{n+3})\mathbf{k}.
\end{aligned}$$

Since $L_n = \alpha^n(1+q^n)$, we obtain

$$\mathbf{V}_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k},$$

which gives the Lucas quaternions K_n (1.2) *(see Horadam (1963)). A similar consideration shows that for $\alpha = 1 + \sqrt{2}$ and $q = -\frac{1}{\alpha^2}$, we obtain the usual Pell quaternions QP_n and Pell-Lucas-quaternions QPL_n given in Cimen and Ipek (2016).*

Example 2 *Let $\alpha = 2$ and $q = -\frac{1}{2}$. Then $q-$Fibonacci quaternion $\mathbf{Q}_n$ and $q-$Lucas quaternion $\mathbf{V}_n$ will be in the following forms:*

$$\begin{aligned}
\mathbf{Q}_n &= \alpha^{n-1}[n]_q + \alpha^n[n+1]_q\mathbf{i} + \\
&\quad \alpha^{n+1}[n+2]_q\mathbf{j} + \alpha^{n+2}[n+3]_q\mathbf{k} \\
\mathbf{Q}_n &= \frac{2^n - (-1)^n}{3} + \frac{2^{n+1} - (-1)^{n+1}}{3}\mathbf{i} + \\
&\quad \frac{2^{n+2} - (-1)^{n+2}}{3}\mathbf{j} + \frac{2^{n+3} - (-1)^{n+3}}{3}\mathbf{k},
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_n &= \alpha^n \frac{[2n]_q}{[n]_q} + \alpha^{n+1}\frac{[2n+2]_q}{[n+1]_q}\mathbf{i} + \\
&\quad \alpha^{n+2}\frac{[2n+4]_q}{[n+2]_q}\mathbf{j} + \alpha^{n+3}\frac{[2n+6]_q}{[n+3]_q}\mathbf{k} \\
\mathbf{V}_n &= 2^n + (-1)^n + 2^{n+1} + (-1)^{n+1}\mathbf{i} + \\
&\quad 2^{n+2} + (-1)^{n+2}\mathbf{j} + 2^{n+3} + (-1)^{n+3}\mathbf{k}.
\end{aligned}$$

For $p = 1, q = 2$ the sequences U_n and V_n defined in (1.3) *and* (1.4) *are called the Jacobsthal sequence $\{J_n\}_n$ and Jacobsthal-Lucas sequence $\{j_n\}_n$, respectively. These sequences are given by the formulæ*

$$J_n = \frac{2^n - (-1)^n}{3}$$

and

$$j_n = 2^n + (-1)^n.$$

Hence we have

$$\begin{aligned}
\mathbf{Q}_n &= \frac{2^n - (-1)^n}{3} + \frac{2^{n+1} - (-1)^{n+1}}{3}\mathbf{i} + \\
&\quad \frac{2^{n+2} - (-1)^{n+2}}{3}\mathbf{j} + \frac{2^{n+3} - (-1)^{n+3}}{3}\mathbf{k} \\
\mathbf{Q}_n &= J_n + J_{n+1}\mathbf{i} + J_{n+2}\mathbf{j} + J_{n+3}\mathbf{k},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{V}_n &= 2^n + (-1)^n + 2^{n+1} + (-1)^{n+1}\mathbf{i} + \\
&\quad 2^{n+2} + (-1)^{n+2}\mathbf{j} + 2^{n+3} + (-1)^{n+3}\mathbf{k} \\
\mathbf{V}_n &= j_n + j_{n+1}\mathbf{i} + j_{n+2}\mathbf{j} + j_{n+3}\mathbf{k},
\end{aligned}$$

which are Jacobsthal quaternions JQ_n and Jacobsthal-Lucas quaternions JLQ_n, respectively. For more details about these quaternions, see Szynal-Liana and Włoch (2016).

Simplification. By taking $\alpha\sqrt{q} = \mathbf{i}$ in Example 1, we can rewrite Q_n and K_n after some calculations and simplifications as

$$Q_n = \frac{q^{-\frac{n+1}{2}}\mathbf{i}^{n-1}}{1-q}[qa_n - a_{n+2}\mathbf{j}],$$

where

$$a_n = 1 - q^n - q^{-1/2}(1 - q^{n+1})$$

and

$$K_n = q^{-n/2-1}\mathbf{i}^n[qb_n - b_{n+2}\mathbf{j}],$$

where

$$b_n = 1 + q^n - q^{-1/2}(1 + q^{n+1}).$$

Thus these quaternions are equal to one of the following forms, according to reduction of the integer n modulo 4 :

$$Q_n = \frac{1}{1-q}\begin{cases} q^{-2k-1/2}[-qa_{4k}\mathbf{i} + a_{4k+2}\mathbf{k}] & , \quad n = 4k \\ q^{-2k-1}[qa_{4k+1} - a_{4k+3}\mathbf{j}] & , \quad n = 4k+1 \\ q^{-2k-3/2}[qa_{4k+2}\mathbf{i} - a_{4k+4}\mathbf{k}] & , \quad n = 4k+2 \\ q^{-2k-2}[-qa_{4k+3} + a_{4k+5}\mathbf{j}] & , \quad n = 4k+3 \end{cases}$$

and

$$K_n = \begin{cases} q^{-2k-1}[qb_{4k} - b_{4k+2}\mathbf{j}] & , \quad n = 4k \\ q^{-2k-3/2}[qb_{4k+1}\mathbf{i} - b_{4k+3}\mathbf{k}] & , \quad n = 4k+1 \\ q^{-2k-2}[-qb_{4k+2} + b_{4k+4}\mathbf{j}] & , \quad n = 4k+2 \\ q^{-2k-5/2}[-qb_{4k+3}\mathbf{i} + b_{4k+5}\mathbf{k}] & , \quad n = 4k+3 \end{cases}.$$

Linearization. Let $\alpha^2 q = -1$. From the Binet formulæ (2.1), for any integers $n \geq 1$ we have

$$\begin{aligned} \mathbf{Q}_n - (\alpha q)\mathbf{Q}_{n-1} &= \frac{\alpha^n\underline{\alpha} - (\alpha q)^n\underline{\gamma}}{\alpha(1-q)} \\ &- (\alpha q)\frac{\alpha^{n-1}\underline{\alpha} - (\alpha q)^{n-1}\underline{\gamma}}{\alpha(1-q)} \\ &= \frac{\alpha^n\underline{\alpha} - (\alpha q)\alpha^{n-1}\underline{\alpha}}{\alpha(1-q)} \\ &= \frac{\alpha^{n-1}(\alpha - (\alpha q))\underline{\alpha}}{\alpha(1-q)} \\ &= \alpha^{n-1}\underline{\alpha} \qquad (2.2) \end{aligned}$$

and

$$\begin{aligned} \mathbf{Q}_n - \alpha\mathbf{Q}_{n-1} &= \frac{\alpha^n\underline{\alpha} - (\alpha q)^n\underline{\gamma}}{\alpha(1-q)} - \\ &\quad \alpha\frac{\alpha^{n-1}\underline{\alpha} - (\alpha q)^{n-1}\underline{\gamma}}{\alpha(1-q)} \\ &= \frac{\alpha(\alpha q)^{n-1}\underline{\gamma} - (\alpha q)^n\underline{\gamma}}{\alpha(1-q)} \\ &= \frac{(\alpha q)^{n-1}(\alpha - (\alpha q))\underline{\gamma}}{\alpha(1-q)} \\ &= (\alpha q)^{n-1}\underline{\gamma}. \qquad (2.3) \end{aligned}$$

Multiplying equation (2.2) by α and equation (2.3) by αq we obtain the linearization of $\{\mathbf{Q}_n\}$

$$\begin{aligned} \alpha^n\underline{\alpha} &= \alpha\mathbf{Q}_n + \mathbf{Q}_{n-1}, \\ (\alpha q)^n\underline{\gamma} &= \alpha q\mathbf{Q}_n + \mathbf{Q}_{n-1}. \end{aligned}$$

Now we give some summation identities, including the quantum quaternions $\mathbf{Q}_n$ and $\mathbf{V}_n$. For brevity, only the identities related to $\mathbf{Q}_n$ and $\mathbf{V}_n$ are discussed.

Theorem 3 *Let $m, k \in \mathbb{N}$. Then,*

(i) $$\sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}\mathbf{Q}_{2n+k} = \begin{cases} \triangle^{m/2}\mathbf{Q}_{m+k} & , \quad m \text{ is even} \\ \triangle^{(m-1)/2}\mathbf{V}_{m+k} & , \quad m \text{ is odd} \end{cases},$$

(ii) $$\sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}\mathbf{V}_{2n+k} = \begin{cases} \triangle^{m/2}\mathbf{V}_{m+k} & , \quad m \text{ is even} \\ \triangle^{(m+1)/2}\mathbf{Q}_{m+k} & , \quad m \text{ is odd} \end{cases},$$

where $\triangle = [\alpha(1-q)]^2$.

Proof. By using the Binet formulæ (2.1), we have

$$\begin{aligned} S &= \sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}\mathbf{Q}_{2n+k} \\ &= \sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}\left(\frac{\alpha^{2n+k}\underline{\alpha} - (\alpha q)^{2n+k}\underline{\gamma}}{\alpha(1-q)}\right). \end{aligned}$$

Observe that

$$\sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}(\alpha^2)^n = \left(\alpha^2 - \alpha^2 q\right)^m$$

and

$$\sum_{n=0}^{m}\binom{m}{n}(-\alpha^2 q)^{m-n}((\alpha q)^2)^n = \left(\alpha^2 q^2 - \alpha^2 q\right)^m .$$

Since $\alpha^2 - \alpha^2 q = \alpha\sqrt{\triangle}$ and $\alpha^2 q^2 - \alpha^2 q = -\alpha q\sqrt{\triangle}$, we obtain

$$S = (\alpha\sqrt{\triangle})^m \frac{\alpha^k \underline{\alpha}}{\alpha(1-q)} - (-\alpha q\sqrt{\triangle})^m \frac{(\alpha q)^k \underline{\gamma}}{\alpha(1-q)}.$$

If m is even, then

$$\begin{aligned} S &= \triangle^{m/2}\left(\frac{\alpha^{m+k}\underline{\alpha} - (\alpha q)^{m+k}\underline{\gamma}}{\alpha(1-q)}\right) \\ &= \triangle^{m/2}\mathbf{Q}_{m+k}. \end{aligned}$$

If m is odd, then

$$\begin{aligned} S &= \triangle^{(m-1)/2}\left(\alpha^{m+k}\underline{\alpha} + (\alpha q)^{m+k}\underline{\gamma}\right) \\ &= \triangle^{(m-1)/2}\mathbf{V}_{m+k}. \end{aligned}$$

■

Theorem 4 *For $m, k \in \mathbb{N}$, we have*

(i) $\sum_{n=0}^{m}\binom{m}{n}(-1)^n(-\alpha^2 q)^{m-n}\mathbf{Q}_{2n+k} = [-\alpha(1+q)]^m\mathbf{Q}_{m+k},$

(ii) $\sum_{n=0}^{m}\binom{m}{n}(-1)^n(-\alpha^2 q)^{m-n}\mathbf{V}_{2n+k} = [-\alpha(1+q)]^m\mathbf{V}_{m+k}.$

Proof. Applying Binet's formulæ (2.1), we obtain

$$\begin{aligned} S &= \sum_{n=0}^{m}\binom{m}{n}(-1)^n(-\alpha^2 q)^{m-n}\mathbf{Q}_{2n+k} \\ &= \sum_{n=0}^{m}\binom{m}{n}(-1)^n(-\alpha^2 q)^{m-n}\left(\frac{\alpha^{2n+k}\underline{\alpha} - (\alpha q)^{2n+k}\underline{\gamma}}{\alpha(1-q)}\right) \\ &= (-\alpha^2 - \alpha^2 q)^m \frac{\alpha^k\underline{\alpha}}{\alpha(1-q)} - \\ &\quad (-(\alpha q)^2 - \alpha^2 q)^m \frac{(\alpha q)^k\underline{\gamma}}{\alpha(1-q)} \\ &= [-\alpha(1+q)]^m\left(\frac{\alpha^{m+k}\underline{\alpha} - (\alpha q)^{m+k}\underline{\gamma}}{\alpha(1-q)}\right) \\ &= [-\alpha(1+q)]^m\mathbf{Q}_{m+k}. \end{aligned}$$

■

Theorem 5 *Let $m \in \mathbb{N}$. Then,*

(i) $\sum_{n=0}^{m}\binom{m}{n}[\alpha(1+q)]^n(-\alpha^2 q)^{m-n}\mathbf{Q}_n = \mathbf{Q}_{2m},$

(ii) $\sum_{n=0}^{m}\binom{m}{n}[\alpha(1+q)]^n(-\alpha^2 q)^{m-n}\mathbf{V}_n = \mathbf{V}_{2m}.$

Proof. Using Binet's formulæ (2.1), we have

$$\begin{aligned} S &= \sum_{n=0}^{m}\binom{m}{n}[\alpha(1+q)]^n(-\alpha^2 q)^{m-n}\mathbf{Q}_n \\ &= \sum_{n=0}^{m}\binom{m}{n}[\alpha(1+q)]^n(-\alpha^2 q)^{m-n}\left(\frac{\underline{\alpha}\alpha^n - \underline{\gamma}(\alpha q)^n}{\alpha(1-q)}\right) \\ &= \left(\sum_{n=0}^{m}\binom{m}{n}[\alpha^2(1+q)]^n(-\alpha^2 q)^{m-n}\right)\frac{\underline{\alpha}}{\alpha(1-q)} \\ &\quad -\left(\sum_{n=0}^{m}\binom{m}{n}[\alpha^2 q(1+q)]^n(-\alpha^2 q)^{m-n}\right)\frac{\underline{\gamma}}{\alpha(1-q)} \\ &= (\alpha^2)^m\frac{\underline{\alpha}}{\alpha(1-q)} - (\alpha^2 q^2)^m\frac{\underline{\gamma}}{\alpha(1-q)} \\ &= \mathbf{Q}_{2m}. \end{aligned}$$

■

3. $q-$Quaternion polynomials

In this section, we derive quantum quaternion polynomials ($q-$quaternion polynomials) $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$. We also derive the Binet formulæ, the generating functions of these type of polynomials. We then obtain some results of $q-$quaternion polynomial sequences. However, first $Q_n(z)$ and $V_n(z)$ must be defined.

Definition 2 *Let $p(z)$ and $q(z)$ be polynomials with complex coefficients. The $q-$polynomials $Q_n(z)$ and $V_n(z)$ are defined by the recurrence relation*

$$\begin{aligned} Q_{n+2}(z) &= p(z)Q_{n+1}(z) - q(z)Q_n(z) \\ V_{n+2}(z) &= p(z)V_{n+1}(z) - q(z)V_n(z) \end{aligned} \tag{3.1}$$

with the initial conditions $Q_0(z) = 0, Q_1(z) = 1$ and $V_0(z) = 2, V_1(z) = p(z)$, respectively.

Example 3 *(i) Constant type:*
Let $p(z) = \alpha(q+1)$ and $q(z) = \alpha^2 q$, we obtain the polynomials

$$Q_{n+2}(z) = \alpha(q+1)Q_{n+1}(z) - \alpha^2 q Q_n(z)$$

and

$$V_{n+2}(z) = \alpha(q+1)V_{n+1}(z) - \alpha^2 q V_n(z).$$

(ii) Nonconstant type:
Now let $p(z) = h(x)$ be a polynomial with real coefficients and $q(z) = -1$. Then we get

$$Q_{n+2}(x) = h(x)Q_{n+1}(x) + Q_n(x)$$

with the initial conditions

$$Q_0(x) = 0, \quad Q_1(x) = 1.$$

We see that this recurrence gives the $h(x)-$Fibonacci polynomials. For $h(x) = k$, k any real number, we obtain the $k-$Fibonacci numbers. In particular, for $k = 1$ and $k = 2$, we have Fibonacci numbers F_n and Pell numbers P_n, respectively.

Let the roots of the characteristic equation

$$w^2 - p(z)w + q(z) = 0$$

of the recurrences (3.1) be

$$\alpha(w) = \frac{p(z) + \sqrt{p(z)^2 - 4q(z)}}{2}$$

and

$$\beta(w) = \frac{p(z) - \sqrt{p(z)^2 - 4q(z)}}{2}.$$

Then the Binet formulæ for $q-$polynomials $Q_n(z)$ and $V_n(z)$ are

$$Q_n(z) = \frac{\alpha(w)^n - \beta(w)^n}{\alpha(w) - \beta(w)}$$

and

$$V_n(z) = \alpha(w)^n + \beta(w)^n.$$

Now we will define two $q-$quaternion sequences with components taken from the sequences defined above.

Definition 3 *The $q-$quaternion polynomials $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$ are defined by the recurrence relation*

$$\begin{aligned}
\mathbf{Q}_n(z) &= Q_n(z) + Q_{n+1}(z)\mathbf{i} + Q_{n+2}(z)\mathbf{j} + Q_{n+3}(z)\mathbf{k},\\
\mathbf{V}_n(z) &= V_n(z) + V_{n+1}(z)\mathbf{i} + V_{n+2}(z)\mathbf{j} + V_{n+3}(z)\mathbf{k}.
\end{aligned}$$

The initial conditions of the $q-$quaternion polynomial sequence $\mathbf{Q}_n(z)$ are

$$\begin{aligned}
\mathbf{Q}_0(z) &= Q_0(z) + Q_1(z)\mathbf{i} + Q_2(z)\mathbf{j} + Q_3(z)\mathbf{k}\\
&= \mathbf{i} + p(z)\mathbf{j} + (p(z)^2 - q(z))\mathbf{k}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Q}_1(z) &= Q_1(z) + Q_2(z)\mathbf{i} + Q_3(z)\mathbf{j} + Q_4(z)\mathbf{k}\\
&= 1 + p(z)\mathbf{i} + (p(z)^2 - q(z))\mathbf{j} +\\
&\quad (p(z)^3 - 2p(z)q(z))\mathbf{k}.
\end{aligned}$$

For the $q-$quaternion polynomial sequence $\mathbf{V}_n(z)$, the initial conditions are

$$\begin{aligned}
\mathbf{V}_0(z) &= V_0(z) + V_1(z)\mathbf{i} + V_2(z)\mathbf{j} + V_3(z)\mathbf{k}\\
&= 2 + p(z)\mathbf{i} + (p(z)^2 - 2q(z))\mathbf{j} +\\
&\quad (p(z)^3 - 3p(z)q(z))\mathbf{k}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{V}_1(z) &= V_1(z) + V_2(z)\mathbf{i} + V_3(z)\mathbf{j} + V_4(z)\mathbf{k}\\
&= p(z) + (p(z)^2 - 2q(z))\mathbf{i} +\\
&\quad (p(z)^3 - 3p(z)q(z))\mathbf{j}\\
&\quad +(p(z)^4 - 4p(z)^2q(z) + 2q(z)^2)\mathbf{k}.
\end{aligned}$$

Example 4 *Let $p(z) = h(x)$ be a polynomial with real coefficients. As we have seen in Example 3, we get the $h(x)-$ Fibonacci polynomials from the $q-$polynomials $Q_n(z)$, and thus we obtain $h(x)-$Fibonacci quaternion polynomials from the $q-$quaternion polynomials $\mathbf{Q}_n(z)$.*

Theorem 6 *The generating functions for the $q-$quaternion polynomials $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$ are*

$$\mathbf{Q}F(t) = \frac{\mathbf{Q}_0(z) + [\mathbf{Q}_1(z) - p(z)\mathbf{Q}_0(z)]t}{1 - p(z)t + q(z)t^2}$$

and

$$\mathbf{V}F(t) = \frac{\mathbf{V}_0(z) + [\mathbf{V}_1(z) - p(z)\mathbf{V}_0(z)]t}{1 - p(z)t + q(z)t^2},$$

respectively.

Proof. The form of the generating function $\mathbf{Q}F(t)$ for the $q-$quaternion polynomial $\mathbf{Q}_n(z)$ is $\sum_{n=0}^{\infty} \mathbf{Q}_n(z)t^n$. Then the power series expansion of $-p(z)t$ and $q(z)t^2$ will be $\sum_{n=0}^{\infty} -p(z)\mathbf{Q}_n(z)t^{n+1}$ and $\sum_{n=0}^{\infty} q(z)\mathbf{Q}_n(z)t^{n+2}$, respectively. Thus we obtain that

$$\begin{aligned}&(1 - p(z)t + q(z)t^2)\mathbf{Q}F(t)\\ &= \mathbf{Q}_0(z) + [\mathbf{Q}_1(z) - p(z)\mathbf{Q}_0(z)]t,\end{aligned}$$

and so

$$\mathbf{Q}F(t) = \frac{\mathbf{Q}_0(z) + [\mathbf{Q}_1(z) - p(z)\mathbf{Q}_0(z)]t}{1 - p(z)t + q(z)t^2}.$$

Similarly, the generating functions for the $q-$quaternion polynomial $\mathbf{V}_n(z)$ is

$$\mathbf{V}F(t) = \frac{\mathbf{V}_0(z) + [\mathbf{V}_1(z) - p(z)\mathbf{V}_0(z)]t}{1 - p(z)t + q(z)t^2}.$$

■

We can get also the Binet formulæ for these quaternion polynomial sequences.

Theorem 7 *The Binet formulæ of the $q-$quaternion polynomials $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$ are*

$$\mathbf{Q}_n(z) = \frac{\alpha(w)^n \underline{\alpha(w)} - \beta(w)^n \underline{\beta(w)}}{\alpha(w) - \beta(w)} \qquad (3.2)$$

and

$$\mathbf{V}_n(z) = \alpha(w)^n \underline{\alpha(w)} + \beta(w)^n \underline{\beta(w)}, \qquad (3.3)$$

where

$$\begin{aligned}\underline{\alpha(w)} &= 1 + \alpha(w)\mathbf{i} + \alpha(w)^2\mathbf{j} + \alpha(w)^3\mathbf{k},\\ \underline{\beta(w)} &= 1 + \beta(w)\mathbf{i} + \beta(w)^2\mathbf{j} + \beta(w)^3\mathbf{k}.\end{aligned}$$

The following relations can be obtained after some calculations

$$\begin{aligned}\mathbf{Q}_1(z) - \alpha(w)\mathbf{Q}_0(z) &= \underline{\beta(w)},\\ \mathbf{Q}_1(z) - \beta(w)\mathbf{Q}_0(z) &= \underline{\alpha(w)},\\ \mathbf{V}_1(z) - \alpha(w)\mathbf{V}_0(z) &= (\beta(w) - \alpha(w))\, \underline{\beta(w)},\\ \mathbf{V}_1(z) - \beta(w)\mathbf{V}_0(z) &= (\alpha(w) - \beta(w))\, \underline{\alpha(w)}.\end{aligned}$$

Now we give some summation formulæ, including the quaternions $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$.

Theorem 8 *For $\mathbf{Q}_n(z)$ and $\mathbf{V}_n(z)$, $n \geq 0$, we have the following summation formulæ*

$$(i)\ \sum_{n=0}^{m} \binom{m}{n}(-q(z))^{m-n}p(z)^n\mathbf{Q}_n(z) = \mathbf{Q}_{2n}(z),$$

$$(ii)\ \sum_{n=0}^{m} \binom{m}{n}(-q(z))^{m-n}p(z)^n\mathbf{V}_n(z) = \mathbf{V}_{2n}(z).$$

Proof. For (i), applying Binet's formulæ (3.2) we get

$$\begin{aligned}&\sum_{n=0}^{m}\binom{m}{n}(-q(z))^{m-n}p(z)^n\mathbf{Q}_n(z)\\ =\ &\sum_{n=0}^{m}\binom{m}{n}(-q(z))^{m-n}p(z)^n\frac{\alpha(w)^n\underline{\alpha(w)} - \beta(w)^n\underline{\beta(w)}}{\alpha(w) - \beta(w)}\\ =\ &\left(\sum_{n=0}^{m}\binom{m}{n}(-q(z))^{m-n}p(z)^n\alpha(w)^n\right)\frac{\underline{\alpha(w)}}{\alpha(w) - \beta(w)}\\ &-\left(\sum_{n=0}^{m}\binom{m}{n}(-q(z))^{m-n}p(z)^n\beta(w)^n\right)\frac{\underline{\beta(w)}}{\alpha(w) - \beta(w)}\\ =\ &(-q(z) + p(z)\alpha(w))^m\frac{\underline{\alpha(w)}}{\alpha(w) - \beta(w)}\\ &-(-q(z) + p(z)\beta(w))^m\frac{\underline{\beta(w)}}{\alpha(w) - \beta(w)}\\ =\ &\frac{\alpha(w)^{2m}\underline{\alpha(w)} - \beta(w)^{2m}\underline{\beta(w)}}{\alpha(w) - \beta(w)}\\ =\ &\mathbf{Q}_{2n}(z).\end{aligned}$$

Using Binet formulæ (3.3), (ii) can be reached in the same way. ■

4. Applications of the Fibonacci quaternions

In this section we give some applications of time evolution and rotation, including Hamilton's quaternions with components from some special integer sequences.

4.1 Time evolution

Quaternion differentiation's formula connects the time derivative of the component of quaternion $\mathbf{q}(t)$ with the component of the angular velocity $\mathbf{w}(t)$. For a more in depth discussion, see Rotella (2014). We can write the angular velocity $\mathbf{w}(t)$

$$\begin{aligned}\mathbf{w}(t) &= w_x(t)\mathbf{i} + w_y(t)\mathbf{j} + w_z(t)\mathbf{k} \\ &= (0, w_x(t), w_y(t), w_z(t))\end{aligned}$$

as a quaternion with zero scalar part. Then the derivative of unit quaternion $\mathbf{q}(t)$ will be

$$\frac{d\mathbf{q}(t)}{dt} = \frac{1}{2}\mathbf{w}(t)\mathbf{q}(t). \tag{4.1}$$

We will now derive a relation between the velocity vector

$$\mathbf{w}(t) = (0, \sin\theta t, \sin\theta t, \sin\theta t)$$

and the quaternion

$$\mathbf{q}_1(t) = (tF_n, tF_{n+1}, tF_{n+2}, tF_{n+3})$$

time derivative. By (4.1), we have

$$\begin{aligned}\frac{d\mathbf{q}_1(t)}{dt} &= \frac{1}{\sqrt{N(\mathbf{q}_1(t))}}\begin{pmatrix} -tF_{n+3}\sin\theta t \\ \frac{t}{2}F_{n+2}\sin\theta t \\ -\frac{t}{2}F_{n+1}\sin\theta t \\ tF_n\sin\theta t\end{pmatrix}^T \\ &= \frac{1}{\sqrt{N(Q_n)}}\begin{pmatrix} -F_{n+3}\sin\theta t \\ \frac{1}{2}F_{n+2}\sin\theta t \\ -\frac{1}{2}F_{n+1}\sin\theta t \\ F_n\sin\theta t\end{pmatrix}^T,\end{aligned}$$

where

$$Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}$$

is the n^{th} Fibonacci quaternion. For

$$\mathbf{w}(t) = (0, \sin\theta t, \sin\theta t, \sin\theta t)$$

and the quaternion

$$\mathbf{q_2}(t) = (tF_n, F_{n+1}, F_{n+2}, F_{n+3}),$$

we have

$$\begin{aligned}&\frac{d\mathbf{q}_2(t)}{dt} \\ &= \frac{1}{\sqrt{N(\mathbf{q}_2(t))}}\begin{pmatrix} -F_{n+3}\sin\theta t \\ \frac{1}{2}(F_{n+1}\sin\theta t + tF_n\sin\theta t) \\ \frac{1}{2}(F_{n+2}\sin\theta t + tF_n\sin\theta t) \\ \frac{1}{2}(F_n\sin\theta t + tF_n\sin\theta t)\end{pmatrix}^T.\end{aligned}$$

4.2 Rotation

Quaternions with zero real parts are used to represent vectors in $\mathbb{R}^3$. So a vector $\mathbf{v}$ is represented by $v_0 = (0, \mathbf{v})$. For a unit quaternion q, consider the transformation

$$L_q(\mathbf{v}) = q\mathbf{v}q^*.$$

It can be easily seen that the operator $L_q(\mathbf{v})$ is linear over $\mathbb{R}^3$. For any vector $\mathbf{v}$, the action of this operator on $\mathbf{v}$ is equivalent to a rotation of the vector through an angle θ about $\widehat{\mathbf{u}}$ as the axis of rotation. We can see that q is preserved by the rotation and hence is along the axis of rotation $\widehat{\mathbf{u}}$. After some calculations q can be written as

$$q = \cos\frac{\theta}{2} + \widehat{\mathbf{u}}\sin\frac{\theta}{2}.$$

Rotating a vector $\mathbf{v}$ about the axis $\widehat{\mathbf{u}}$ through the angle θ we obtain that

$$L_q(\mathbf{v}) = (\cos\theta)\mathbf{v} + (1-\cos\theta)(\widehat{\mathbf{u}}\cdot\mathbf{v})\widehat{\mathbf{u}} + \sin\theta(\widehat{\mathbf{u}}\times\mathbf{v}). \tag{4.2}$$

Note that $(-q)$ represents the same rotation and composition of rotations that correspond to the multiplication of quaternions.

1. For a real number r and the generalized Fibonacci sequence $W_n(a, b; p, q)$, let $q = (r, r, r, r)$ and

$$Q_n = W_n + W_{n+1}\mathbf{i} + W_{n+2}\mathbf{j} + W_{n+3}\mathbf{k}.$$

Then

$$qQ_nq^{-1} = (W_n, W_{n+2}, W_{n+1}, W_{n+3}).$$

2. For the Fibonacci sequence $\{F_n\}_{n>0}$, let $Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}$. We want to find the resulting quaternion after rotation, that is

$$Q_kQ_nQ_k^{-1}$$

for all nonnegative integers n and k. Let $Q_kQ_nQ_k^{-1} = (a, b, c, d)$. Then it can be seen that

$$\begin{aligned} a &= F_n, \\ b &= \frac{1}{k!3F_{2k+3}}\frac{d^k f(n, x)}{dx}|_{x=0} \end{aligned}$$

for

$$f(n, x) = \frac{U_{n,1} + V_{n,1}x + T_{n,1}x^2}{1 - 2x - 2x^2 + x^3}$$

with Fibonacci sequences $U_{n,1}$, $V_{n,1}$ and $T_{n,1}$, where

$$\begin{aligned} U_{n,1} &= 6F_{n+2}, \\ V_{n,1} &= W_n(-5, -9; 1, 1), \\ T_{n,1} &= -F_{n+1}, \end{aligned}$$

$$c = \frac{1}{k!3F_{2k+3}}\frac{d^k g(n, x)}{dx}|_{x=0}$$

for

$$g(n, x) = \frac{U_{n,2} + V_{n,2}x + T_{n,2}x^2}{1 - 2x - 2x^2 + x^3}$$

with Fibonacci sequences $U_{n,2}$, $V_{n,2}$ and $T_{n,2}$, where

$$\begin{aligned} U_{n,2} &= 6F_{n+1}, \\ V_{n,2} &= W_n(13, 18; 1, 1), \\ T_{n,2} &= W_n(-7, -10; 1, 1), \end{aligned}$$

and

$$d = \frac{1}{k!3F_{2k+3}}\frac{d^k h(n, x)}{dx}|_{x=0}$$

for

$$h(n, x) = \frac{U_{n,3} + V_{n,3}x + T_{n,3}x^2}{1 - 2x - 2x^2 + x^3}$$

with Fibonacci sequences $U_{n,3}$, $V_{n,3}$ and $T_{n,3}$, where

$$\begin{aligned} U_{n,3} &= 6F_{n+3}, \\ V_{n,3} &= W_n(2, 9; 1, 1), \\ T_{n,3} &= W_n(-2, -5; 1, 1). \end{aligned}$$

3. Let $\mathbf{v} = (F_{4k+2}, F_{4k+3}, F_{4k+4})$, where k is a nonnegative integer. We consider a rotation about an axis defined by $(1, 1, 1)$ through an angle of π. So it will also hold for $\pi + 2n\pi$. We define a unit vector $\widehat{\mathbf{u}} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Then by (4.2), we have $L_q(\mathbf{v})$

$$= (F_{4k+4} - \frac{1}{3}F_{4k}, F_{4k+2} + \frac{1}{3}F_{4k+4}, \frac{1}{3}F_{4k+4})$$
$$= \frac{1}{3}(3F_{4k+4} - F_{4k}, 3F_{4k+2} + F_{4k+4}, F_{4k+4}).$$

For $\theta = \frac{\pi}{2}$ and $\mathbf{v} = (F_{4k+2}, F_{4k+3}, F_{4k+4})$, a rotation about an axis defined by $(1, 1, 1)$ through θ will be
$L_q(\mathbf{v})$

$$= \frac{2}{3}F_{4k+4}(1, 1, 1) + \frac{1}{\sqrt{3}}(F_{4k+2}, -F_{4k+3}, F_{4k+1}).$$

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الأرباع: نهج حساب التفاضل والتكامل الكمومي مع التطبيقات

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الملخص

في هذا البحث، نقدم نوعين من المتواليات الرباعية مع عناصر تشمل أعداد كمية. ونقدم كذلك كثيرات الحدود الرباعية. علاوة على ذلك، نعرض بعض الخصائص والهويات لهذه الأرباع الكمومية ومتعددة الحدود. وأخيراً، نقدم تطبيقات لتطور وتناوب الوقت لبعض المتواليات الرباعية المحددة. يمكن تحويل التطبيقات إلى نماذج عددية في ظروف مناسبة مع اعتبارات مماثلة.