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On central boolean rings and boolean type fuzzy ideals

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Abstract

In this paper, we introduce the concepts of central Boolean rings and near-rings. We obtain conditions under which central Boolean near-rings are commutative. We study derivations in central Boolean rings and near-rings. Finally, we introduce Boolean type fuzzy ideals of left, right and central Boolean rings and near-rings.

Keywords: Boolean near-ring; Boolean ring; fuzzy ideal; near-ring; Ring.

Mathematics Subject Classification (2010) :16Y30

1. Introduction

An algebraic system N along with binary operations addition and mutiplication that satisfies the axioms of a ring, with an exception of commutativity of addition and one of the distributive laws is a near-ring. In this paper, N denotes a right near-ring.

A ring R that satisfies the condition $x^2 = x$ for all $x \in R$ is a Boolean ring. A Boolean near-ring is a near-ring where all the elements are idempotent. (Reddy, 2017) studied recent developments in Boolean near-rings.
(Kedukodi *et al.*, 2009) studied equiprime, 3-prime and c-prime fuzzy ideals of near-rings. (Jagadeesha *et al.*, 2016) studied homomorphic images of interval-valued L-fuzzy ideals and proved isomorphism theorems. (Nayak *et al.*, 2018) introduced left and right Boolean rings and near-rings and discussed derivation of them. (Ma & Zhan, 2014) studied concepts of fuzzy soft Γ-hemirings.

For more information on recent developments in near-rings and Boolean near-rings research, see (Kedukodi *et al.*, 2017), (Bhavanari *et al.*, 2010), (Nayak *et al.*, 2018), (Koppula *et al.*, 2018), (Davvaz & Sadrabadi, 2014) and (Zulfiqar & Shabir, 2015). These authors provide basic definitions. (Bhavanari & Kuncham, 2013), (Pilz, 1996), (Jagadeesha *et al.*, 2016) and (Jagadeesha *et al.*, 2016).

The paper is divided into three sections. In Section 3, we introduce central Boolean near-rings with motivating examples, and we obtain interrelations with left (resp. right) Boolean near-rings. In Section 4, we study derivations on central Boolean rings and near-rings. In Section 5, we introduce Boolean type fuzzy ideals of central Boolean rings and near-rings.

2. Preliminaries

Definition 2.1 (Nayak *et al.*, 2018) Let N be a near-ring. N is called a *left (resp. right) Boolean near-ring* if there exists $n \in N$ such that $x^2 = nx$ (resp. $x^2 = xn$) for all $x \in N$. If N is a ring satisfying $x^2 = nx$ (resp. $x^2 = xn$) for all $x \in N$, then N is called a *left (resp. right) Boolean ring.*

For computations in near-rings, we use the GAP package SONATA (Aichinger *et al.*, 2012).

Example 2.1 (Nayak *et al.*, 2018)
Let $M = \{\begin{bmatrix} x & (0,0) \\ (0,0) & x \end{bmatrix} | x \in N\}$, where $(N, +) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Take $(0,0) = e, (0,1) = a, (1,0) = b$ and $(1,1) = c$. Let $\cdot$ be defined on N as follows

$$x \cdot y = \begin{cases} a & \text{if } \quad x \in \{b, c\} \text{ and } y = c \\ e & \quad \text{otherwise.} \end{cases}$$

Then M is a left Boolean near-ring with $n = \begin{bmatrix} c & e \\ e & c \end{bmatrix}$. Note that M is neither a right Boolean near-ring nor a commutative near-ring.

Definition 2.3 (Plasser, 1974) An ideal I of N is said to be *IFP ideal* if for every $a, b \in N$, $ab \in I$ implies $anb \in I$ for all $n \in N$. N is called an *IFP near-ring* if for $a, b \in N$, $ab = 0$ implies $anb = 0$ for all $n \in N$. N is said to have *strong IFP* if all ideals N are IFP ideals.

Theorem 2.4 (Kedukodi *et al.*, 2009) Let μ be a fuzzy ideal of N. Then μ is an equiprime (3-prime and c-prime, respectively) fuzzy ideal of N if and only if for every $t \in (\alpha, \beta]$, the level subset μ_t is an equiprime (3-prime and c-prime, respectively) ideal of N.

Corollary 2.5 (Kamal & Al-Shaalan, 2012) Let R be a 3-prime near-ring with a non-zero semigroup right (left) ideal U and a non-zero semigroup ideal V. If R admits a non-zero derivation d such that $d(vu) = d(uv)$ for all $v \in V$, $u \in U$, then R is a commutative ring.

Corollary 2.6 (Kamal & Al-Shaalan, 2012) Let R be a 3-prime near-ring with a non-zero derivation d and a non-zero semigroup ideal U.
(i) If $d(uy) = -d(yu)$ for all $u \in U$, $y \in A$, where A is a subgroup of R, then $A \subseteq Z(R)$ and R is of characteristic 2 or $d(A) = \{0\}$.
(ii) If $d(vu) = -d(uv)$ for all $v \in V$, $u \in U$, where V is a non-zero semigroup left ideal of R, then R is a commutative ring.
(iii) If $d(vu) = -d(uv)$ for all $v \in V$, $u \in U$, where V is a non-zero right R-subgroup of R, then R is a commutative ring of characteristic two.

3. Central Boolean rings and near-rings

Definition 3.1 Let N be a near-ring. N is called a *central Boolean near-ring* if there exist $n, m \in N$, such that $x^2 = nxm$ for all $x \in R$. If N is a ring satisfying $x^2 = nxm$ for all $x \in N$, then N is called a *central Boolean ring.*

We give an example of a central Boolean ring which is not a Boolean ring.

Example 3.2 Consider $(R, +) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\cdot$ be defined on N as follows

$$x \cdot y = \begin{cases} (1,0) & \text{if } x \in \{(1,0),(1,1)\} \\ & \quad \text{and} \\ & \quad y \in \{(1,0),(1,1)\}. \\ (0,0) & \quad \text{otherwise.} \end{cases}$$

Let $n = (1,0)$ and $m = (1,1)$. Then R is a central Boolean ring.
We have $(0,1)^2 = 0 \neq (0,1)$. Hence, R is not a Boolean ring.

Example 3.3 Let R be a central Boolean ring. In R^4, define addition componentwise and multiplication by

$(x_1, y_1, z_1, w_1)(x_2, y_2, z_2, w_2)$
$= (x_1x_2, 0, 0, y_1z_2 - y_2z_1)$. Then R^4 is a central Boolean ring.

Example 3.4 Let $(N, +) \cong D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$. For the purpose of completeness, we explicitly mention $+$ and $\cdot$ operations in the following tables, wherein we take $e = 0, r = 3, r^2 = 5, r^3 = 6, s = 1, rs = 7, r^2s = 4, r^3s = 2$.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	6	0	4	3	7	1	5
3	3	7	1	5	2	6	0	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	2	4	0	7	3	5	1
7	7	3	5	1	6	2	4	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	5	5	0	0	5	5
2	0	5	0	5	5	0	5	0
3	0	5	5	0	5	0	0	5
4	0	0	5	5	0	0	5	5
5	0	0	0	0	0	0	0	0
6	0	5	5	0	5	0	0	5
7	0	5	0	5	5	0	5	0

In this example, we see that N is both a left and right Boolean near-ring for $n = 5$, and N is a central Boolean near-ring for all $n, m \in N$.

Remark 3.5 (i) Every zero square ring is a central Boolean ring for all $n, m \in R$.
(ii) Near-ring M defined in Example 2.1 is a left Boolean near-ring and not a central Boolean near-ring.
(iii) A left(resp. right) Boolean near-ring with right(resp. left) identity is a central Boolean near-ring.

Now, we give an example of a central Boolean near-ring which is not a right Boolean near-ring.

Example 3.6 Let $(N, +) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $\cdot$ be defined as follows:

.	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, 1)	(0, 0)	(0, 0)	(0, 0)	(0, 1)
(1, 0)	(0, 0)	(0, 0)	(1, 0)	(1, 0)
(1, 1)	(0, 0)	(0, 0)	(1, 0)	(1, 1)

Then N is a central Boolean near-ring with $n = (1, 1) = m$. N is also a left Boolean near-ring with $n = (1, 1)$. However, N is not a right Boolean near-ring.

Proposition 3.7 Let R be a central Boolean ring. If n, m are not zero divisors then R is commutative.
Proof. We have $(x + y)^2 = n(x + y)m = nxm+nym$. Then $(x+y)(x+y) = nxm+nym \Rightarrow x(x + y) + y(x + y) = nxm + nym \Rightarrow x^2 + xy + yx + y^2 = nxm + nym \Rightarrow nxm + xy+yx+nym = nxm+nym \Rightarrow xy+yx = 0 \Rightarrow xy = -yx$....(1). Also, $(x+x)^2 = n(x+x)m = nxm + nxm \Rightarrow x^2 + x^2 + x^2 + x^2 = nxm + nxm \Rightarrow n(x + x)m = 0 \Rightarrow x = -x$....(2). By (1) and (2), we get $xy = yx$.

Proposition 3.8 Let R be a central Boolean ring with $|R| \geq 3$. If $0 \neq n, 0 \neq m$ are not zero divisors, then R has a proper zero divisor.
Proof. Let $x, y \in R$ such that $0 \neq x \neq y \neq 0$. Note that $x + y \neq 0$. If $xy = 0$, then x is a proper zero divisor. Let $xy \neq 0$. $(xy)(x + y) = xyx + xyy = yx^2 + xy^2$(because R is commutative when n and m are not zero-divisors) $= ynxm + xnym = xnym + xnym = 0$. Thus xy is a proper zero-divisor.

Proposition 3.9 Let $f : R_1 \to R_2$ be an onto ring homomorphism. If R_1 is a central Boolean ring then R_2 is a central Boolean ring.

Proof. The proof is obvious.

Proposition 3.10 Let $R \neq 0$ be a domain with no zero divisors and a central Boolean ring. Then $nm = 1$.
Proof. The proof is obvious.

Theorem 3.11 Let N be a central Boolean near-ring and P be a c-prime ideal of N. If n and m are not zero divisors then P is maximal.
Proof. We have, $x^2 = nxm \Rightarrow x^2 - nxm = 0 \in P \Rightarrow x(x - nxm) \in P \Rightarrow x \in P$ or $x - nxm \in P \Rightarrow x \in P$ or $x \in nxm + P \Rightarrow N/P = \{P, nxm + P\} \Rightarrow N/P$ is a field. Hence P is maximal.

Corollary 3.12 Let N be a central Boolean near-ring and P be an equiprime ideal that has IFP. If n and m are not zero divisors then P is maximal.
Proof. Proof follows from Theorem 3.21 of (Kedukodi *et al.*, 2009) and Theorem 3.11.

Definition 3.13 Let I be an ideal of N. I is called a *central Boolean type* if there exist $n, m \in N$ such that $x^2 - nxm \in I$ for all $x \in N$.

Proposition 3.14 Let I be an ideal of N. Then N/I is a central Boolean near-ring if and only if I is a central Boolean type.
Proof. We have $x^2 - nxm \in I \Leftrightarrow x^2 + I = nxm + I \Leftrightarrow (x + I)^2 = (n + I)(x + I)(m + I)$. Hence, N/I is a central Boolean near-ring if and only if I is a central Boolean type.

Remark 3.15 If $I = \{0\}$ is a central Boolean type, then N is a central Boolean near-ring.

Theorem 3.16 Let R be a central Boolean near-ring. If n is a distributive element and has a left identity e, then R is a zero symmetric near-ring. Furthermore, if there exists a left ideal I such that $e \in I$ and (i) $nxm \in I \Rightarrow x \in I$; (ii) $[X, Y] \cap I = \{0\}$, then R is a commutative ring.

Proof. Since e is left distributive, the equation $(e + e)^2 = n(e + e)m = nem = nem \Rightarrow (e + e)(e+e) = nem+nem \Rightarrow e(e+e)+e(e+e) = nem+nem \Rightarrow e^2+e^2+e^2+e^2 = nem+nem \Rightarrow nem+nem = 0 \Rightarrow n(e+e)m = 0 \Rightarrow e+e = 0$. If x is in R, then $x + x = (e + e)x = 0.x = 0 \Rightarrow x + x = 0$.
Let w be an arbitrary element in R. Then $(e + w)^2 = n(e + w)m = nem + nwm \Rightarrow (e + w)(e + w) = nem + nwm \Rightarrow e(e + w) + w(e+w) = nem+nwm \Rightarrow e^2+w+w(e+w) = nem + nwm \Rightarrow w(e + w) = -w + nwm = w + w^2 = (e + w)w$.

$$w(e + w) - (e + w)w = 0 - - - (1)$$

Now, $w(e+w)0 = (e+w)w0 \Rightarrow w(e0+w0) = ew0 + ww0 \Rightarrow ww0 = ew0 + ww0 \Rightarrow w0 = 0$. Thus R is a zero symmetric near-ring.
Replacing $w = ab$ and $w = ba$ in equation (1), we get $(e+ab)ab = ab(e+ab) \Rightarrow ab+(ab)^2 = ab(e + ab)$. Now we have

$$ab = ab(e + ab) - nabm.$$

Similarly, $ba = ba(e + ba) - nbam.$

Hence $ab - ba = ab(e + ab) - nabm - [ba(e + ba) - nbam] \in I$.
We have $ab - ba \in [X, Y] = \{xy - yx | x \in X, y \in Y\}$ and $ab - ba \in I \Rightarrow ab - ba = 0 \Rightarrow ab = ba$.

Corollary 3.17 Let R be a central Boolean near-ring. Let n be a distributive non zero divisor of R, and let I be a left ideal such that $[X, Y] \cap I = \{0\}$. Then $(R, \leq)$ is a partially ordered set with $\leq$ defined by $x \leq y$ if $xy = nxm$. In addition, if $nxm \in I \Rightarrow x \in I$ and R has a left identity e such that $e \in I$, then $(R, \leq)$ is a lattice with meet and join operations given respectively by $x \wedge y = xy$ and $x \vee y = x + y + xy$.
Proof. It is straightforward to verify that $(R, \leq)$ is a partially ordered set. The rest of the proof follows from Theorem 3.16.

Definition 3.18 Let N be a central Boolean near-ring and I be an ideal of N. N is said to satisfy a *weak commutative property* with respect to ideal I, if for all $a, b, c \in N$, $abc - acb \in I$.

Lemma 3.19 Let N be central Boolean near-ring. If there exists an ideal I in N such that (i) $nxm \in I \Rightarrow x \in I$ and (ii) $xIx \subseteq I$ for all $x \in N$, then $ab - aba \in I$ for all $a, b \in I$.
Proof. We have $(ab - aba)^2 = n(ab - aba)m$. Now $(ab - aba)^2 = (ab - aba)(ab - aba) = ab(ab - aba) - aba(ab - aba) = ab(ab - i_1) - i_2$ [where $i_1 = aba$ and $i_2 = aba(ab - aba)$] $= i_3 + abab - i_4 \in I \Rightarrow n(ab - aba)m \in I \Rightarrow ab - aba \in I$.

Theorem 3.20 Let N be a central Boolean near-ring. If there exists an ideal I in N such that (i) $nxm \in I \Rightarrow x \in I$ and (ii) $xIx \subseteq I$ for all $x \in N$, then N satisfies the weak commutative property with respect to I.
Proof. We have $abc - acb = abc - a(cbc + i_1)$[because $cb - cbc \in I \Rightarrow cb = cbc + i_1$] $= abc - i_2 - acbc = abc - acbc - i_2 = (a - ac)bc - i_2 = ((a - ac)b(a - ac) + i_3)c - i_2 \in I$.

Corollary 3.21 Let N be a central Boolean near-ring with ideals I such that (i) $nxm \in I \Rightarrow x \in I$ and (ii) $xIx \subseteq I$ for all $x \in N$. Then N has a strong IFP.
Proof. Let $ab \in I$. By Theorem 3.20, $axb = abx + i \in I$. This implies N has a strong IFP.

Theorem 3.22 Let N be a central Boolean near-ring, and I be an ideal of N such that (i) $nxm \in I \Rightarrow x \in I$; (ii) $xIx \subseteq I$ for all $x \in N$. If L is any left ideal of N containing I, then L is an ideal of N.
Proof. Let L be a left ideal of N. To show that L is an ideal, it suffices to show that $LN \subseteq L$. Let $l \in L, n \in N$. $l = l_0 + l_c$ and $n = n_0 + n_c$ are Pierce decompositions, where $l_0, n_0 \in N_0$; $l_c, n_c \in N_c$-(1). Since L is a left ideal, we have that $N_0L \subseteq L$, for $m_0l' = m_0(0 + l') - m_00 \in L$ for all $m_0 \in N_0, l \in L$-(2). Now, $nlm = l^2 = (l_0 + l_c)l = l_0l + l_cl = l_0l + l_c$. By (2) $l, l_0l \in L$, and it follows $l_c \in L$, and hence, $l_0 \in L$. We have $ln = (l_0 + l_c)n = l_0n + l_cn = l_0n + l_c = l_0(n_0 + n_c) + l_c = l_0(n_0 + n_c)l_0 + i + l_c = l_0(n_0l_0 + n_c) + i + l_c$. We have $l_0n_c = l_0n_c0 = l_00n_c + i = 0 + i \in I$. Since $n_0, l_0 \in L$, we have $l_0(n_0l_0 + n_c) = l_0n_c + i_2 \in I \subseteq L$. Thus, $ln = l_0(n_0l_0 + n_c) + l_c \in L$. Hence, L is an ideal of N.

4. Derivations

Definition 4.1 (Bell, 1987) A *derivation* on N is defined as an additive endomorphism satisfying the product rule

$$D(xy) = xD(y) + D(x)y$$

for all $x, y \in N$.

Theorem 4.2 Let R be a central Boolean ring and D be the derivation on R with $D(nxm) = 0$ for all $x \in R$. Then $D(xy) = -D(yx)$.
Proof. Let R be a central Boolean ring. Then $x^2 = nxm$. We have $D(x^2) = D(nxm) = 0. \Rightarrow xD(x) + D(x)x = 0 \Rightarrow D(x)x = -xD(x)$. Now we have $(x + y)^2 = n(x + y)$. Then $D(x + y)^2 = D(n(x + y)m) = 0 \Rightarrow D((x + y)(x + y)) = 0 \Rightarrow (x + y)D(x + y) + D(x + y)(x + y) = 0 \Rightarrow xD(x) + xD(y) + yD(x) + yD(y) + D(x)x + D(y)x + D(x)y + D(y)y = 0 \Rightarrow xD(y) + yD(x) + D(y)x + D(x)y = 0 \Rightarrow D(xy) = -D(yx)$.

Corollary 4.3 1. If $x + x = 0$, then $D(xy) = D(yx)$.
2. If R is a zero square ring, then $D(xy) = -D(yx)$.

Theorem 4.4 Let R be a central Boolean ring without zero divisors. Let D be the commuting derivation on R. Then either $D(n) = D(m)$ or $D(n + m) = n - m$.
Proof. We have $x^2 = nxm$. Then we have the following: $(D(n))^2 = nD(n)m$ and $D(m)^2 = nD(m)m$. We get $D(n)D(n) - D(m)D(m) = (nD(n) - nD(m))m$
$\Rightarrow (D(n) - D(m))(D(n) + D(m)) = n(D(n) - D(m))m$. Since D is commuting,
$(D(n) - D(m))(D(n) + D(m) - n + m) = 0$.

Hence, $D(n) = D(m)$ or $D(n+m) = n - m$.

Corollary 4.5 Let R be a central Boolean ring without zero divisors. Let D be the comm uting derivation on R. Then,
1. $D(n) = 0$ or $D(n) = nm$.
2. $D(m) = 0$ or $D(m) = nm$.
Proof. The proof is obvious.

Theorem 4.6 Let R be a central Boolean ring without zero divisors. Let $n^2 + m^2 \neq 0$ and D be the derivation. If $D(n) \neq 0$ and $D(m) \neq 0$ then D is one-one.
Proof. We have $x^2 = nxm$. Then $D(x^2) = D(nxm) \Rightarrow xD(x) + D(x)x = nD(xm) + xmD(n) \Rightarrow n(xD(m)+mD(x))+xmD(n) = 0 \Rightarrow nxD(m) + nmD(x) + xmD(n) = 0 \Rightarrow nD(x)m = nxD(m) + mxD(n) \Rightarrow (D(x))^2 = (nD(m) + D(n)m)x = (nnm + nmm)x = (n^2 + m^2)x$. Now if $D(x) = 0 \Rightarrow (D(x))^2 = 0 \Rightarrow (n^2 + m^2)x = 0 \Rightarrow x = 0$. Hence, D is one-one.

Definition 4.7 A near-ring N is said to be 2-torsion if $x + x = 0$ for all $x \in N$.

Theorem 4.8 Let N be a 2 torsion central Boolean near-ring without zero divisors. Let $n^2 + m^2 \neq 0$ and D be the derivation. If $D(n) \neq 0$ and $D(m) \neq 0$, then D is one-one.

Proposition 4.9 Let N be a central Boolean near-ring with a nilpotent element x.
1. If n is not a zero divisor, $n^2 = 0$ and $n = m$, then $x0 = 0$.
2. If N is a near-field and $m = n^{-1}$, then $n0 = 0$.
Proof. 1. We have $(x^k)^2 = 0$. Now $(x^2)^k = (nxn)^k = (nxn)(nxn)...$
$(nxn) = nx0$. As n is not a zero divisor, we get $x0 = 0$.
2. We have $(x^k)^2 = 0$. Also, $(x^2)^k = (nxn^{-1})^k = (nxn^{-1})(nxn^{-1})...$
$(nxn^{-1}) = nx^k n^{-1} = n0n^{-1} = n0$. Hence, $n0 = 0$.

Example 4.10 Let N be a near-ring defined in Example 3.4. Define $D : N \to N$ by $D(x) = 2x$ for all $x \in N$. Then D is a derivation on N. Note that in Example 3.4, $n0 = 0$.

Remark 4.11 In Example 3.4, as $xNy = 0$, we observe that $D(x) = x^2$ is a derivation on N.

Theorem 4.12 Let R be a 3-prime central Boolean ring. If D is the derivation on R with $D(nxm) = 0$ for all $x \in R$, then R is a commutative ring.
Proof. Proof follows from Corollary 4.4 (ii) of (Kamal & Al-Shaalan, 2012) and Theorem 4.2.

Corollary 4.13 Let R be a 3-prime central Boolean ring with $x + x = 0$ for all $x \in R$. If D is the derivation on R such that $D(nxm) = 0$, then R is a commutative ring.
Proof. Proof follows from Corollary 3.2 of (Kamal & Al-Shaalan, 2012) and (1) of Corollary 4.3.

5. Boolean type fuzzy ideals

Definition 5.1 (Davvaz, 2006) Let $\alpha, \beta \in [0,1]$ and $\alpha < \beta$. Let μ be a fuzzy subset of a near-ring N. Then μ is called a *fuzzy ideal* with thresholds of N, if $\forall\ x, y, i \in N$ the following holds:

(1) $\alpha \vee \mu(x+y) \geq \beta \wedge \mu(x) \wedge \mu(y)$,
(2) $\alpha \vee \mu(-x) \geq \beta \wedge \mu(x)$,
(3) $\alpha \vee \mu(y + x - y) \geq \beta \wedge \mu(x)$,
(4) $\alpha \vee \mu(xy) \geq \beta \wedge \mu(x)$,
(5) $\alpha \vee \mu(x(y+i) - xy) \geq \beta \wedge \mu(i)$,

wherein $\wedge$ and $\vee$ denote the usual meet and join operations on the lattice (chain) $L = [0,1]$.

Definition 5.2 A left (resp. right, resp. central) Boolean type ideal I of near-ring N is called a *strong left (resp. right, resp. central) Boolean type ideal* if for all $n \in N$, $x^2 - nx \in I$

implies $x \in I$.

Definition 5.3 Let N be a near-ring and μ be a fuzzy ideal of N. Let α and β be lower and upper thresholds, respectively. Then μ is called a *strong left (resp. right, resp. central) Boolean type* if for $n \in N$, for all $x \in N$, $\alpha, \beta \in [0,1]$ and $\alpha < \beta$,

$$\alpha \vee \mu(x) \geq \beta \wedge \inf_{n \in N} \mu(x^2 - nx)$$

$$resp.\ \alpha \vee \mu(x) \geq \beta \wedge \inf_{n \in N} \mu(x^2 - xn)$$

$$resp.\ \alpha \vee \mu(x) \geq \beta \wedge \inf_{n,m \in N} \mu(x^2 - nxm)$$

Definition 5.4 (Kedukodi *et al.*, 2009) Let μ be a fuzzy subset of near-ring N. For $t \in [0,1]$, the set $\mu_t = \{x \in N | \mu(x) \geq t\}$ is called a level subset of N.

Example 5.5 Consider $\mathbb{Z}_{12} = \{0,1,2,...,11\}$. Let $\alpha = 0.1$ and $\beta = 0.7$. Define a fuzzy subset $\mu : \mathbb{Z}_{12} \rightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if} \quad x = 6 \\ 0.8 & \text{if} \quad x = 0 \\ 0.7 & \text{if} \quad x \in \{3,9\} \\ 0.1 & \quad \text{elsewhere} \end{cases}$$

Then μ is a left as well as right Boolean type fuzzy ideal of $\mathbb{Z}_{12}$ (Take $n = 3$). Also, μ is a central Boolean type fuzzy ideal of $\mathbb{Z}_{12}$ (Take $n = 3$ and $m = 1$).

Theorem 5.6 Let μ be a fuzzy subset of N. Then μ is a strong left (resp. right) Boolean type fuzzy ideal of N if and only if the level subset μ_t is a strong left (resp. right) Boolean type ideal of N for all $t \in (\alpha, \beta]$.

Proof. Let μ be a strong left Boolean type fuzzy ideal of N. Take $t \in (\alpha, \beta], x \in N$ such that $x^2 - nx \in \mu_t$ for all $n \in N$. Then $\mu(x^2 - nx) \geq t$ for all $n \in N$. Hence, $\inf_{n \in N} \mu(x^2 - nx) \geq t$. By definition of a Boolean type fuzzy ideal, we have $\alpha \vee \mu(x) \geq \beta \wedge \inf_{n \in N} \mu(x^2 - nx) \geq \beta \wedge t = t$. Hence, we get $\mu(x) \geq t \Rightarrow x \in \mu_t$. Therefore, μ_t is a strong left Boolean type ideal of N.
Conversely, assume that there exists $x \in N$ such that

$$\alpha \vee \mu(x) < \beta < \inf_{n \in N} \mu(x^2 - nx).$$

Choose t such that

$$\alpha \vee \mu(x) < t < \beta \wedge \inf_{n \in N} \mu(x^2 - nx).$$

This implies $\mu(x) < t$ and $\inf_{n \in N} \mu(x^2 - nx) > t$ $\Rightarrow x \notin \mu_t$ and $x^2 - nx \in \mu_t$ for all $n \in N$, a contradiction to the assumption that μ_t is a strong Boolean type fuzzy ideal of N for every $t \in (\alpha, \beta]$. The proof is similar for a strong right Boolean type fuzzy ideal.

Theorem 5.7 Let μ be a fuzzy subset of N. Then μ is a strong central Boolean type fuzzy ideal of N if and only if level subset μ_t is a strong central Boolean type ideal of N for all $t \in (\alpha, \beta]$.

Proof. Let μ be a strong central Boolean type fuzzy ideal of N. Take $t \in (\alpha, \beta], x \in N$ such that $x^2 - nxm \in \mu_t$ for all $n, m \in N$. Then $\mu(x^2 - nxm) \geq t$ for all $n, m \in N$. Hence $\inf_{n,m \in N} \mu(x^2 - nxm) \geq t$. By definition of Boolean type fuzzy ideal, we have

$$\alpha \vee \mu(x) \geq \beta \wedge \inf_{n,m \in N} \mu(x^2 - nxm)$$

$$\geq \beta \wedge t = t.$$

This implies $\mu(x) \geq t$. Hence, $x \in \mu_t$. Therefore, μ_t is a strong central Boolean type ideal of N.
Conversely, assume that there exists $x \in N$ such that

$$\alpha \vee \mu(x) < \beta < \inf_{n,m \in N} \mu(x^2 - nxm).$$

Choose t such that

$$\alpha \vee \mu(x) < t < \beta \wedge \inf_{n,m \in N} \mu(x^2 - nxm).$$

This implies $\mu(x) < t$ and $\inf_{n,m \in N} \mu(x^2 - nxm) > t \Rightarrow x \notin \mu_t$ and $x^2 - nxm \in \mu_t$ for all $n, m \in N$, a contradiction to the assumption that μ_t is a strong Boolean type fuzzy ideal of N for every $t \in (\alpha, \beta]$.

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حول الحلقات البوليانية المركزية والمثاليات الضبابية من النوع البولياني

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الملخص

في هذا البحث، نقدم مفاهيم عن الحلقات البوليانية (Boolean) المركزية والحلقات المقتربة. وقد حصلنا على الظروف التي تكون فيها الحلقات البوليانية المركزية والحلقات المقتربة تبادلية، كما قمنا بدراسة الاشتقاقات في تلك الحلقات. وأخيراً، قمنا بتقديم مثاليات ضبابية من النوع البولياني للحلقات البوليانية والحلقات المقتربة اليسرى واليمنى والمركزية.