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Limits and colimits in the variety of dynamical systems

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Abstract

In this paper, we prove the class of dynamical system is a variety. Also the variety of dynamical systems can be seen as a category. Then in this area we pay attention to some limits, colimits and their existence or lack of existence in the mentioned category.

Keywords : Congruence relation; colimit; extended morphism; limit; subdynamical system.

1. Introduction

Category as a field in mathematics plays an important role in recognizing the other fields, meaning it can help us to search about objects and relations between them as morphisms.

Dynamical systems can be seen by algebraic and topological instrument. (Nezhad & Davvaz, 2010; Molaei, 2005; Vries, 1993). In this paper, based on this view, we want to see the class of dynamical systems as a categorical window. Now, we pay attention to considering the definition of category in dynamical systems.

Category of dynamical systems is a class of objects, where, there are dynamical systems (denoted A, B, instead of $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$, $\{B, \{\psi^i\}_{i\in\mathbb{Z}}\}$, respectively) together with a class of disjoint sets, denoted $\hom(\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}, \{B, \{\psi^i\}_{i\in\mathbb{Z}}\})$ where $\mathbb{Z}$ is the set of integer numbers. This category is denoted by $\mathcal{C}$. An element of $\hom(A, B)$ is an extended morphism from A to B and is denoted by $f : A \to B$. In this paper, $\hom(A, B)$ is denoted by $ex-Mor(A, B)$. Motivated by the relations between groups in the category of groups and the definition of product and coproduct in groups theory, we try to define product and coproduct in the category of dynamical systems. More than this, we define some limits and colimits in this category. Also their existence or lack of existence is considered.

In section 2, we define extended morphism and congruence in dynamical systems. Then in an important theorem, we see the generated congruence by a relation. Also, we see the class of dynamical systems is a variety. Limits and colimits are two items to continue in the next two sections. In this way, the basis of work is on the existence and lack of existence of limits and colmits.

2. The class of dynamical systems is a variety

Dynamical systems as an applied field in mathematics can be helpful (Ahmad, 1988). Using the properties of dynamical system defined in (Alligood *et al.*, 1996) we see a dynamical system as following: $\{A,\{\varphi^i\}_{i\in\mathbb{Z}}\}$ where A is a nonempty set. Also $\varphi^i : A \to A$ is a mapping such that $\varphi^i o\, \varphi^j = \varphi^{i+j}$ and $\varphi^0 = id_A$. Continuously we introduce a congruence relation, which is generated by a subset of Cartesian product of state space by itself of the dynamical system.

Definition 2.1. Let $\{A,\{\varphi^i\}_{i\in\mathbb{Z}}\}$ be a dynamical system and ρ be an equivalence relation on A. We say ρ is a congruence relation on the above dynamical system, if for all $a, a' \in A$ where $a\ \rho\ a'$, we have $\varphi^i(a)\ \rho\ \varphi^i(a')$ for every $i \in \mathbb{Z}$.

Example 2.2. Let $\varphi^i : \mathbb{R} \to \mathbb{R}$, where $\theta \in \mathbb{R}$ is a constant, such that $x \mapsto xe^{i\theta}$.

Then $\{\mathbb{R},\{\varphi^i\}_{i\in\mathbb{Z}}\}$ is a dynamical system. We define an equivalence relation $\equiv$ on $\mathbb{R}$ as the following

x $\equiv$ y if and only if $x, y \geq 0$ or $x, y < 0$. It is obvious that x $\equiv$ y $\Leftrightarrow$ $\varphi^i(x) \equiv \varphi^i(y)$, for all i. Therefore $\equiv$ is a congruence relation.

It is clear that the intersection of congruence relations is a congruence relation.

Let $\{A,\{\varphi^i\}_{i\in\mathbb{Z}}\}$ be a dynamical system. The smallest congruence relation, which is included $X \subseteq A \times A$ is denoted by $\rho(X)$. We denote the set of congruence relations on A with $con(A)$. A very good question can be this:

what is the generated congruence relation by $X \subseteq A \times A$?

The following theorem can be the answer.

Theorem 2.3. Let $\{A,\{\varphi^i\}_{i\in\mathbb{Z}}\}$ be a dynamical system, $X \subseteq A \times A$ and $\rho = \rho(X)$ be the generated congruence relation by X. Then for every $a, b \in A$, $a\ \rho\ b$ if and only if $a = b$ or there exist a natural number n and a sequence

$$a = \varphi^{t_1}(c_1), \varphi^{t_1}(d_1) = \varphi^{t_2}(c_2), \varphi^{t_2}(d_2) = \varphi^{t_3}(c_3), \ldots, \varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n),\ \varphi^{t_n}(d_n) = b$$

such that $t_1, \ldots, t_n \in \mathbb{Z}$ and $((c_i, d_i) \in X\ or\ (d_i, c_i) \in X)$, for all $i \in \{1, 2, \ldots, n\}$

We show that the above defined congruence relation is the smallest congruence relation, which is included X. It is clear to see ρ is reflexive and symmetric. Also if $a\ \rho\ b$ and $b\ \rho\ c$, then there exist natural numbers n and k and two sequences

$a = \varphi^{t_1}(c_1), \varphi^{t_1}(d_1) = \varphi^{t_2}(c_2), \varphi^{t_2}(d_2) = \varphi^{t_3}(c_3), ..., \varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n), \varphi^{t_n}(d_n) = b,$

and

$\mathrm{b} = \varphi^{t'_1}(c'_1), \varphi^{t'_1}(d'_1) = \varphi^{t'_2}(c'_2), \varphi^{t'_2}(d'_2) = \varphi^{t'_3}(c'_3), ..., \varphi^{t'_{k-1}}(d'_{k-1}) = \varphi^{t'_k}(c'_k), \varphi^{t'_k}(d'_k) = c$

Such that for all $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., k\}$, $t_i, t'_j \in \mathbb{Z}$ and $((c_i, d_i) \in X$ $or\,(d_i, c_i) \in X)$. Also $((c'_j, d'_j) \in X \;\; or \;\; (d'_j, c'_j) \in X)$.

Now by choosing $d_{n+j} := d'_j$, $c_{n+j} := c'_j$ and $t_{n+j} := t'_j$ we have the following sequence

$$a = \varphi^{t_1}(c_1)$$

$$\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2), ..., \varphi^{t_n}(d_n) = \varphi^{t_{n+1}}(c_{n+1}), ..., \varphi^{t_{n+k-1}}(d_{n+k-1}) = \varphi^{t_{n+k}}(c_{n+k}),$$

$$\varphi^{t_{n+k}}(d_{n+k}) = c$$

such that for $i \in \{1, 2, ..., n+k\}$, $t_i \in \mathbb{Z}$ and $((c_i, d_i) \in X \;\; or \;\; (d_i, c_i) \in X)$.

It shows $a\ \rho\ c$. Also if a = b or b = c then $a\ \rho\ c$. Hence ρ is an equivalence relation. It is clear that if $a\ \rho\ b$ then $\varphi^i(a)\ \ \rho\ \ \varphi^i(b)$, for all $i \in \mathbb{Z}$.

Let θ be a congruence relation on A which is included X and $(a, b) \in \rho(X)$.

If a = b then by the reflexivity property $(a, b) \in \theta$. Now let $a = \varphi^{t_1}(c_1)$, $\varphi^{t_1}(d_1) = \varphi^{t_2}(c_2), ..., \varphi^{t_{n-1}}(d_{n-1}) = \varphi^{t_n}(c_n), \varphi^{t_n}(d_n) = b$ where for all $i \in \{1, 2, ..., n\}$, $((c_i, d_i) \in X \;\; or \;\; (d_i, c_i) \in X)$ and $t_i \in \mathbb{Z}$. Hence $(\varphi^{t_1}(c_1), \varphi^{t_2}(d_2)) \in \theta$.

Now let $(\varphi^{t_1}(c_1), \varphi^{t_{i-1}}(d_{i-1})) \in \theta$ for all $2 \le i \le n$. Because of $\varphi^{t_n}(c_n)\theta\varphi^{t_n}(d_n)$ and $\varphi^{t_n}(c_n) = \varphi^{t_{n-1}}(d_{n-1})$ and transitivity's property of θ, $(a, b) \in \theta$.

Example 2.4. We define $\varphi^i : \mathbb{R} \to \mathbb{R}$, such that

$$x \mapsto i + x.$$

Clearly $\{\mathbb{R}, \{\varphi^i\}_{i \in \mathbb{Z}}\}$ is a dynamical system. Let $X = \{(1,2), (1,3), (6,4)\}$.

Since $2 = \varphi^1(1), \varphi^1(2) = \varphi^2(1), \varphi^2(2) = \varphi^0(4), \varphi^0(6) = 6$ we have $(2,6) \in \rho(X)$.

Definition 2.5. A congruence relation ρ is called finitely generated, if there exists a finite subset $X \subseteq A \times A$ such that $\rho = \rho(X)$. ρ is called cyclic, if it is generated by $\{(x, y)\}$ and is denoted by $\rho(x, y)$.

Definition 2.6. Let $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{B, \{\psi^i\}_{i\in\mathbb{Z}}\}$ be dynamical systems.

The mapping $f : A \to B$ is called an extended morphism between two dynamical systems $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{B, \{\psi^i\}_{i\in\mathbb{Z}}\}$ if $f(\varphi^i(a)) = \psi^i(f(a))$ for all $i \in \mathbb{Z}$ and $a \in A$.

If f is one to one or onto, then it is called an extended monomorphism or an extended epimorphism, respectively. If f is one to one and onto, then it is called an extended isomorphism.

Now we can define Kernel of f, which is denoted by Ker$f$ as the following:

$$a \ \ker f \ a' \Leftrightarrow f(a) = f(a')$$

Obviously, Ker$f$ is an equivalence relation.

Example 2.7. We define $\psi : (0,1) \to (0,1)$ by $\psi(x) = 1 - x$. It is obvious that $\{(0,1), \{\psi^i\}_{i\in\mathbb{Z}}\}$ is a dynamical system, where, $\psi^i = \psi o \psi o \ldots o \psi$, i times (let $\psi^0 = id$).

Also $\{\mathbb{R}, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ is a dynamical system, where $\varphi^i : \mathbb{R} \to \mathbb{R}$ is defined by

$$\varphi^i(x) = i + x.$$

We define $f : R \to (0,1)$ such that $f(x) = \frac{1}{2}$. Then f is an extended morphism between two dynamical systems $\{\mathbb{R}, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{(0,1), \{\psi^i\}_{i\in\mathbb{Z}}\}$.

Remark : If $\{A, \{\varphi^i : A \to A\}_{i\in\mathbb{Z}}\}$ is a dynamical system and ρ is a congruence relation on A, then $\{A/\rho, \{\psi^i : A/\rho \to A/\rho\}_{i\in\mathbb{Z}}\}$ is a dynamical system, where $A/\rho = \{[a]_\rho, a \in A\}$ and $\psi^i([a]_\rho) = [\varphi^i(a)]_\rho$ for all $[a]_\rho \in A/\rho$.

Motivated by Homomorphism theorem in groups theory in Hungerford, 1974, we can consider Morphism theorem as an important theorem in the following:

Theorem 2.8. (Morphism theorem in dynamical systems) Let $f : A \to B$ be an extended morphism between dynamical systems $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{B, \{\psi^i\}_{i\in\mathbb{Z}}\}$ and ρ a congruence relation such that $\rho \subseteq \ker f$. Then there is an unique extended morphism $f' : A/\rho \to B$ such that $f'\gamma = f$, where $\gamma : A \to A/\rho$ is a canonical extended morphism. If $\rho = \ker f$ then f' is one to one. If f is onto then, f' is onto too.

Proof. We define

$$f': A/\rho \to B$$

$$[x]_\rho \mapsto f(x).$$

Then f' is an extended morphism between dynamical systems such that $f'\gamma = f$. If $g: A/\rho \to B$ is an extended morphism between dynamical systems such that $g\gamma = f$ then

$$g([a]_\rho) = g(\gamma(a)) = f(a) = f'\gamma(a) = f'([a]_\rho), \ for \ all \ a \in A.$$

Remarks:

(1) The composition of extended morphisms between dynamical systems is an extended morphism.

(2) The image of a dynamical system under an extended morphism is a dynamical system. Meaning, if $f: A \to B$ is an extended morphism between dynamical systems $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{B, \{\psi^i\}_{i\in\mathbb{Z}}\}$ then, $\{f(A), \{\psi^i|_{f(A)}\}_{i\in\mathbb{Z}}\}$ is a dynamical system.

Definition 2.9. Let $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ be a dynamical system and $A' \subseteq A$. Then $\{A', \{\varphi^i|_{A'}\}_{i\in\mathbb{Z}}\}$ is called a subdynamical system of the dynamical system $\{A, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ if $\varphi^i(a) \in A'$ for all i and for all $a \in A'$.

Example 2.10. $\{R^+, \{\varphi^i|_{R^+}\}_{i\in\mathbb{Z}}\}$ is a subdynamical system of dynamical system in example 2.2.

If a dynamical system does not have any subdynamical system, it is called simple.

Example 2.11. If $A = \{x\}$ then a dynamical system $\{A, \{\varphi^i|_A\}_{i\in\mathbb{Z}}\}$ is simple.

We define product of dynamical systems as the following:

Let $\{A_j, \{\varphi_j^i\}_{i\in\mathbb{Z}}\}_{j\in J}$ be a family of dynamical systems. Then $\{\prod_{j\in J} A_j, \{(\prod_{j\in J}\varphi_j)^i\}_{i\in\mathbb{Z}}\}$ is a dynamical system, where

$$(\prod_{j\in J}\varphi_j)^i : \prod_{j\in J} A_j \to \prod_{j\in J} A_j$$

$$\{a_j\}_{j\in J} \mapsto \{\varphi_j^i(a_j)\}_{j\in J}.$$

In the next section, we see more about product of dynamical systems.

Theorem 2.12. The class of dynamical systems is a variety, meaning it is closed with respect to Cartesian product, the image of a dynamical system under an extended morphism is a dynamical system and also any subdynamical system of a dynamical system is a dynamical system too.

Using the definition of limit and colimit (Awodey, 2006), in the next two sections we consider some limits and colimits in the category of dynamical systems like Terminal objects, Product, Equalizer, Pullback, Initial objects, Coproduct and Coequalizer. Then we pay attention to the existence or lack of existence of them in this category.

3. Limit in dynamical systems

3.1 Terminal objects

Definition 3.1.1. An object $A \in \mathcal{C}$ is called a terminal object if $\left| ex - Mor(C, A) \right| = 1$ for all $C \in \mathcal{C}$.

Theorem 3.1.2. Every dynamical system with singleton state space is a terminal object in the category of dynamical systems.

3.2 Product

Definition 3.2.1. Let $\{ A_i \mid i \in I \}$ be a family of objects of $\mathcal{C}$. A product for the family $\{ A_i \mid i \in I \}$ is an object P of $\mathcal{C}$ together with a family of extended morphisms $\{ p_i : P \to A_i \mid i \in I \}$ such that for any object B and family of extended morphisms $\{ \varphi_i : B \to A_i \mid i \in I \}$ there is an unique extended morphism $\varphi : B \to P$ such that $p_i \, o \, \varphi = \varphi_i$ for all $i \in I$.

Theorem 3.2.2. (The existence of product of dynamical systems)

Let $\{A_i, \{\varphi_i^j\}_{j \in \mathbb{Z}}\}_{i \in I}$ be a family of dynamical systems. Then,

$\{\prod_{i \in I} A_i, \{(\prod_{i \in I} \varphi_i)^j\}_{j \in \mathbb{Z}}\}$ is the product of the above family, where $\prod_{i \in I} A_i =$ $\{ (x_i)_{i \in I} \mid x_i \in A_i \}$ and

$$(\prod_{i \in I} \varphi_i)^j : \prod_{i \in I} A_i \to \prod_{i \in I} A_i$$

$$(x_i)_{i \in I} \mapsto \{\varphi_i^j (x_i)\}_{i \in I}$$

Proof. It is clear $\{\prod_{i \in I} A_i, \{(\prod_{i \in I} \varphi_i)^j\}_{j \in \mathbb{Z}}\}$ is a dynamical system. For every dynamical system $\{ Q, \{\tau^i\}_{i \in \mathbb{Z}}\}$ and every family $\{ q_i : Q \to A_i \}_{i \in I}$ of extended morphisms, we define

$$q : Q \to \prod_{i \in I} A_i$$

$$v \mapsto (q_i(v))_{i \in I} \, .$$

q is an unique extended morphism such that $p_i q = q_i$, for all $i \in I$, where $p_i : \prod_{i \in I} A_i \to A_i$ is i th projection.

3.3 Equalizer

Definition 3.3.1. Consider an equalizer situation for two extended morphisms $f_1, f_2 : X \to Y$ in $\mathcal{C}$. Let $\{E, \{\tau^i\}_{i \in \mathbb{Z}}\}$ be a dynamical system and $e : E \to X$ be an extended morphism in $\mathcal{C}$. A pair (E, e) is called an equalizer of f_1 and f_2 in $\mathcal{C}$ and is denoted by $Eq(f_1, f_2)$ if

(1) $f_1 e = f_2 e$, and

(2) the following universal property is fulfilled in $\mathcal{C}$:

For every extended morphism $h : H \to X$ with $f_1 h = f_2 h$ there exists an unique extended morphism $h^- : H \to E$ such that $h = eh^-$.

Theorem 3.3.2. By notations of Definition 3.3.1, Let $f_1, f_2 : X \to Y$ be extended morphisms. Then there exists $Eq(f_1, f_2)$, if and only if $E_s = \{x \in X \mid f_1(x) = f_2(x)\} \neq \varnothing$.

Proof. Let $(M, m) = Eq(f_1, f_2)$. Since every state space of dynamical systems is nonempty, we have $M \neq \varnothing$. Then for $x \in M$ we have $f_1(m(x)) = f_2(m(x))$. Conversely, let $E_s \neq \varnothing$. Then $(E_s, id) = Eq(f_1, f_2)$. Condition 1 of Definition 3.3.1 is straightforward. Also if there exists (H, h) such that $f_1 h = f_2 h$ then, $h(H) \subseteq E_s$. Hence it is enough $h^- := h$.

3.4 Pullback

Definition 3.4.1. Consider the pullback situation of two extended morphisms $f_i : X_i \to Y$ in $\mathcal{C}, i = 1,2$. The pair $(P, (p_1, p_2))$ with two extended morphisms $p_i : P \to X_i, i = 1,2$, in $\mathcal{C}$ is called a pullback of the pair (f_1, f_2) if

(1) $f_1 p_1 = f_2 p_2$, and

(2) For any pair $(P', (p'_1, p'_2))$ with two extended morphisms $p'_i : P' \to X_i, i =$

$1,2,$ and $f_1p_1' = f_2p_2'$ there exists exactly one extended morphism $p : P' \to P$ such that $p_i p = p_i'$, $i = 1,2$.

Theorem 3.4.2. Let $\{X_1, \{\varphi^j\}_{j\in\mathbb{Z}}\}$, $\{X_2, \{\psi^j\}_{j\in\mathbb{Z}}\}$ and $\{Y, \{\rho^i\}_{i\in\mathbb{Z}}\}$ be dynamical systems and $f_i : X_i \to Y$, $i = 1,2$ be extended morphisms in the category $\mathcal{C}$. Then the pullback $(P_s, (p_1, p_2))$ of (f_1, f_2) exists in $\mathcal{C}$ if and only if $P_s = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\} \neq \varnothing$.

where $p_i : P_s \to X_i$, $i = 1,2$ are i th projection.

Proof. Let $(M, (m_1, m_2))$ be the pullback of (f_1, f_2) in the category $\mathcal{C}$. Since $M \neq \varnothing$, then $(m_1(x), m_2(x)) \in P_s$, for all $x \in M$.

Conversely, let $P_s \neq \varnothing$. Consider $\varphi^i : X_1 \to X_1$ and $\psi^i : X_2 \to X_2$ for all $i \in \mathbb{Z}$. Now by considering $(\varphi\times\psi)^i\big|_{P_s} := (\varphi^i \times \psi^i)\big|_{P_s} : P_s \to P_s$, $for\ all\ \ i \in \mathbb{Z}$ and $p_i : P_s \to X_i$, $i = 1,2$ be i th projections which are extended morphisms.

For all $(x_1, x_2) \in P_s$ since f_1 and f_2 are extended morphisms we have

$$f_1(\varphi^i(x_1)) = \rho^i(f_1(x_1)) = \rho^i(f_2(x_2)) = f_2(\psi^i(x_2)).$$

This means $(\varphi^i(x_1), \psi^i(x_2)) \in P_s$, for all $i \in \mathbb{Z}$. Now it is clear $\{P_s, \{(\varphi\times\psi)^i\big|_{P_s}\}_{i\in\mathbb{Z}}\}$ is a subdynamical system of $\{X_1 \times X_2, \{(\varphi\times\psi)^i\}_{i\in\mathbb{Z}}\}$.

Also the condition (1) of Definition 3.4.1 is satisfied. If there exists $(P', (p_1', p_2'))$ such that $f_1p_1' = f_2p_2'$, then for all $z \in P'$, $(p_1'(z), p_2'(z)) \in P_s$. We can define

$$p : P' \to P_s$$

$$z \mapsto (p_1'(z), p_2'(z))$$

where $p_i p = p_i'$.

Then $(P_s, (p_1, p_2))$ is the pullback of (f_1, f_2) in $\mathcal{C}$.

4. Colimit in dynamical systems

4.1 Coproduct

Definition 4.1.1. $\{X_j, \{\varphi_j^i\}_{i\in\mathbb{Z}}\}_{j\in I}$ be a family of dynamical systems in $\mathcal{C}$. A pair $(C, (u_i)_{i\in I})$ is called a coproduct of $(X_i)_{i\in I}$ in $\mathcal{C}$, if

(1) $C \in \mathcal{C}$ and $u_i \in ex-Mor_{\mathcal{C}}(X_i, C)$ for every $i \in I$ and

(2) $(C, (u_i)_{i \in I})$ is universal in the following:

For every K $\in \mathcal{C}$ and for every family $(k_i \in ex-Mor_{\mathcal{C}}(X_i, K))_{i \in I}$ there exists a unique $k \in ex-Mor_{\mathcal{C}}(C, K)$ such that $ku_i = k_i$ for all $i \in I$.

Theorem 4.1.2. (Existence of Coproduct in $\mathcal{C}$)

Let $\{A_j, \{\varphi_j^i\}_{i \in \mathbb{Z}}\}_{j \in I}$ be a family of dynamical systems and $\bigcup_{j \in I} A_j$ is disjoint union of the family of $\{A_j\}_{j \in I}$. $\{\bigcup_{j \in I} A_j, \{(\bigcup \varphi_j)^i\}_{i \in \mathbb{Z}}\}$ is a dynamical system, where $(\bigcup \varphi_j)^i : \bigcup_{j \in I} A_j \to \bigcup_{j \in I} A_j$ such that

$$x_k \to \varphi_k^i(x_k),$$

for $i \in \mathbb{Z}$, where $x_k \in A_k$. We define

$$u_k : A_k \to \bigcup_{j \in I} A_j$$

as an inclusion. $(\bigcup_{j \in I} A_j, (u_j)_{j \in I})$ is coproduct of $\{A_j, \{\varphi_j^i\}_{i \in \mathbb{Z}}\}_{j \in J}$.

4.2 Initial object

Definition 4.2.1. An object A $\in \mathcal{C}$ is called an initial object if

$|ex-Mor(A, D)| = 1$, for all D $\in \mathcal{C}$.

Theorem 4.2.2. There is not any initial object in the category $\mathcal{C}$.

Proof. Let $\{\{\theta\}, \{\varphi_i\}_{i \in \mathbb{Z}}\}$ and $\{\{\theta'\}, \{\psi_i\}_{i \in \mathbb{Z}}\}$ be dynamical systems, where $\theta \neq \theta'$. We consider $\{\{\theta, \theta'\}, \{(\varphi \bigcup \psi)^i\}_{i \in \mathbb{Z}}\}$ is a dynamical system, where $\bigcup$ is disjoint union. Now for all A $\in \mathcal{C}$ there exist two extended morphism $f_1, f_2 : A \to \{\theta, \theta'\}$ such that $f_1 \neq f_2$.

4.3 Coequalizer

Definition 4.3.1. Consider two extended morphisms $f_1, f_2 : X \to Y$ in $\mathcal{C}$. A pair (c, C) with an extended morphism $c : Y \to C$ in $\mathcal{C}$ is called a coequalizer of f_1 and f_2 in $\mathcal{C}$ if

(1) $cf_1 = cf_2$, and

(2) the following universal property is fulfilled in $\mathcal{C}$:

For every extended morphism $k : Y \to K$ with $kf_1 = kf_2$ there exists an unique extended morphism $k^- : C \to K$ such that $k = k^- c$.

Theorem 4.3.2. Let $\{X, \{\varphi^i\}_{i\in\mathbb{Z}}\}$ and $\{Y, \{\psi^i\}_{i\in\mathbb{Z}}\}$ be dynamical systems and $f_1, f_2 : X \to Y$ be extended morphisms. Also θ be the generated congruence by $\{(f_1(x), f_2(x)) \mid x \in X\}$. Then $(\prod, Y/\theta)$ is coequlizer of (f_1, f_2) in the category of dynamical systems, $\mathcal{C}$, where $\prod : Y \to Y/\theta$ is natural extended morphism.

Proof. Since $(f_1(x), f_2(x)) \in \theta$, we have $[f_1(x)]_\theta = [f_2(x)]_\theta$, for all $x \in X$. Then the condition 1 of Definition 3.3.1 is satisfied. Also, if there exists a dynamical system $\{K, \{\rho^i\}_{i\in\mathbb{Z}}\}$ and extended morphism $k : Y \to K$ where $kf_1 = kf_2$. We define extended morphism $\overline{k} : Y/\theta \to K$ by $\overline{k}([y]_\theta) = k(y)$. $\overline{k}$ is well-defined, because by Theorem 2.3 if

$$[y_1]_\theta = [y_2]_\theta \Rightarrow y_1 \theta y_2 \Rightarrow y_1 = y_2 \ or$$

$$y_1 = \psi^{t_1}(f_1(x_1)), \psi^{t_1}(f_2(x_1)) = \psi^{t_2}(f_1(x_2)), \psi^{t_2}(f_2(x_2)) = \psi^{t_3}(f_1(x_3)), \ldots,$$

$\psi^{t_{n-1}}(f_2(x_{n-1})) = \psi^{t_n}(f_1(x_n)), \psi^{t_n}(f_2(x_n)) = y_2$, for some $x_i \in X$, $i = 1, 2, \ldots, n$.

Then

$$k(y_1) = k(\psi^{t_1}(f_1(x_1))), \ldots, k(\psi^{t_n}(f_2(x_n))) = k(y_2) \ .$$

Hence

$$k(y_1) = \rho^{t_1}(kf_1(x_1)), \rho^{t_1}(kf_2(x_1)) = \rho^{t_2}(kf_1(x_2)), \ldots, \rho^{t_{n-1}}(kf_2(x_{n-1})) = \rho^{t_n}$$
$$(kf_1(x_n)), \ \rho^{t_n}(kf_2(x_n)) = k(y_2).$$

Since $kf_1 = kf_2$ we have $k(y_1) = k(y_2)$. Also $\overline{k}$ is unique, meaning if there exists $f : Y/\theta \to K$ such that $f \prod = k$ then $f = \overline{k}$.

5. Conclusion and future research

In this paper, based on some properties of dynamical systems, we paid attention to the definition of some limits and colimits in a variety of dynamical systems. After that, we considered their existence or maybe lack of existence.

In future research, we might consider this category of dynamical systems of other view.

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النهايات و النهايات المشاركة في متنوعة الأنظمة الديناميكية

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خلاصة

ندرس في هذا البحث صنفاً من الأنظمة الديناميكية في متنوعة. و يمكن النظر إلى متنوعة الأنظمة الديناميكية كطائفة. ونركز اهتمامنا في هذا المقام، على بعض النهايات و النهايات المشاركة من حيث وجودها أو عدم وجودها في الطائفة المذكورة.