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Slant null scrolls in Minkowski 3-space E_1^3

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Abstract

In this study, we introduce a new type of null scrolls called slant null scrolls in Minkowski 3-space E_1^3. We define three types of slant null scrolls called α-slant, β-slant and γ-slant null scrolls and give characterizations for a null scroll to be a slant null scroll. Moreover, for β-slant null scrolls, we obtain some nonlinear differential equations and we give some results by using their analytical solutions. Finally, we give some examples for the obtained results.

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Keywords: B -scroll; null scroll; slant null scroll.

1. Introduction and Preliminaries

In the space, a continuously moving of a straight line generates a surface, which is called a ruled surface. These surfaces are one of the most important topics of surface theory. Because of this position of ruled surfaces, many mathematicians have studied these surfaces in different spaces (Ekici & Özüsağlam, 2012; Karadağ *et al.,* 2014; Karger & Novak, 1978; Liu, 2009a; Liu, 2009b). Especially, the frames of ruled surfaces are used widely in the study of theory of such surfaces. A ruled surface S in the Euclidean 3-space E^3 is introduced by the parametrization

$$\vec{r}(s,v)=\vec{c}(s)+v\,\vec{q}(s),\quad \left\|\vec{q}(s)\right\|=1\,,$$

where $\vec{c}(s)$ is striction curve of the surface. The frame of the surface along $\vec{c}(s)$ is given by $\left\{\vec{q},\vec{h},\vec{a}\right\}$ where $\vec{q}$, $\vec{h}$, $\vec{a}$ are called ruling, central normal vector and central tangent vector, respectively, and called Frenet frame of the surface (Karger & Novak, 1978). The classifications and characterizations of ruled surfaces are studied by considering this frame and its invariants. Önder (2013) has used this frame to define new types of ruled surfaces. He has defined these surfaces as slant ruled surfaces. A surface S is called a q-slant (h-slant or a-slant, respectively) ruled surface, if the ruling vector $\vec{q}$ (the vector $\vec{h}$ or the vector $\vec{a}$, respectively) makes a constant angle with a fixed non-zero direction $\vec{u}$ in the space (Önder, 2013). Later, differential equation characterizations for slant ruled surfaces have been studied by Önder & Kaya (2013). They have also defined a new type of slant ruled surfaces as Darboux slant ruled surfaces and given characterizations for this type of slant ruled surfaces (Önder & Kaya, 2015).

Moreover, in the Minkowski 3-space, a ruled surface with a lightlike ruling has more different and more interesting properties than the other ruled surfaces generated by a timelike or a spacelike ruling. Such a surface S given by the parametrization $\vec{x}(u,v)=\vec{a}(u)+v\vec{b}(u)$ in Minkowski 3-space E_1^3 with inner product

$$\left\langle \vec{a},\vec{b}\right\rangle = a_1b_1 + a_2b_2 - a_3b_3,$$

and vector product

$$\vec{a}\times\vec{b} = \left(-\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right),$$

of two vectors $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)\in E_1^3$ is called a null-scroll if $\vec{a}(u)$ and $\vec{b}(u)$ satisfy $\langle \vec{a}',\vec{a}'\rangle = \langle \vec{b},\vec{b}\rangle = 0$ and $\langle \vec{a}',\vec{b}\rangle = 1$ (Liu, 2009b). On the other hand, for a vector $\vec{a}\in E_1^3$, if $\langle \vec{a},\vec{a}\rangle > 0$ or $\vec{a}=0$ ($\langle \vec{a},\vec{a}\rangle < 0$ or $\langle \vec{a},\vec{a}\rangle = 0$, $\vec{a}\neq 0$ respectively), then the vector $\vec{a}$ is called a spacelike vector (a timelike vector or a null vector, respectively).

Null scrolls are an interesting and important type of ruled surfaces with lightlike ruling, because of a ruled surface with lightlike ruling can be written as a null scroll (Liu, 2009b). In this definition, substituting $\vec{\alpha}=\vec{a}'$, $\vec{\gamma}=\vec{b}$ and $\vec{\beta}=\vec{\gamma}\times\vec{\alpha}$, we have

$$\langle \vec{\alpha},\vec{\alpha}\rangle = \langle \vec{\gamma},\vec{\gamma}\rangle = \langle \vec{\alpha},\vec{\beta}\rangle = \langle \vec{\gamma},\vec{\beta}\rangle = 0\,,\ \langle \vec{\alpha},\vec{\gamma}\rangle = 1\,.$$

Assuming that $\vec{\alpha}'(u)=\lambda(u)\vec{\beta}(u)$ and $\vec{\gamma}'(u)=\mu(u)\vec{\beta}(u)$ it follows,

$$\begin{cases} \vec{\alpha}'(u) = \lambda(u)\vec{\beta}(u), \\ \vec{\beta}'(u) = -\mu(u)\vec{\alpha}(u) - \lambda(u)\vec{\gamma}(u), \\ \vec{\gamma}'(u) = \mu(u)\vec{\beta}(u), \end{cases} \tag{1}$$

where $\lambda=\lambda(u)$, $\mu=\mu(u)$ are curvatures of the null scroll and are smooth functions (Liu, 2009b). Especially, if $\lambda\neq 0$ and $\mu=$ constant, then the surface is called a B-scroll, which is introduced by Graves (1979). Since B-scrolls are related to cone curves (Liu, 2004), these surfaces form a much larger class of ruled surfaces in Minkowski

3-space and appear in many classification results of some special ruled surfaces (Alias *et al.,* 1994; Alias *et al.,* 1998; Ekici & Özüsağlam, 2012; Ferrandez & Lucas, 1992; Ferrandez & Lucas, 2000; Kim & Kim, 2003; Kim *et al.,* 2002).

In this paper, we study slant null scrolls. We define three types of slant null scrolls called α-slant, β-slant and γ-slant null scrolls and give characterizations for a null scroll to be a slant null scroll.

2. γ-Slant null scrolls

In this section, we define γ-slant null scrolls and give characterizations.

Definition 1. Let the surface S given by the parameterization

$$\vec{x}(u,v) = \vec{a}(u) + v\vec{b}(u) \text{ ,}$$

be a null scroll with frame $\left\{\vec{\alpha},\vec{\beta},\vec{\gamma}\right\}$ and $\vec{d}$ be a non-zero fixed line in Minkowski 3-space E_1^3. If the condition

$$\left\langle \vec{\gamma},\vec{d} \right\rangle = c = \text{constant} \neq 0,$$

is satisfied for the vectors $\vec{\gamma}$ and $\vec{d}$, then the surface S is called a γ-slant null scroll. Here, the vector $\vec{d}$ is called the axis of γ-slant null scroll.

Then we can give the following characterizations for γ-slant null scrolls:

Theorem 1. If the null scroll S is a γ-slant null scroll with $\mu \neq 0$, then

$$\left\langle \vec{\alpha}(u),\vec{d} \right\rangle = -c\frac{\lambda(u)}{\mu(u)},$$

is satisfied for a non-zero fixed line $\vec{d}$ where $c = \left\langle \vec{\gamma},\vec{d} \right\rangle = \text{constant} \neq 0$.

Proof. Since S is a γ-slant null scroll in E_1^3, from Definition 1, we have

$$\left\langle \vec{\gamma},\vec{d} \right\rangle = \text{constant}.$$

By differentiating last equation with respect to u and using the fact that $\vec{d} = \text{constant}$, from (1) we have

$$\mu(u)\left\langle \vec{\beta}(u),\vec{d} \right\rangle = 0.$$

Since $\mu \neq 0$, we get

$$\left\langle \vec{\beta}(u),\vec{d} \right\rangle = 0 .$$

Differentiating last equality gives

$$\left\langle \vec{\gamma}(u), \vec{d} \right\rangle = -\frac{\mu(u)}{\lambda(u)} \left\langle \vec{\alpha}(u), \vec{d} \right\rangle.$$

From Definition 1, we have the desired equality.

Theorem 2. Let S be a γ-slant null scroll with $\mu \neq 0$ and axis $\vec{d}$. Then, the axis $\vec{d}$ is given by

$$\begin{cases} \vec{d} = c\left(\vec{\alpha} - \dfrac{\lambda}{\mu}\vec{\gamma}\right), & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,} \\ \vec{d} = c\vec{\alpha},\ \lambda = 0, & \text{if } \vec{d} \text{ is a null vector,} \end{cases}$$

where $c = \left\langle \vec{\gamma}, \vec{d} \right\rangle = \text{constant} \neq 0$.

Proof. Let S be a γ-slant null scroll with axis $\vec{d}$ and write the vector $\vec{d}$ as follows

$$\vec{d} = x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma}, \tag{2}$$

where $x = x(u),\ y = y(u),\ z = z(u)$ are smooth functions. Taking the inner product of $\vec{d}$ and $\vec{\beta}$, and considering that $\left\langle \vec{\beta}(u), \vec{d} \right\rangle = 0$ in Theorem 1, it is easy to see that $y = 0$. Thus, (2) is reduced to

$$\vec{d} = x\vec{\alpha} + z\vec{\gamma}. \tag{3}$$

Analogue to the previous process, by taking inner product of $\vec{d}$ with $\vec{\alpha}$ and $\vec{\gamma}$, respectively, we get

$$\begin{cases} z = \left\langle \vec{\alpha}, \vec{d} \right\rangle, \\ x = \left\langle \vec{\gamma}, \vec{d} \right\rangle = c. \end{cases} \tag{4}$$

From Theorem 1, we have

$$\left\langle \vec{\gamma}(u), \vec{d} \right\rangle = -\frac{\mu(u)}{\lambda(u)} \left\langle \vec{\alpha}(u), \vec{d} \right\rangle.$$

Then, Equation (3) becomes

$$\vec{d} = c\left(\vec{\alpha} - \frac{\lambda}{\mu}\vec{\gamma}\right). \tag{5}$$

Let now investigate the Lorentzian casual character of $\vec{d}$. From (5) we get

$$\left\langle \vec{d}, \vec{d} \right\rangle = -2c\frac{\lambda}{\mu}. \tag{6}$$

Then we have that $\vec{d}$ is a spacelike or a timelike vector, if $\lambda \neq 0$; and $\vec{d}$ is a null vector, if $\lambda = 0$, i.e., the surface is a degenerate surface, which finishes the proof.

If $\vec{d}$ is a null vector, from the definition of null scrolls we get $\vec{\alpha}' = 0$. Since $\vec{a}' = \vec{\alpha}$, we have that $\vec{a}'$ is a constant vector. Then, we have the following corollary:

Corollary 1. The axis of a γ-slant null scroll with $\mu \neq 0$ is a null line if and only if the base curve is a null line and also it is the axis of the surface.

Corollary 2. Null scroll S is a γ-slant null scroll with $\mu \neq 0$ if and only if the curvatures satisfy the followings:

$$\begin{cases} \dfrac{\lambda}{\mu} = \text{constant} \neq 0, & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,} \\ \lambda = 0, & \text{if } \vec{d} \text{ is a null vector.} \end{cases}$$

Proof. Let S be a γ-slant null scroll. From Theorem 2, if $\vec{d}$ is a spacelike or a timelike vector, we have

$$\vec{d} = c\left(\vec{\alpha} - \frac{\lambda}{\mu}\vec{\gamma}\right).$$

By differentiating the last equation and considering that $\vec{d}$ is constant, it follows

$$\left(\frac{\lambda}{\mu}\right)' \vec{\gamma} = 0,$$

which means that $\dfrac{\lambda}{\mu} = \text{constant}$. If $\vec{d}$ is null, from Theorem 2, it is clear that $\lambda = 0$.

Conversely, in the case of spacelike or timelike, let $\dfrac{\lambda}{\mu}$ be a constant and $\lambda \neq 0$. We define

$$\vec{d} = c\left(\vec{\alpha} - \frac{\lambda}{\mu}\vec{\gamma}\right), \tag{7}$$

with respect to the signs of $\dfrac{\lambda}{\mu}$ and c, $\vec{d}$ can be a spacelike or a timelike vector. Taking the inner product of (7) with $\vec{\gamma}$ gives $\left\langle \vec{\gamma}, \vec{d} \right\rangle = c = \text{constant}$ and from the derivative of (7) it is obtained that $\vec{d}$ is constant. Therefore, from the Definition 1, S is a γ-slant null scroll.

If $\vec{d}$ is a null vector and $\lambda = 0$, then, we define

$$\vec{d} = c\vec{\alpha}.$$

Then it is clear that $\langle \vec{\gamma}, \vec{d} \rangle = c =$ constant and $\vec{d}$ is a constant vector since $\vec{d}' = 0$. Therefore, S is a γ-slant null scroll.

Corollary 3. S is a γ-slant null scroll if and only if $\det(\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''') = 0$.

Proof. From the equalities given in (1), we have $\vec{\gamma}' = \mu\vec{\beta}$ and differentiating this equality twice, we obtain

$$\begin{cases} \vec{\gamma}'' = -\mu^2\vec{\alpha} + \mu'\vec{\beta} - \lambda\mu\vec{\gamma}, \\ \vec{\gamma}''' = -3\mu\mu'\vec{\alpha} + \left(\mu'' - 2\lambda\mu^2\right)\vec{\beta} + \left(-\lambda'\mu - 2\lambda\mu'\right)\vec{\gamma}, \end{cases}$$

respectively. Then, it follows

$$\det(\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''') = -\mu^5\left(\frac{\lambda}{\mu}\right)'.$$

So, we have that $\det(\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''') = 0$ holds if and only if $\frac{\lambda}{\mu}$ is constant or $\lambda = 0$. Considering Corollary 2, it follows that S is a γ-slant null scroll if and only if $\det(\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''') = 0$ holds.

Corollary 4. S is a γ-slant null scroll with $\mu \neq 0$ if and only if $\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = 0$ holds.

Proof. From (1), we get $\vec{\alpha}' = \lambda\vec{\beta}$ and differentiating it twice, it follows,

$$\begin{cases} \vec{\alpha}'' = -\lambda\mu\vec{\alpha} + \lambda'\vec{\beta} - \lambda^2\vec{\gamma}, \\ \vec{\alpha}''' = -\left(2\lambda'\mu + \lambda\mu'\right)\vec{\alpha} + \left(\lambda'' - 2\lambda^2\mu\right)\vec{\beta} - 3\lambda\lambda'\vec{\gamma}. \end{cases}$$

respectively, and we get

$$\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = -\lambda^3\mu^2\left(\frac{\lambda}{\mu}\right)'.$$

Then we have that $\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = 0$ if and only if $\frac{\lambda}{\mu}$ is constant or $\lambda = 0$. Considering Corollary 2, it follows that S is a γ-slant null scroll if and only if $\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = 0$.

3. $\boldsymbol{\beta}$-slant null scrolls

In this section, we define the second type of slant null scrolls called β-slant null scrolls and give the characterizations of β-slant null scrolls. First, we give the following definition.

Definition 2. Let the surface S be a null scroll with frame $\left\{\vec{\alpha},\vec{\beta},\vec{\gamma}\right\}$ and $\vec{d}$ be a non-zero fixed line in Minkowski 3-space E_1^3. If the condition

$$\left\langle \vec{\beta},\vec{d}\right\rangle = m = \text{constant} \neq 0,$$

is satisfied for the vectors $\vec{\beta}$ and $\vec{d}$, then the surface S is called a β-slant null scroll. Here, the vector $\vec{d}$ is called the axis of β-slant null scroll.

Now, we can give the following characterizations for β-slant null scroll.

Theorem 3. Let the surface S be a null scroll with frame $\left\{\vec{\alpha},\vec{\beta},\vec{\gamma}\right\}$ and curvatures $\mu \neq 0$, $\lambda \neq 0$ and $\mu\lambda > 0$. Then, S is a β-slant null scroll with axis $\vec{d}$ if and only if

$$\begin{cases} \dfrac{\lambda'\mu-\lambda\mu'}{(\lambda\mu)^{3/2}} = \text{constant}, & \text{if } \vec{\mathrm{d}} \text{ is a spacelike or a timelike vector,} \\ \dfrac{\lambda'\mu-\lambda\mu'}{(\lambda\mu)^{3/2}} = \pm 2\sqrt{2}, & \text{if } \vec{\mathrm{d}} \text{ is a null vector,} \end{cases}$$

holds.

Proof. Since S is a β-slant null scroll, from Definition 2, we have

$$\left\langle \vec{\beta},\vec{d}\right\rangle = m = \text{constant} \neq 0 .$$

We define the axis of surface as follows,

$$\vec{d} = x\vec{\alpha} + m\vec{\beta} + y\vec{\gamma}, \tag{8}$$

where $x = x(u)$, $y = y(u)$ are smooth functions of u. Since $\vec{d}$ is constant, differentiation (8) gives following system,

$$\begin{cases} x' - m\mu = 0, \\ x\lambda + y\mu = 0, \\ y' - m\lambda = 0. \end{cases} \tag{9}$$

On the other hand, from (8), we have

$$\left\langle \vec{d},\vec{d}\right\rangle = m^2 + 2xy. \tag{10}$$

Let us now consider Lorentzian casual character of $\vec{d}$.

i) Let $\vec{d}$ be a spacelike or a timelike vector. Without loss of generality, we can assume that $\vec{d}$ is unit. Then, from (10), we get

$$m^2 + 2xy = \pm 1. \tag{11}$$

By using (11) and the second equation of the system (9), we obtain

$$x = \pm A\sqrt{\frac{\mu}{\lambda}}\,. \tag{12}$$

where $A = \sqrt{\left|\dfrac{m^2 \mp 1}{2}\right|} = \text{constant}$. Substituting (12) in the first equation of system (9), it follows

$$\frac{\lambda'\mu - \lambda\mu'}{(\lambda\mu)^{3/2}} = \frac{2m}{A} = \text{constant}.$$

ii) Let $\vec{d}$ be a null vector. Then, from (10), we have

$$m^2 + 2xy = 0. \tag{13}$$

By using (13) and the second equation of system (9), we get

$$x = \pm\frac{\sqrt{2}}{2}m\sqrt{\frac{\mu}{\lambda}}.$$

Substituting the last equation in the first equation of system (9), it follows

$$\frac{\lambda'\mu - \lambda\mu'}{(\lambda\mu)^{3/2}} = \pm 2\sqrt{2}.$$

Conversely, for the case that $\vec{d}$ is spacelike or timelike, let

$$\frac{\lambda'\mu - \lambda\mu'}{(\lambda\mu)^{3/2}} = 2p = \text{constant} \neq 0\,, \tag{14}$$

holds where p is a non-zero real constant. We define the vector $\vec{d}$ as

$$\vec{d} = -\sqrt{\frac{\mu}{\lambda}}\vec{\alpha} + p\vec{\beta} + \sqrt{\frac{\lambda}{\mu}}\vec{\gamma}. \tag{15}$$

From (15), it is obvious that $\langle \vec{\beta}, \vec{d} \rangle = \text{constant}$. On the other hand, from the derivative of (15) and by using (14), we get $\vec{d} = \text{constant}$ which means that S is a β-slant null scroll.

For the case that $\vec{d}$ is null, let the equality

$$\frac{\lambda'\mu-\lambda\mu'}{(\lambda\mu)^{3/2}}=\pm 2\sqrt{2}, \tag{16}$$

holds. We define the null vector $\vec{d}$ as

$$\vec{d}=-\sqrt{\frac{\mu}{\lambda}}\vec{\alpha}\pm\sqrt{2}\vec{\beta}+\sqrt{\frac{\lambda}{\mu}}\vec{\gamma}. \tag{17}$$

From (17), it is obvious that $\langle\vec{\beta},\vec{d}\rangle=\text{constant}$. On the other hand, from the derivative of (17) and by using (16), we get $\vec{d}=\text{constant}$ which means that S is a β-slant null scroll.

Corollary 5. S is a β-slant B-scroll if and only if the following nonlinear differential equations are satisfied:

$$\begin{cases}\lambda'-\dfrac{2m}{A}\sqrt{|\mu|}\lambda^{3/2}=0, & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,}\\ \lambda'\mp 2\sqrt{2}\sqrt{|\mu|}\lambda^{3/2}=0, & \text{if } \vec{d} \text{ is a null vector,}\end{cases}$$

where $A=\sqrt{\left|\frac{m^2\mp 1}{2}\right|}=\text{constant}$.

Proof. Since S is a B-scroll, i.e., $\mu=\text{constant}$ and the proof is clear from Theorem 3.

By solving the differential equations in Corollary 5, we have the followings:

i) If we take $\vec{d}$ as a spacelike or a timelike vector, the nonlinear differential equation

$$\lambda'-\frac{2m}{A}\sqrt{|\mu|}\lambda^{3/2}=0, \tag{18}$$

is satisfied. Hence, for the general solution of (18), we obtain

$$\lambda=\frac{4}{\left(\dfrac{2m}{A}\sqrt{|\mu|}u+n_1\right)^2},$$

where $A=\sqrt{\left|\frac{m^2\mp 1}{2}\right|}$ and n_1 is the integration constant.

ii) Similarly, if $\vec{d}$ is a null vector, then we have

$$\lambda'\mp 2\sqrt{2}\sqrt{|\mu|}\lambda^{3/2}=0, \tag{19}$$

and for the solution of (19), we get

$$\lambda=\frac{4}{\left(n_2\pm 2\sqrt{2}u\right)^2},$$

where n_2 is the integration constant. Therefore we can give the following Corollary:

Corollary 6. S is a β-slant B-scroll if and only if λ satisfies the following conditions:

$$\begin{cases}\lambda=\dfrac{4}{\left(\dfrac{2m}{A}\sqrt{|\mu|}u+n_1\right)^2}, & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,}\\ \lambda=\dfrac{4}{\left(n_2\pm 2\sqrt{2}u\right)^2}, & \text{if } \vec{d} \text{ is a null vector,}\end{cases}$$

where n_1, n_2 are integration constants.

Theorem 4. Let S be a null scroll in E_1^3. If S is a β-slant null scroll and $\lambda=\text{constant}=n$, then, the following nonlinear differential equation is satisfied

$$(\mu')^2\mp R\mu^3=0,$$

where $R=\dfrac{8m^2n}{m^2\mp 1}$ is a non-zero constant.

Proof. Let S be a β-slant null scroll and $\lambda=\text{constant}=n$. Then, from Definition 2, we have

$$\left\langle\vec{\beta},\vec{d}\right\rangle=m=\text{constant}\neq 0.$$

By differentiating the last equation and using (1), we obtain

$$\left\langle\vec{\alpha},\vec{d}\right\rangle=-\frac{n}{\mu}\left\langle\vec{\gamma},\vec{d}\right\rangle, \tag{20}$$

and putting $\left\langle\vec{\gamma},\vec{d}\right\rangle=q$, it follows

$$\left\langle\vec{\alpha},\vec{d}\right\rangle=-\frac{nq}{\mu}.$$

We define the vector $\vec{d}$ as follows

$$\vec{d}=q\vec{\alpha}+m\vec{\beta}-\frac{nq}{\mu}\vec{\gamma}. \tag{21}$$

In the case that $\vec{d}$ is a spacelike or a timelike vector, without loss of generality, we can assume that $\vec{d}$ is unit, i.e., $\left\langle\vec{d},\vec{d}\right\rangle=\pm 1$. Hence, from (21), we get

$$m^2 - \frac{2nq^2}{\mu} = \pm 1,$$

and from the last equation, we obtain

$$q = \pm\sqrt{\left|\frac{m^2 \mp 1}{2n}\mu\right|}. \tag{22}$$

On the other hand, by differentiating the equality $\langle \vec{\beta}, \vec{d} \rangle = m = \text{constant} \neq 0$ two times, we get

$$q = \frac{2\mu^2 m}{\mu'}. \tag{23}$$

Finally, from (22) and (23), we obtain

$$(\mu')^2 \mp \frac{8m^2 n}{m^2 \mp 1}\mu^3 = 0,$$

which is desired.

From the previous theorem we can give the following corollary:

Corollary 7. Let S be a null scroll in E_1^3. If S is a β-slant null scroll and $\lambda = \text{constant}$, then, μ is given by

$$\mu = 4\left(\sqrt{|R|}u + k\right)^{-2},$$

where k is the integration constant.

Proof. Let S be a β-slant null scroll and $\lambda = n = \text{constant}$. From Theorem 4, we have

$$(\mu')^2 \mp R\mu^3 = 0, \tag{24}$$

where $R = \dfrac{8m^2 n}{m^2 \mp 1}$ and from the solution of (24), we get

$$\mu = 4\left(\sqrt{|R|}u + k\right)^{-2},$$

where $R = \dfrac{8m^2 n}{m^2 \mp 1}$ and k is the integration constant.

4. α-slant null scrolls

In this section, we define the third type of slant null scrolls called α-slant null scrolls and give the characterizations for these surfaces.

Definition 3. Let the surface S be a null scroll with frame $\{\vec{\alpha},\vec{\beta},\vec{\gamma}\}$ and $\vec{d}$ be a non-zero fixed line in E_1^3. If the condition

$$\langle\vec{\alpha},\vec{d}\rangle = r = \text{constant} \neq 0,$$

is satisfied for the vectors $\vec{\alpha}$ and $\vec{d}$, then S is called an α-slant null scroll. Here, the vector $\vec{d}$ is called the axis of α-slant null scroll.

Theorem 5. Let S be an α-slant null scroll with frame $\{\vec{\alpha},\vec{\beta},\vec{\gamma}\}$ and curvature $\lambda \neq 0$. Then, the axis $\vec{d}$ is given by

$$\begin{cases} \vec{d} = r\left(-\dfrac{\mu}{\lambda}\vec{\alpha}+\vec{\gamma}\right), & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,} \\ \vec{d} = r\vec{\gamma}, \ \ \mu = 0, & \text{if } \vec{d} \text{ is a null vector,} \end{cases} \tag{25}$$

where $r = \langle\vec{\alpha},\vec{d}\rangle = \text{constant} \neq 0$,.

Proof. Let S be an α-slant null scroll with axis $\vec{d}$. Then, from Definition 3, we have

$$\langle\vec{\alpha},\vec{d}\rangle = \text{constant} = r \neq 0.$$

Differentiating last equation gives

$$\lambda\langle\vec{\beta},\vec{d}\rangle = 0.$$

Since $\lambda \neq 0$, it follows

$$\langle\vec{\beta},\vec{d}\rangle = 0. \tag{26}$$

By differentiating (26), we obtain

$$\langle\vec{\alpha},\vec{d}\rangle = -\frac{\lambda}{\mu}\langle\vec{\gamma},\vec{d}\rangle. \tag{27}$$

Let us write the vector $\vec{d}$ as

$$\vec{d} = x\vec{\alpha} + y\vec{\beta} + z\vec{\gamma}, \tag{28}$$

where $x = x(u)$, $y = y(u)$, $z = z(u)$ are smooth functions of u. Taking the inner product of $\vec{d}$ with $\vec{\beta}$, it is easy to see that $y = 0$. Thus, (28) is reduced to

$$\vec{d} = x\vec{\alpha} + z\vec{\gamma}. \tag{29}$$

Analogue to the previous process, taking the inner product of $\vec{d}$ with $\vec{\gamma}$ and $\vec{\alpha}$, respectively, we get

$$\begin{cases} x = \langle \vec{\gamma}, \vec{d} \rangle, \\ z = \langle \vec{\alpha}, \vec{d} \rangle = r. \end{cases} \tag{30}$$

By using (27) and (30), equation (29) becomes

$$\vec{d} = r\left(-\frac{\mu}{\lambda}\vec{\alpha} + \vec{\gamma} \right). \tag{31}$$

Let us now investigate the Lorentzian casual character of $\vec{d}$. From (31) we get

$$\langle \vec{d}, \vec{d} \rangle = -2r^2 \frac{\mu}{\lambda}.$$

Then we have that $\vec{d}$ is a spacelike or a timelike vector, if $\mu \neq 0$; and $\vec{d}$ is a null vector, if $\mu = 0$. Therefore we obtain (25).

Corollary 8. S is an α-slant null scroll if and only if

$$\begin{cases} \dfrac{\mu}{\lambda} = \text{constant} \neq 0, & \text{if } \vec{d} \text{ is a spacelike or a timelike vector,} \\ \mu = 0, & \text{if } \vec{d} \text{ is a null vector,} \end{cases}$$

holds.

Proof. Let S be an α-slant null scroll. From Theorem 5, if $\vec{d}$ is a spacelike or a timelike constant vector, we have

$$\vec{d} = r\left(-\frac{\mu}{\lambda}\vec{\alpha} + \vec{\gamma} \right),$$

where $r = \text{constant}$. By differentiating the last equation, it follows

$$\left(\frac{\mu}{\lambda} \right)' \vec{\alpha} = 0,$$

which means that $\frac{\mu}{\lambda} = \text{constant}$. If $\vec{d}$ is a null vector, from Theorem 5, it is clear that $\mu = 0$.

Conversely, let $\vec{d}$ be a spacelike or a timelike constant vector and $\frac{\mu}{\lambda}$ be a non-zero constant. We define

$$\vec{d} = r\left(-\frac{\mu}{\lambda}\vec{\alpha} + \vec{\gamma} \right). \tag{32}$$

Then, we obtain that $\langle \vec{\alpha}, \vec{d} \rangle = r = \text{constant}$ and from the derivative of (32) it follows that $\vec{d}$ is a constant vector. Therefore, from Definition 3, S is an α-slant null scroll.

Similarly, if $\vec{d}$ is a null vector and $\mu = 0$, then we define

$$\vec{d} = r\vec{\gamma}.$$

From the last equation, it is clear that $\langle \vec{\alpha}, \vec{d} \rangle = r =$ constant and from its derivative we get $\vec{d}$ is a constant vector. Therefore, from the Definition 3, S is an α-slant null scroll.

Corollary 9. Let S be a non-degenerate surface. Then S is an α-slant null scroll if and only if $\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = 0$ holds.

Proof. From the equalities given by (1), we get $\vec{\alpha}' = \lambda\vec{\beta}$ and from its derivatives, we obtain

$$\begin{cases} \vec{\alpha}'' = -\lambda\mu\vec{\alpha} + \lambda'\vec{\beta} - \lambda^2\vec{\gamma}, \\ \vec{\alpha}''' = (-2\lambda'\mu - \lambda\mu')\vec{\alpha} + (\lambda'' - 2\lambda^2\mu)\vec{\beta} - 3\lambda\lambda'\vec{\gamma}. \end{cases}$$

Thus, we get

$$\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = \lambda^5 \left(\frac{\mu}{\lambda}\right)'.$$

From Corollary 8, it is clear that $\det(\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}''') = 0$ holds if and only if S is an α-slant null scroll.

5. Examples

In this section we give two examples for the obtained results. We will consider the general examples introduced in Liu (2009b) for some special chosen constant given in examples and show that these surfaces are slant surfaces.

Example 1. Let us consider the B-scroll S given by the parameterization

$$\vec{x}_1(u,v) = \vec{a}_1(u) + v\vec{b}_1(u),$$

where

$$\begin{cases} a_1(u) = (0, -s, -s), \\ b_1(u) = \left(s, \dfrac{1}{2}\left(s - \dfrac{1}{s}\right), \dfrac{1}{2}\left(s + \dfrac{1}{s}\right)\right), \end{cases}$$

(Figure 1). The curvatures of S are obtained as $\lambda = 0$, $\mu = -1$, respectively. From Corollary 3, we see that $\det(\vec{\gamma}', \vec{\gamma}'', \vec{\gamma}''') = 0$ holds. Therefore, S is a γ-slant null scroll. Moreover, from Theorem 2, the axis of surface coincides with $\vec{\alpha}$.

Example 2. Let us consider the B-scroll S given by the parameterization

$$\vec{x}_2(u,v)=\vec{a}_2(u)+v\vec{b}_2(u),$$

where

$$\begin{cases} a_2(u)=\left(\frac{1}{12}(2+\log s)^3-\frac{1}{4}\log s,\ -\frac{1}{4}(2+\log s)^2,\ -\frac{1}{12}(2+\log s)^3-\frac{1}{4}\log s\right), \\ b_2(u)=s\left(\frac{1}{2}\left(1-\log^2 s\right),\ \log s,\ \frac{1}{2}\left(1+\log^2 s\right)\right), \end{cases}$$

(Figure 2). The curvatures of S are $\lambda=-\frac{1}{2s^2}$, $\mu=-1$, respectively. Then, we see that

$$\frac{\lambda'\mu-\lambda\mu'}{(\lambda\mu)^{3/2}}=\pm 2\sqrt{2}\ .$$

Therefore, from Theorem 3, S is a β-slant null scroll with a null axis.

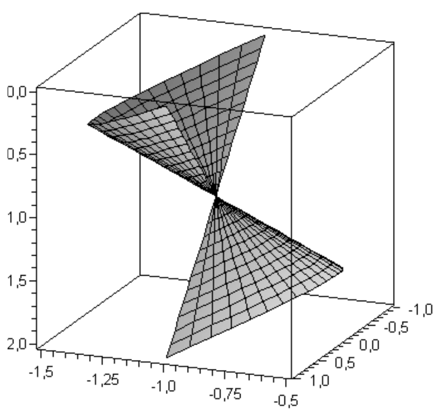

Fig. 1. B-scroll surface $\vec{x}_1$.

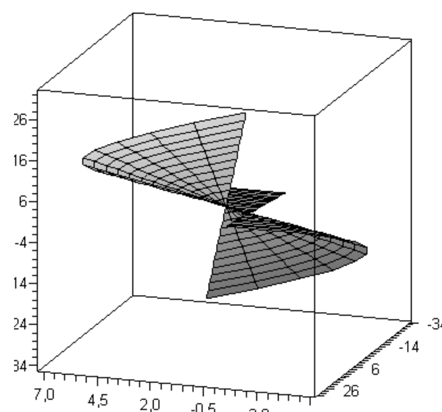

Fig. 2. B-scroll surface $\vec{x}_2$.

6. Conclusions

Since the characterizations of ruled surfaces with lightlike rulings are not so similar to Euclidean case, these surfaces have an important role in Minkowski 3-space. Generally, these surfaces are classified as null scrolls and B-scrolls and used in many classification results of some special ruled surfaces. This is the reason why the study of null scrolls and B-scrolls are important. By considering this importance, three new types of null scrolls are defined in this study. Generally, these surfaces are called slant null scrolls and characterizations for slant null scrolls are introduced.

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لفائف جانبية صفرية في فضاء مينكوفسكي الثلاثي E_1^3

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خلاصة

نقدم في هذه الدراسة نوعاً جديداً من اللفائف الصفرية في فضاء مينكوفسكي الثلاثي و نسميها باللفائف الصفرية الجانبية. ونعَرف ثلاثة أنواع من هذه اللفائف ونسميها بجانبية α، جانبية β وجانبية γ. كما نعطي وصفاً يميز متى تكون اللفيفة الصفرية لفيفة جانبية. أما بالنسبة لجانبيات β فإننا نحصل على معادلات تفاضلية غير خطية و نعطي بعض النتائج باستخدام حلولها التحليلية و نعطي في الآخر بعض الأمثلة على ما حصلنا عليه من نتائج.