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A numerical framework for solving high-order pantograph-delay Volterra integro-differential equations

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Abstract

In this paper, we present an efficient numerical method to solve Volterra integro-differential equations of pantograph-delay type. We use the Euler polynomials to approximate the solutions. The proposed method is discussed in detail and compared by solving some numerical examples. Moreover, the error estimates of the proposed method is given. Special cases of the main results are also mentioned.

Keywords: Accuracy; euler polynomials; high-order; integro-differential equations; pantograph-delay.

1. Introduction

Integro-differential equations have a major importance in modeling of some phenomena in sciences and engineering. Functional-differential equations with proportional delays are usually referred to as pantograph equations or generalized pantograph equations. Integro-differential equations play a cardinal role in numerous fields of Mathematics, Chemistry, Physics and Engineering such as population dynamics, dynamical systems, control and biology; see the various references cited in Jerri (1999), Ockendon & Tayler (1971) and Spiridonov (1995).

Since the mentioned problems are usually difficult to solve analytically, a numerical method is required. Several numerical methods were used such as the Kanwal & Liu (1989); Nas *et al.*(2000); Yalçinbas & Sezer (2000); Lü & Cui (2008); Muroya *et al.*(2003); Jiang & Ma (2013); Tang*et al.*(2008); Ali *et al*. (2009); Tohidi & Kiliçman (2014); Keskin *et al*. (2007); Evans & Raslan (2005); Liu & Jiang (2013); Yüzbasi & Sezer (2013); Abubakar & Taiwo (2014); Yi & Wang (2014); Fazeli & Hojjati (2015); Jiang & Tian (2015) and Yi & Huang (2015).

Among global methods, polynomial approximations are well-known for researchers in applied mathematics. In this field of approximation, some expressions may be used to simplify the algebraic computations. For instance, operational matrices of integration and differentiation can be used in computations for simplifying the associated algebraic equations. Applying these operational matrices also make the software implementation easy. The solution of integro-differential equations by using operational matrices has been performed by some researchers. Numerical methods which are based on operational matrices are used by many researchers in the case of Horvat (1999); Gülsu & Sezer (2011); Bhrawy *et al.* (2012); Heydari *et al.* (2013); Gokdogan & Merdan (2013); Ghany & Hyder (2014); Bellour & Bousselsal (2014) and Yüzbasi (2014). In this paper, using both of the operational matrices of differentiation and delay in the case of Euler polynomials, we will obtain the numerical solution of the following pantograph-delay type Volterra integro-differential equations

$$u^{(n)}(t)+\sum_{i=0}^{n_0}y_i(t)u^{(i)}(t)+\sum_{i=0}^{n_0}y_i^*(t)u^{(i)}(\tau_i t+\nu_i)=f(t)+\sum_{i=0}^{n_0}\left(\int_0^t k_{1,i}(t,x)u^{(i)}(x)dx+\int_0^{h(t)}k_{2,i}(t,x)u^{(i)}(x)dx\right), \quad (1)$$

with the initial conditions

$$\sum_{i=0}^{n-1}c_{i,j}u^{(i)}(0)+d_{i,j}u^{(i)}(\xi)=\mu_i, \quad j=0,1,\ldots,n-1, \quad 0\le a\le\xi\le b, \tag{2}$$

where $n\in \mathrm{N}$, $0\le n_0\le n-1$; $a\le\xi\le b$; $y_i(t), y_i^*(t), f(t), k_{1,i}(t,x)$ and $k_{2,i}(t,x)$ are analytic functions; τ_i, ν_i and $c_{i,j}$, $d_{i,j}$ a, b are appropriate constants.

The aim of this study is to give an approximate solution of the problem (1) and (2) in the form

$$u_N(t)=\sum_{m=0}^{N}\alpha_m E_m(t), \quad 0\le a\le t\le b, \tag{3}$$

where $E_m(t), m=0,1,\ldots,N$ are Euler polynomials of degree m and $\alpha_m, m=0,1,\ldots,N$ are the coefficients to be determined.

The Euler polynomials $E_m(t)$ and Euler numbers E_m are respectively determined by the power expansions

$$\frac{2e^{tx}}{e^x+1}=\sum_{m=0}^{\infty}E_m(t)\frac{x^n}{n!}, \qquad \frac{2e^{\frac{t}{2}}}{e^t+1}=\sum_{m=0}^{\infty}\frac{E_m}{m!}\left(\frac{t}{2}\right)^m. \tag{4}$$

Also an explicit formula for the Euler polynomials is given by

$$E'_m(t)=mE_{m-1}(t), \quad m=1,2,\ldots. \tag{5}$$

Recently, the Euler polynomials have been used to solve the linear and nonlinear Fredholm integro-differential equations, Volterra integral equations and Mirzaee & Bimesl (2014) have obtained some positive results.

2. Basic matrices

First of all, we can write the Euler polynomials in the matrix form as follows:

$$u_N(t) = \sum_{m=0}^{N} \alpha_m E_m(t) = \mathbf{\Phi(t)}\boldsymbol{\alpha}^T, \quad \text{and} \quad u_N^{(i)}(t) = \mathbf{\Phi}^{(i)}(t)\,\boldsymbol{\alpha}, \tag{6}$$

where the Euler coefficient vector $\boldsymbol{\alpha}$ and the Euler vector $\mathbf{\Phi(t)}$ are given by

$$\mathbf{\Phi(t)} = [E_0(t), E_1(t), \ldots, E_N(t)], \quad \boldsymbol{\alpha} = [\alpha_0, \alpha_1, \ldots, \alpha_N].$$

Using the differentiation property (5), we get

$$\mathbf{\Phi}^{(i)}(t) = \mathbf{\Phi}(t)(\mathbf{P^T})^i, \tag{7}$$

where

$$P = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & N & 0 \end{bmatrix}. \tag{8}$$

Substituting the relation (7) into (6), we get the matrix form

$$u_N^{(i)}(t) = \mathbf{\Phi}^{(i)}(t)\boldsymbol{\alpha} = \mathbf{\Phi}(t)(\mathbf{P^T})^i\boldsymbol{\alpha}. \tag{9}$$

Mirzaee & Bimesl (2015) established the following matrix relation

$$\mathbf{\Phi}(t) = \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T, \tag{10}$$

where

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \cdots & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{T}(t) = [1, t, t^2, \ldots, t^N]. \tag{11}$$

Substituting (10) and (11) into (9) gives

$$u_N^{(i)}(t) = \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P^T})^i\boldsymbol{\alpha}. \tag{12}$$

By placing

$$t \to \tau_i t + \nu_i, \quad i = 0,1,\ldots,n-1,$$

in (12), we get the relation

$$u_N^{(i)}(\tau_i t + \nu_i) = \mathbf{T}(\tau_i t + \nu_i)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha} = \mathbf{T}(t)\mathbf{C}_{\tau_i,\nu_i}(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha}, \tag{13}$$

where

$$\mathbf{C}_{\tau_i,\nu_i} = (c_{s,r})_{(N+1)\times(N+1)}; \qquad c_{s,r} = \begin{cases} \binom{r-1}{s-1}\tau_i^{s-1}\nu_i^{r-s} & s \le r, \\ 0 & s > r, \end{cases} \qquad 0 \le s,r \le N. \tag{14}$$

The kernel functions $k_{1,i}(t,x)$ can be approximated by the truncated Euler series

$$k_{1,i}(t,x) = \sum_{s=0}^{N}\sum_{r=0}^{N} k_{s,r}^{1,i} E_s(t)E_r(x) = \mathbf{\Phi}(t)\mathbf{K}^{1,i}\mathbf{\Phi}(x)^T, \quad \mathbf{K}^{1,i} = (k_{s,r}^{1,i})_{(N+1)\times(N+1)}, \quad 0 \le s,r \le N,$$

By substituting the above matrix forms into the firs integral part of (1), we have

$$\begin{aligned} \int_0^t k_{1,i}(t,x)u^{(i)}(x)dx &= \int_0^t \mathbf{\Phi}(t)\mathbf{K}^{1,i}\mathbf{\Phi}(x)^T\mathbf{\Phi}(x)(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha} dx \\ &= \mathbf{\Phi}(t)\mathbf{K}^{1,i}\left(\int_0^t \mathbf{\Phi}(x)^T\mathbf{\Phi}(x)dx\right)(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha} \\ &= \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{1,i}\left(\int_0^t \mathbf{\Lambda}^{-1}\mathbf{T}^T(x)\mathbf{T}(x)(\mathbf{\Lambda}^{-1})^T dx\right)(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha} \\ &= \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{1,i}\mathbf{\Lambda}^{-1}\left(\int_0^t \mathbf{T}^T(x)\mathbf{T}(x)dx\right)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha} \\ &= \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{1,i}\mathbf{\Lambda}^{-1}\mathbf{B_1}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha}. \end{aligned} \tag{15}$$

Where the components of $\mathbf{B_1}(t)$ are written as

$$\mathbf{B_1}(t) = (b_{s,r}^1)_{(N+1)\times(N+1)}, \quad b_{s,r}^1 = \frac{t^{s+r+1}}{s+r+1}, \quad 0 \le s,r \le N.$$

As in the previous part, we can approach to the kernel functions $k_{2,i}(t,x)$ by the truncated Euler series

$$k_{2,i}(t,x) = \sum_{s=0}^{N}\sum_{r=0}^{N} k_{s,r}^{2,i} E_s(t)E_r(x) = \mathbf{\Phi}(t)\mathbf{K}^{2,i}\mathbf{\Phi}(x)^T, \quad \mathbf{K}^{2,i} = (k_{s,r}^{2,i})_{(N+1)\times(N+1)}, \quad 0 \le s,r \le N.$$

Similarly to the second integral part of (1), another relation can be obtained as

$$\int_0^{h(t)} k_{2,i}(t,x)u^{(i)}(x)dx = \mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{2,i}\mathbf{\Lambda}^{-1}\mathbf{B_2}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i \boldsymbol{\alpha}, \tag{16}$$

where

$$\mathbf{B_2}(t) = (b_{s,r}^2)_{(N+1)\times(N+1)}, \quad b_{s,r}^2 = \frac{(h(t))^{s+r+1}}{s+r+1}, \quad 0 \le s,r \le N.$$

3. Proposed method of the solution

In this part of paper, we convert the main equation (1) to an equivalent system of algebraic equations. Firstly, let us substitute the matrix forms (12), (13), (15) and (16) into equation (1) and thus we obtain the matrix equation

$$\mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^n\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i(t)\mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i^*(t_j)\,\mathbf{T}(t)\,\mathbf{C}_{\tau_i,v_i}(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}=f(t)$$

$$+\sum_{i=0}^{n_0}(\mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{1,i}\mathbf{\Lambda}^{-1}\mathbf{B_1}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}+\mathbf{T}(t)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{2,i}\mathbf{\Lambda}^{-1}\mathbf{B_2}(t)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}). \quad (17)$$

By substituting the collocation points $t_j=\dfrac{(b-a)j}{N-n}$, $j=0,1,\ldots,N-n$ into equation (17), we have a system of matrix equations as

$$\mathbf{T}(t_j)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^n\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i(t_j)\mathbf{T}(t_j)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i^*(t_j)\,\mathbf{T}(t_j)\,\mathbf{C}_{\tau_i,v_i}(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}=f(t_j)$$

$$+\sum_{i=0}^{n_0}(\mathbf{T}(t_j)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{1,i}\mathbf{\Lambda}^{-1}\mathbf{B_1}(t_j)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}+\mathbf{T}(t_j)(\mathbf{\Lambda}^{-1})^T\mathbf{K}^{2,i}\mathbf{\Lambda}^{-1}\mathbf{B_2}(t_j)(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}). \quad (18)$$

This system can be written in the matrix form

$$(\mathbf{T}\overline{\mathbf{\Lambda}}(\mathbf{P}^{\mathbf{T}})^n+\sum_{i=0}^{n_0}\mathbf{Y}_i\mathbf{T}\overline{\mathbf{\Lambda}}(\mathbf{P}^{\mathbf{T}})^i+\sum_{i=0}^{n_0}\mathbf{Y}_i^*\mathbf{T}\mathbf{C}_{\tau_i,v_i}(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i-\sum_{i=0}^{n_0}(\mathbf{T}\overline{\mathbf{\Lambda}}\,\overline{\mathbf{K}}^{1,i}\,\overline{\mathbf{B}}_\mathbf{1}\,\overline{\mathbf{P}}^i+\mathbf{T}\overline{\mathbf{\Lambda}}\,\overline{\mathbf{K}}^{2,i}\,\overline{\mathbf{B}}_\mathbf{2}\,\overline{\mathbf{P}}^i))\,\boldsymbol{\alpha}=\mathbf{F}, \quad (19)$$

$$\underbrace{(\mathbf{T}\overline{\mathbf{\Lambda}}(\mathbf{P}^{\mathbf{T}})^n+\sum_{i=0}^{n_0}\mathbf{Y}_i\mathbf{T}\overline{\mathbf{\Lambda}}(\mathbf{P}^{\mathbf{T}})^i+\sum_{i=0}^{n_0}\mathbf{Y}_i^*\mathbf{T}\mathbf{C}_{\tau_i,v_i}(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i-\sum_{i=0}^{n_0}(\mathbf{T}\overline{\mathbf{\Lambda}}\,\overline{\mathbf{K}}^{1,i}\,\overline{\mathbf{B}}_\mathbf{1}\,\overline{\mathbf{P}}^i+\mathbf{T}\overline{\mathbf{\Lambda}}\,\overline{\mathbf{K}}^{2,i}\,\overline{\mathbf{B}}_\mathbf{2}\,\overline{\mathbf{P}}^i))}_{\mathbf{W}_{(N-n+1)\times(N+1)}}\,\boldsymbol{\alpha}=\mathbf{F} \quad (20)$$

where

$$T=\begin{bmatrix}1 & t_0 & t_0^2 & \cdots & t_0^N\\ 1 & t_1 & t_1^2 & \cdots & t_1^N\\ \vdots & & & & \\ 1 & t_{N-n-1} & t_{N-n-1}^2 & \cdots & t_{N-n-1}^N\\ 1 & t_{N-n} & t_{N-n}^2 & \cdots & t_{N-n}^N\end{bmatrix}=\begin{bmatrix}T(t_0)\\ T(t_1)\\ \vdots\\ T(t_{N-n-1})\\ T(t_{N-n})\end{bmatrix},\quad \overline{\mathbf{\Lambda}}=\begin{bmatrix}(\mathbf{\Lambda}^{-1})^T & & 0_{1\times(N-n)}\\ & \ddots & \\ & & \ddots \\ 0_{1\times(N-n)} & & (\mathbf{\Lambda}^{-1})^T\end{bmatrix},$$

$$\mathbf{Y}_i=\begin{bmatrix}y_i(t_0) & & 0_{1\times(N-n)}\\ 0 & y_i(t_1) & 0_{1\times(N-n-1)}\\ & & \ddots \\ 0_{1\times(N-n)} & & y_i(t_{N-n})\end{bmatrix},\quad \mathbf{Y}_i^*=\begin{bmatrix}y^*_i(t_0) & & 0_{1\times(N-n)}\\ 0 & y^*_i(t_1) & 0_{1\times(N-n-1)}\\ & & \ddots \\ 0_{1\times(N-n)} & & y^*_i(t_{N-n})\end{bmatrix},$$

$$\overline{\mathbf{K}}^{1,i}=\begin{bmatrix}\mathbf{K}^{1,i} & & & \mathbf{0}_{1\times(N-n)}\\ & \ddots & & \\ & & \ddots & \\ \mathbf{0}_{1\times(N-n)} & & & \mathbf{K}^{1,i}\end{bmatrix},\quad \overline{\mathbf{K}}^{2,i}=\begin{bmatrix}\mathbf{K}^{2,i} & & & \mathbf{0}_{1\times(N-n)}\\ & \ddots & & \\ & & \ddots & \\ \mathbf{0}_{1\times(N-n)} & & & \mathbf{K}^{2,i}\end{bmatrix},$$

$$\overline{\mathbf{B}}_i=\begin{bmatrix}\mathbf{\Lambda}^{-1}\mathbf{B}_i(t_0)(\mathbf{\Lambda}^{-1})^T & & & \mathbf{0}_{1\times(N-n)}\\ 0 & \mathbf{\Lambda}^{-1}\mathbf{B}_i(t_1)(\mathbf{\Lambda}^{-1})^T & & \mathbf{0}_{1\times(N-n-1)}\\ & & \ddots & \\ \mathbf{0}_{1\times(N-n)} & & & \mathbf{\Lambda}^{-1}\mathbf{B}_i(t_{N-n})(\mathbf{\Lambda}^{-1})^T\end{bmatrix},\qquad i=1,2,$$

$$\overline{P}=\begin{bmatrix}(P^T)^i\\(P^T)^i\\ \vdots\\(P^T)^i\\(P^T)^i\end{bmatrix},\quad \overline{P}^0=\begin{bmatrix}I_{N-n+1}\\I_{N-n+1}\\ \vdots\\ I_{N-n+1}\\ I_{N-n+1}\end{bmatrix},\quad \textit{and}\quad F=\begin{bmatrix}f(t_0)\\f(t_1)\\ \vdots\\ f(t_{N-n-1})\\ f(t_{N-n})\end{bmatrix}.$$

Using (12), the matrix form of conditions can be written as

$$\underbrace{\sum_{i=0}^{n-1}(c_{i,j}\mathbf{T}(0)+\mathbf{T}(\xi))(\mathbf{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i}_{\mathbf{C}_i}\boldsymbol{\alpha}=(\mu_i),\quad i=0,1,\ldots,n-1, \tag{21}$$

where

$$\mathbf{T}(0)=[1,\underbrace{0,0,\ldots,0}_{N\text{times}}],\quad \mathbf{C}_i=[c_{i,0},c_{i,1},\ldots,c_{i,N}]\quad \text{and}\quad \mathbf{T}(\xi)=[1,\xi,\xi^2,\ldots,\xi^N].$$

Substituting the row matrices (21) by the last n rows of the matrix (20) results an augmented matrix

$$[\overline{\mathbf{W}},\overline{\mathbf{F}}]=\begin{bmatrix}\overline{\mathbf{W}}_{(N-n+1)\times(N+1)} & \mathbf{F}_{(N-n+1)\times 1}\\ \mathbf{C}_{n\times(N+1)} & \boldsymbol{\mu}_{n\times 1}\end{bmatrix},$$

where

$$\mathbf{C}_{n\times(N+1)}=\begin{bmatrix}\mathbf{C}_0\\ \mathbf{C}_1\\ \vdots\\ \mathbf{C}_{n-1}\\ \mathbf{C}_n\end{bmatrix}\quad \text{and}\quad \boldsymbol{\mu}_{n\times 1}=\begin{bmatrix}\mu_0\\ \mu_1\\ \vdots\\ \mu_{n-1}\\ \mu_n\end{bmatrix}.$$

Evidently, the unknown Euler coefficients can be found by solving the above matrix equation. If rank $\overline{\mathbf{W}}$ =rank $[\overline{\mathbf{W}},\overline{\mathbf{F}}]=N+1$ then $\boldsymbol{\alpha}=\overline{\mathbf{W}}^{-1}\overline{\mathbf{F}}$.

4. Error estimation of the method

The accuracy of the approximate solution (3) can be easily checked as follows:

Since the truncated Euler series $u_N(t)$ is approximate solution of (1), when the function $u_N(t)$ and its ith derivative are substituted in equation (1), the resulting equation must be satisfied approximately; that is, for $t_j \in [a,b], \quad j=0,1,...$

$$E_N(t_j)=|\mathbf{T}(t_j)(\boldsymbol{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^n\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i(t_j)\mathbf{T}(t_j)(\boldsymbol{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}+\sum_{i=0}^{n_0}y_i^*(t_j)\,\mathbf{T}(t_j)\,\mathbf{C}_{\tau_i,v_i}(\boldsymbol{\Lambda}^{-1})^T(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha}$$

$$-\sum_{i=0}^{n_0}(\mathbf{T}(t_j)(\boldsymbol{\Lambda}^{-1})^T\,\mathbf{K}^{1,i}\,\boldsymbol{\Lambda}^{-1}\mathbf{B}_1(t_j)(\mathbf{P}^{\mathbf{T}})^i\,\boldsymbol{\alpha}+\mathbf{T}(t_j)(\boldsymbol{\Lambda}^{-1})^T\,\mathbf{K}^{2,i}\,\boldsymbol{\Lambda}^{-1}\,\mathbf{B}_2(t_j)(\mathbf{P}^{\mathbf{T}})^i\boldsymbol{\alpha})-f(t_j)|\cong 0, \quad (22)$$

where $E_N(t_j)\le 10^{-\varepsilon_j}$ (ε_j is a positive integer). If max $10^{-\varepsilon_j}=10^{-\varepsilon}$ (ε is a positive integer) is prescribed,then the truncation limit N is increased until the difference $E_N(t_j)$ at each of the points becomes smaller than the prescribed $10^{-\varepsilon}$. On the other hand, the error can be estimated by the function $E_N(t)$. If $E_N(t)\to 0$ when N is sufficiently large enough, then the errors decrease.

5. Numerical experiments

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in MAPLE. In this regard, we have reported in tables and figures, the values of the exact solution $u(t)$, the approximate solution $u_N(t)$, the absolute error function $E_N(t)=|u(t)-u_N(t)|$ and the maximum absolute error $E_N=\|u(t)-u_N(t)\|_\infty$ at the selected points of the given interval.

Example 1. Let us first consider the pantograph-delay Volterra integro-differential equation

$$u^{(2)}(t)=\frac{3}{4}u(t)-u(\frac{t}{2})+t\cos(t)+\sin(\frac{t}{2})-\frac{11}{4}\sin(t)+\int_0^t xu(x)dx, \quad 0\le t\le 1, \quad (23)$$

with initial conditions $u(0)=0$, $u^{(')}(0)=1$. It can be verified that the exact solution of(23) is $u(t)=\sin(t)$.

We obtain the approximate solutions of the problem for $N=5$, 10 and 15, respectively,

$u_5(t)=t+(9.2601e-04)t^2-0.1691t^3+(3.1041e-03)t^4+(6.7097e-03)t^5$,

$u_{10}(t)=t-(1.1266e-10)t^2-0.1667t^3-(4.9397e-08)t^4+(8.3335e-03)t^5$

$-(3.7118e-07)t^6-(1.9784e-04)t^7-(5.7604e-07)t^8+(3.1334e-06)t^9-(1.4474e-07)t^{10}$, and

$$u_{15}(t) = t + (1.5850e-16)t^2 - 0.16667t^3 + (2.0448e-14)t^4 + (8.3333e-03)t^5 + (5.3439e-13)t^6 - (1.9841e-4)t^7 + (3.8166e-12)t^8 + (2.7556e-06)t^9 + (7.9713e-12)t^{10} - (2.5059e-08)t^{11} + (4.2496e-12)t^{12} + (1.5901e-10)t^{13} + (2.7591e-13)t^{14} - (7.6008e-13)t^{15}.$$

We give the numerical results of the error functions and CPU times used for different values of Nin Figure 1. This shows that the accuracy increases as N is increased. The CPU times and maximum absolute errors obtained by the present method with $N = 10,16$ besides results of Yüzbasi (2014) with $(N,M) = (7,7),(N,M) = (10,10)$ are depicted in Figure 2. The numerical results obtained by the present method are compared with the Gülsu & Sezer (2011) for $N = 10$ in Table 1. Comparisons show better accuracy of our method.

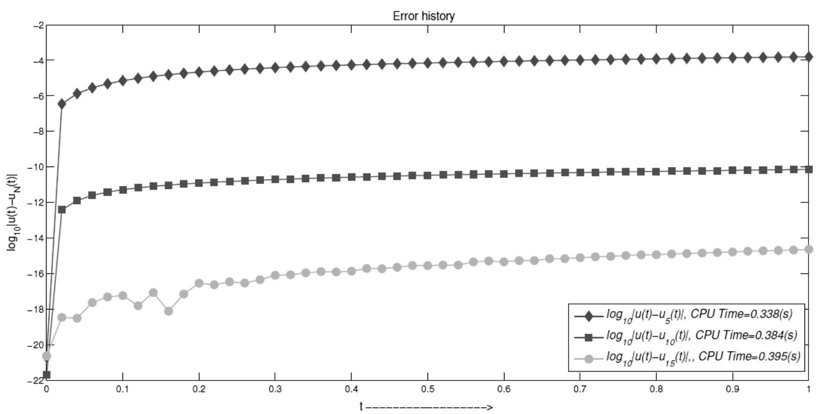

Fig. 1. Graph of $\log_{10}|u(t)-u_N(t)|$ for Example 1 with $N = 5,10,15$.

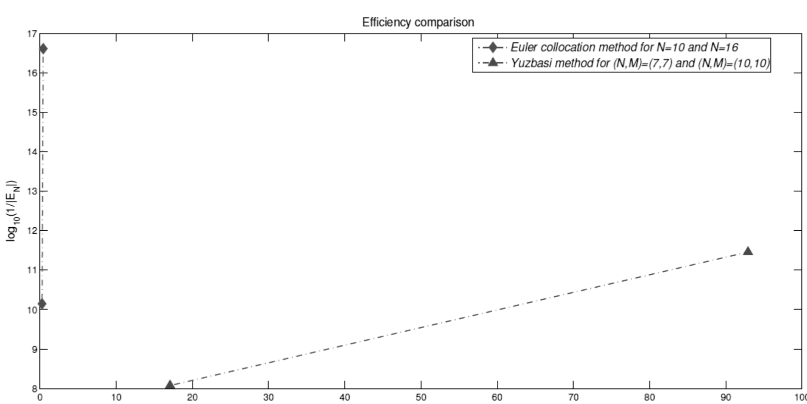

Fig. 2. Graph of efficiency comparison of Example 1.

Table 1. Comparison of the absolute errors for Example 1.

x_i	**Exact solution**	**Present method** $N=10$	**Gülsu-Sezer method** $N=10$
0.0	0	0	0
0.2	0.1987	1.2407e-11	7.00e-10
0.4	0.3895	2.6605e-11	1.50e-09
0.6	0.5646	4.1178e-11	2.40e-09
0.8	0.7174	5.6414e-11	3.00e-09
1.0	0.8415	7.2807e-11	1.06e-07

Example 2. Consider the following pantograph-delay Volterra integro-differential equation

$$u'(t) = 2\exp(1-t) - 3u(t) - \int_0^t (3u(x)+u'(x))dx + \int_0^{t-1} (3u(x)+u'(x))dx, \quad 0 \le t \le 2, \quad (25)$$

with initial conditions $u(0)=0$. It can be verified that the exact solution of (25) is $u(t)=\exp(-t)$. We have plotted the graph of the absolute errors in Figure 3 for $N=6,12,18$. Integro-differential equation (25) is solved inYüzbasi (2014) and Horvat (1999), respectively by Laguerre approach and Spline method. The numerical results presented in Figure 4 and Table 2 show that the method presented here gives more accurate solutions using a fewer number of basis functions.

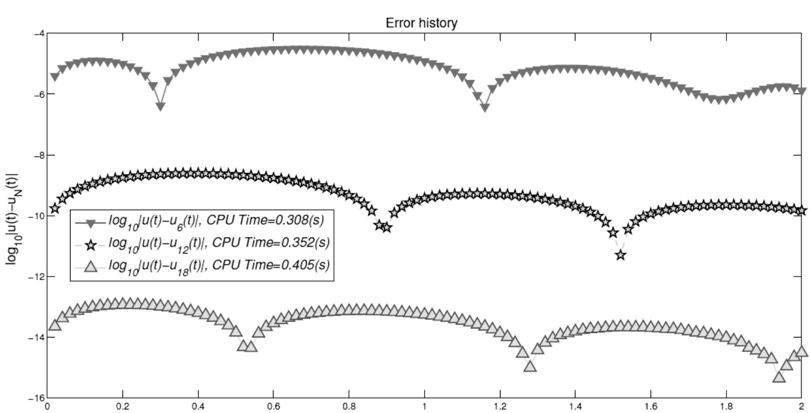

Fig. 3. Graph of $\log_{10}|u(t)-u_N(t)|$ for Example 2 with $N=6,12,18$.

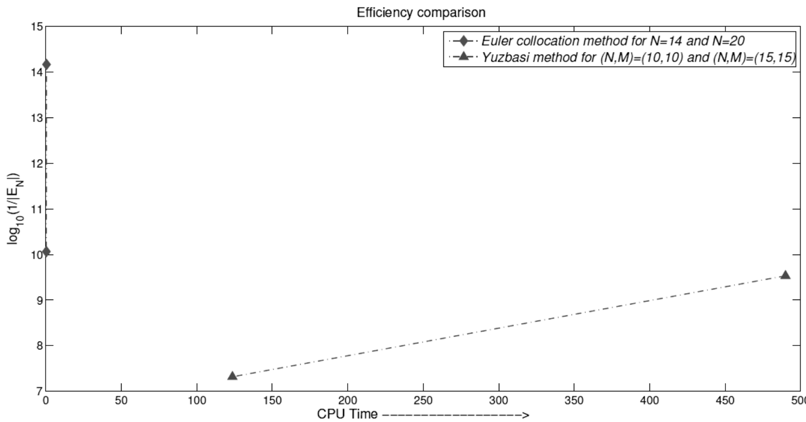

Fig. 4. Graph of efficiency comparison of Example 2.

Table 2. Comparison of the maximum absolute errors for Example 2.

	Maximum absolute error
Horvat method	
Using shifted Gauss points	$8.33e-09$
Using shifted Radau II points	$1.43e-07$
Using shifted Lobatto points	$1.81e-06$
Using points identified as other	$4.25e-06$
Present method	
For $N=10$	$1.6889e-09$

Example 3. Let us consider the following pantograph-delay Volterra integro-differential equation

$$u'(t) = f(t) + \int_0^t (\cos(t+x+1)+2)u(x)dx + \int_0^{t-1} (\cos(t+x+1)+2)u(x)dx, \quad 0 \le t \le 3, \quad (26)$$

where

$$f(t) = \frac{12\cos(t)+\cos(3t+1)-\cos(3t-1)-8\cos(t-1)+2\sin(t+1)+4\sin(2t)-4\sin(2t-1)-8}{4}.$$

The exact solution is $u(t)=\sin(t)+1$ and the initial condition is $u(0)=1$. From Figure 5, the efficiency of our method with respect to the Yüzbasi (2014), can be founded. For comparing the accuracy of our presented method with that of Bellour & Bousselsal (2014), we provide Table 3.

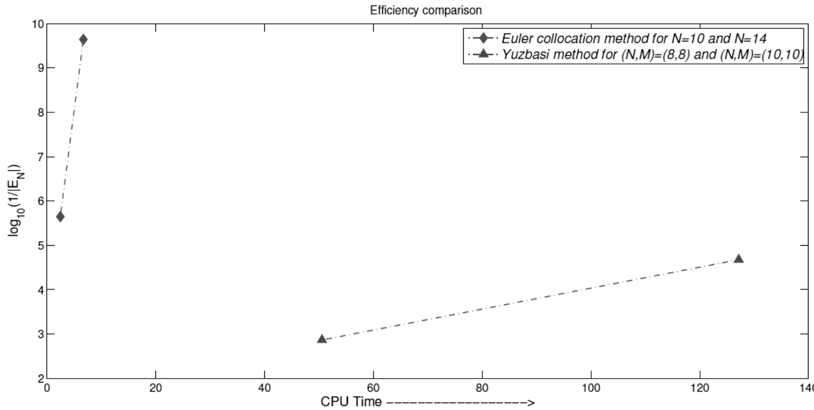

Fig. 5. Graph of efficiency comparison of Example 3.

Table 3. Comparison of the absolute errors for Example 3.

N	Bellour-Bousselsal method	Bellour-Bousselsal method	Present method	Present method
	$(N,m) = (4,4)$	$(N,m) = (5,5)$	$N = 16$	$N = 25$
0	0	0	0	0
0.5	$1.04e-06$	$1.00e-09$	$5.1213e-13$	$1.2961e-21$
1.0	$2.13e-06$	$7.00e-09$	$6.9585e-13$	$2.2015e-21$
1.5	$3.03e-06$	$1.30e-08$	$1.0171e-12$	$3.0886e-21$
2.0	$4.12e-06$	$2.60e-08$	$1.6600e-12$	$5.0652e-21$
2.5	$6.19e-06$	$4.70e-08$	$3.3125e-12$	$1.0102e-20$
3.0	$1.00e-05$	$8.90e-08$	$7.1278e-12$	$2.1740e-20$

Example 4. For our final example we consider the following pantograph-delay Volterra integro-differential equation

$$u'(t) = u(t-1) + \int_0^t u(x)dx - \int_0^{t-1} u(x)dx, \quad u(0) = 1, \quad 0 \le t \le 1. \tag{27}$$

The corresponding exact solution is given by $u(t) = \exp(t)$. The efficiency of our method with respect to theYüzbasi (2014), can be founded from Figure 6. For instance, by our presented method, we reach the maximum absolute error $2.71e-06$ in $0.281s$, meanwhile theYüzbasi (2014), reach to the error $3.23e-03$ in $3.276s$. Also, the computed errors $E_N = \| u(t) - u_N(t) \|_\infty$ obtained by the present method besides results of Gülsu & Sezer (2011), Bellour & Bousselsal (2014) and Horvat (1999) for Examples $1-3$ are given in Table 4. Comparisons show better accuracy of our method using a fewer number of basis functions.

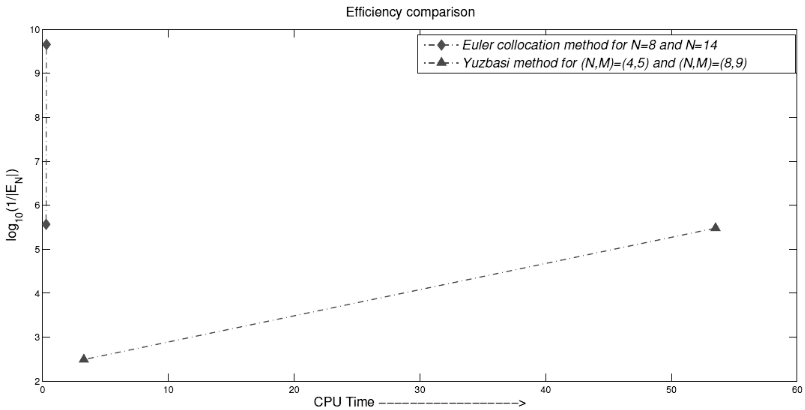

Fig. 6. Graph of efficiency comparison of Example 4.

Table 4. Comparison of the maximum absolute errors.

Methods	Examples 1	Examples 2	Examples 3
Horvat method			
N=30	1.81e-06	4.25e-06	1.43e-07
N=60	1.13e-07	5.85e-07	4.67e-09
Bellour method			
N=16	3.37e-05	2.46e-07	1.00e-05
N=25	1.06e-07	7.64e-09	8.90e-08
Gülsu method			
N=5	4.26e-05	4.21e-05	1.43e-05
N=6	5.02e-06	5.85e-06	7.13e-07
N=7	1.13e-07	1.51e-07	3.74e-08
Present method			
N=5	1.78e-05	1.29e-06	5.56e-06
N=6	5.35e-07	1.97e-07	4.76e-07
N=7	4.59e-08	2.10e-08	1.70e-08
N=8	2.05e-09	1.11e-08	8.62e-09
N=9	7.28e-10	3.58e-09	6.43e-10

6. Conclusion

The properties of the Euler bases together with operational matrices have been utilized to numerically solve a class of Volterra integro-differential equations of pantograph-delay type. The proposed approach reduces the main problem to the corresponding systems of algebraic equations. The method is based on expanding the existing functions in terms of Euler polynomials.The effort required to implement the method is very low, while the accuracy is high. Based upon the numerical results, when the solution is sufficiently smooth, a small number of basis functions is enough to obtain a high accuracy solution. At the end, we note that the method can be easily extended and applied to multi-dimensional integro-differential equations. This will be subject of our future research.

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خلاصة

نقوم في هذا البحث بتقديم طريقة عددية فعالة لحل معادلات فولتيرا التكامل – تفاضلية ذات التأخير المنسافي . نناقش هذه الطريقة بالتفصيل كما نقارنها عن طريق حل بعض الأمثلة العددية. و نعطي علاوة على ذلك تقديرات الخطأ للطريقة المقترحة. ثم نذكر بعض الحالات الخاصة لنتائجنا الأساسية.