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On the existence of Baer triples $(V, G, \mathscr{K})_{\mathbb{F}_2}$ of type 2

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Abstract

The aim of this paper is to construct a full spread $\mathscr{K}$ of length 17 such that $(V, G, \mathscr{K})_{\mathbb{F}_2}$ is a Baer triple of type 2, where $G \cong 2^2$.

Keywords: Baer triple; partial spread
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1. Introduction

Let $V = V(2n, q)$ be an even dimensional vector space over a finite field $\mathbb{F}_q$ and $\mathscr{K}$ a spread on V i.e $\mathscr{K}$ is a set of subspaces $U \leq V$ with $\dim(U) = \frac{1}{2}\dim(V)$, $U \cap W = \{0\}$ for distinct $U, W \in \mathscr{K}$ and $V = \bigcup_{U \in \mathscr{K}} U$. This work was motivated by the following question. Can we find for any finite group G^*, a vector space V with a spread $\mathscr{K}$ such that $G^* \leq G_{\mathscr{K}}$ where $G_{\mathscr{K}}$ is the translation complement group $G_{\mathscr{K}}$ i.e $G_{\mathscr{K}} = \{g \in GL(V) | g$ leaves $\mathscr{K}$ invariant $\}$. The fact that $G_{\mathscr{K}}$ leaves a spread $\mathscr{K}$ invariant, imposes quite strong restrictions on the action of involution of $G_{\mathscr{K}}$ on V and on the elementary abelian 2-subgroups of $G_{\mathscr{K}}$ e.g $\dim C_V(j) = \frac{1}{2}\dim(V)$ or $\dim(X \cap C_V(j)) = \{0\}$ for any $X \in \mathscr{K}$, or the elementary abelian groups tend to act freely V, or that the $G_{\mathscr{K}}$ is quite "small". This situation was demonstrated with examples for $E \leq G_{\mathscr{K}}$, where $E \cong Z_2 \times Z_2$, see Bani-Ata *et al.* (2014), Bani-Ata (2011), Alazemi *et al.* (2015).

A Baer triple of type 2 is a triple $(V, G, \mathscr{K})_{\mathbb{F}_2}$, where G is an elementary abelian 2-group, $G \cong 2^2$ and V is of free rank 2 over a field $K \cong \mathbb{F}_2$, i.e. V is isomorphic as a G-module to a direct sum of the regular G-module $\mathbb{F}_2[G]$ as a group ring, i.e. $V \cong \mathbb{F}_2[G] \oplus \mathbb{F}_2[G]$, and $\mathscr{K}$ is a G-invariant spread on V and if $L \in \mathscr{K}$, then $N_G(L)/C_G(L)$ acts freely on L, where $N_G(L)$ and $C_G(L)$ are the normalizer and the centralizer of L in G respectively. The length of a spread $\mathscr{K}$ is defined to be the order of the spread.

The importance of this study comes from the fact that Baer triple of type 2 (i.e translation planes of order 16 admitting a Baer 4-group) were done on computer, whereas our construction is purely computational where no computer programmes are being used.

2. Notations and earlier results

A Baer triple $(V, G, \mathscr{K})_{\mathbb{F}_2}$, where $G \cong 2^n$, $n \geq 2$, is called a Baer triple of type n.

A group X acts freely on a vector space V over a field K if V is isomorphic as an X-module to a direct sum of the regular X-module $K[X]$, ($K[X]$ is a group ring). It can be easily seen that if X acts freely on V, then there exists an element $b \in V$ such that $\{b^g \,|\, g \in X\}$ is a K-basis of V, see Bani-Ata (2011).

If $(V, G, \mathscr{K})_{\mathbb{F}_2}$ is a Baer triple of type n.

Then $(V,G,\mathscr{K}^g)_{\mathbb{F}_2}$ is a Baer triple too, where $\mathscr{K}^g = \{L^g \mid L \in \mathscr{K}\}$ and $g \in N_{GL(V)}(G)$.

Proposition 2.1 (Huppert & Blackburn (2007))**.** *Let $G \cong 2^n$, $n > 1$ be an elementary abelian 2-group, and*

$$R = \mathbb{F}_2[G] = \left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in \mathbb{F}_2 \right\}.$$

Then, R contains a unique maximal ideal R_0, where $R_0 = \{\sum_{g\in G} \lambda_g g \mid \sum_{g\in G} \lambda_g = 0\} = \{x \in R \mid x^2 = 0\} = \{x \mid xe = 0\}$, *where* $e = \sum_{g\in G} g$.

Theorem 2.1. *Let G be a Baer triple of type n, and $V = R \oplus R$ for $R = \mathbb{F}_2[G]$. Then*

1. *The G-invariant subspaces isomorphic to R have the form vR, where $v \in (R \oplus R) - (R_0 \oplus R_0)$.*
2. *If $X \leq G$, then* $\dim(C_V(X)) = 2|G/X|$, *where $C_V(X) = \{v \in V | v^x = v$ for all $x \in X\}$.*
3. *Let $L \in \mathscr{K}$ and $X \leq G$, then $L^X = L$ if and only if $L \cap C_V(X) \neq \{0\}$ if and only if* $\dim(L \cap C_V(X)) = \frac{1}{2}\dim(C_V(X)) = |G/X|$.
4. *Let $V_0 = [V,G]$, then $V_0 = R_0 \oplus R_0$ and if $V = V_0 + U$, then $V = \bigoplus_{g\in G} U^g$, where* $[V,G] = \{v - gv | g \in G, v \in V\}$.
5. *Let L be a subspace of $V = R \oplus R$ such that $L \oplus L^g = V$ for some $g \in G$, then*
   (a) *$V = [V,G] + L$.*
   (b) *If $V = U \oplus [V,G]$, then $L = \bigoplus_{g\in N_G(l)} U^g$.*

Proof. See Bani-Ata *et al.* (2007).

Note 2.1. *Let $(V,G,\mathscr{K})_{\mathbb{F}_2}$ be a Baer triple of type n, and $X \leq G$. If* $\dim(C_V(X)) = 2m$ *where $m = |G/X|$ then $|\mathscr{K}_{\mathscr{X}}| = 2^m + 1$, where $\mathscr{K}_X = \{\{0\} \neq L \cap C_V(X) | L \in \mathscr{K}\}$.*

Remark 2.1. *If $|G| = 4$, then $|\mathscr{K}| = 2^4 + 1 = 17$. G fixes 3 components, hence there are 3 orbits of length 1. If $1 \neq X \leq G$, then X fixes 5 components, so there are $5 - 3 = 2$ components with stabilizer X, and there are 3 orbits of length 2. It remains $17 - 3 - 2.2 = 8$ components with trivial stabilizer. Hence there are two orbits each of length 4.*

3. Baer Triples of type 2

From Remark 2.1 above, it is clear that the existence of Baer triples of type 2 is equivalent to the construction of a spread $\mathscr{K}$ of length 17 such that the triple $(V,G,\mathscr{K})_{\mathbb{F}_2}$ satisfies the following properties:

1. V is a vector space of dimension $2|G| = 8$ over $\mathbb{F}_2$.
2. G is an elementary abelian 2-group, $G \cong 2^2$ and $G \leq GL(V)$, such that $V \cong \mathbb{F}_2[G] \oplus \mathbb{F}_2[G]$ as a G-module.
3. G leaves a spread $\mathscr{K}$ invariant, such that for all subgroups $X \leq G$ and all $L \in \mathscr{K}$ it holds: $L \cap C_V(X) = \{0\}$ or $\dim(L \cap C_V(X)) = \frac{1}{2}\dim(C_V(X))$, where $C_V(X) = \{v \in V \mid v^x = v \text{ for all } x \in X\}$.

Our plane for the construction goes as follows: First: We construct a partial spread $\{L_v | 0 \neq v \in \mathbb{F}_2^3\}$ of lengths 3. Second: We extend this partial spread to a partial spread of length 9 denoted by $\mathscr{K}$ (X,Y,Z). Third: We extend the partial spread $\mathscr{K}$ (X,Y,Z) to a full spread of length 17.

3.1. Orbits of length 1 and 2

Remark 3.1 (Bani-Ata *et al.* (2007))**.** *We can assume without loss of generality that $G \leq \langle \sigma, \tau \rangle \cong 2^2$, $G \leq GL_8(2)$, where*

$$\sigma = \left[\begin{array}{c|c} I_4 & 0 \\ \hline I_4 & I_4 \end{array}\right], \quad \tau = \left[\begin{array}{cc|cc} I_2 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ I_2 & I_2 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & I_2 & 0 \\ & & I_2 & I_2 \end{array}\right],$$

$$\sigma\tau = \left[\begin{array}{cc|cc} I_2 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ I_2 & I_2 & & \\ \hline I_2 & 0 & I_2 & 0 \\ I_2 & I_2 & I_2 & I_2 \end{array}\right], \quad \textit{and the } 3$$

fixed components of $\mathscr{K}$ by G are $L_v = \{(\lambda_1 v, \lambda_2 v, \lambda_3 v, \lambda_4 v) \mid \lambda_i \in \mathbb{F}_2\}$, $0 \neq v \in \mathbb{F}_2^2$.

Proposition 3.1. *Let $(V, G, \mathscr{K})_{\mathbb{F}_2}$ be a Baer triple of type 2. Then:*

(i) The subspaces $U \leq V$, such that U^G is a partial spread of length 2 with $U^\sigma = U$, $U \cap L_v = \{0\}$ for all $0 \neq v \in \mathbb{F}_2^2$ are precisely the subspaces $\begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}$, where X is an arbitrary 2×2 matrix, $M^2 + M + I = 0$ and there are 16 partial spreads of length 2 fixed by σ.

(ii) The subspaces $U \leq V$, such that U^G is a partial spread of length 2, with normalizer $\langle\tau\rangle$ and $U \cap L_v = \{0\}$ for all $0 \neq v \in \mathbb{F}_2^2$ are exactly the subspaces $U = \begin{bmatrix} M & 0 & I & 0 \\ Y & M & 0 & I \end{bmatrix}$, where Y is an arbitrary 2×2 matrix, $M^2 + M + I = 0$ and there are 16 partial spreads of length 2 fixed by τ.

(iii) The subspaces $U \leq V$, such that U^G is a partial spread of length 2, with stabilizer $\sigma\tau$, $U \cap U^\sigma = \{0\}$, $U^{\sigma\tau} = U$ and $U \cap L_v = \{0\}$ for all $0 \neq v \in \mathbb{F}_2^2$ are exactly the subspaces $U = \begin{bmatrix} M & I & I & 0 \\ Z & M+I & 0 & I \end{bmatrix}$, where Z is an arbitrary 2×2 matrix, $M^2 + M + I = 0$ and there are 16 partial spreads of length 2 fixed by $\sigma\tau$.

Proof. 1. As $N_G(U) = \langle\sigma\rangle$, then by Theorem 2.1, $U \cap U^\tau = \{0\}$, if and only if $U \cap C_V(\tau) = \{0\}$ if and only if $V = U \oplus C_V(\tau)$, where $C_V(\tau) = \{(x, 0, y, 0) \mid x, y \in \mathbb{F}_2^2\}$. From this it follows that all subspaces U with $U \cap C_V(\tau) = \{0\}$ can be generated by matrices $U = \begin{bmatrix} A & I & B & 0 \\ C & 0 & D & I \end{bmatrix}$ for 2×2 matrices A, B, C, and D. This means that the rows of 4×8 matrices are a base of U, and U has a base consisting of the matrix above. In this case

$$U^\sigma = \begin{bmatrix} A+B & I & B & 0 \\ C+D & I & D & I \end{bmatrix} = \begin{bmatrix} A+B & I & B & 0 \\ A+B+C+D & 0 & B+D & I \end{bmatrix}.$$

For convenience we identify matrices with subspaces spanned by their rows if no confusion occurs. Hence $U = U^\sigma$ if and only if $U = \begin{bmatrix} A & I & 0 & 0 \\ C & 0 & A & I \end{bmatrix}$.

Moreover $U \cap L_v = \{0\}$, for all $0 \neq v \in \mathbb{F}_2^2$, if and only if $vA \neq v$. This is equivalent to $A^2 + A + I = 0$, where $A \in GL_2(2)$, and order $A = 3$. So, putting all together, the subspaces $U \leq V$ such that $U^\sigma = U$, $U^G = \{U, U^\tau\}$, $0 \neq v \in \mathbb{F}_2^2$ are precisely the subspaces $\begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}$ where $M^2 + M + I = 0$ i.e. $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ or $M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The number of these subspaces is 32. But as

$$\begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}^\tau = \begin{bmatrix} M+I & I & 0 & 0 \\ X+I & 0 & M+I & I \end{bmatrix},$$

it follows that there are exactly 16 partial spreads fixed by σ.

The proof of (ii) and (iii) can be argued similarly. This completes the proof.

As a first step, we classify (up to conjugation in $C_{GL(V)}(G)$) partial spreads of length 9 in Baer triples of type 2 where these partial spreads are invariant under G, and G has 3 fixed components and 3 orbits of length 2. So, let $V_0 = [V, G] = \{(x_1, x_2, x_3, 0) : x_i \in \mathbb{F}_2^2\}$. The subspaces U with $V = V_0 \oplus U$ are of shape $U = [A, B, C, I]$ for 2×2 matrices A, B, and C. If L^G is a partial spread with $N_G(L) = \langle\sigma\rangle$, then $L = U + U^\sigma$ (by Theorem 2.1), and $U + U^\sigma = \begin{bmatrix} A & B & C & I \\ A+C & B+I & C & I \end{bmatrix} = \begin{bmatrix} C & I & 0 & 0 \\ X & 0 & C & I \end{bmatrix}$, where X is an arbitrary 2×2 matrix.

If L^G is a partial spread with $N_G(L) = \langle\tau\rangle$, then $L = U + U^\tau$ and if $U = [A, B, C, I]$, then $U+U^{\tau}=\begin{bmatrix} A & B & C & I \\ A+B & B & C+I & I \end{bmatrix}=\begin{bmatrix} B & 0 & I & 0 \\ Y & B & 0 & I \end{bmatrix}$, where Y is an arbitrary 2×2 matrix.

If L^G is a partial spread with $N_G(L)=\langle\sigma,\tau\rangle$, then $L=U+U^{\sigma\tau}$ and if $U=[A,B,C,I]$, then

$U+U^{\sigma\tau}$

$=\begin{bmatrix} A & B & C & I \\ A+B+C+I & B+I & C+I & I \end{bmatrix}$

$=\begin{bmatrix} V & I & I & 0 \\ Z & V+I & 0 & I \end{bmatrix}$, where V is an arbitrary 2×2 matrix. Putting all these results together, we get Proposition 3.2.

Proposition 3.2. *(a) Partial spreads L^G with $N_G(L)=\langle\sigma\rangle$ are generated by subspaces $L=\begin{bmatrix} A & I & 0 & 0 \\ X & 0 & A & I \end{bmatrix}$, for 2×2 matrices A and X.*

(b) Partial spreads L^G with $N_G(L)=\langle\tau\rangle$ are generated by subspaces $L=\begin{bmatrix} B & 0 & I & 0 \\ Y & B & 0 & I \end{bmatrix}$, for 2×2 matrices B and Y.

(c) Partial spreads L^G with $N_G(L)=\langle\sigma\tau\rangle$ are generated by subspaces $L=\begin{bmatrix} C & I & I & 0 \\ Z & C+I & 0 & I \end{bmatrix}$, for 2×2 matrices C and Z.

Remark 3.2. *In Theorem 2.1(a), it is proved that if $L\le V$ is a G-invariant subspace such that G acts freely on L, then $L=vR$ for some $V\backslash V_0$. Furthermore, if L_1, L_2, and L_3 are 3 such subspaces with $L_i\cap L_j=\{0\}$ for $i\neq j$, then there exists $x\in C_{GL(V)}(G)\cong GL_2(R)$ such that $L_1^x=(1,0)R, L_2^x=(0,1)R, L_3^x=(1,1)R$. In particular $C_{GL(V)}(G)$ is transitive on such triples (L_1,L_2,L_3).*

In our notation above, the subspaces $L_v=\{(\lambda_1 v,\lambda_2 v,\lambda_3 v,\lambda_4 v)\,|\,\lambda_i\in\mathbb{F}_2\}, 0\neq v\in\mathbb{F}_2^2$, form a triple of such subspaces, this will be fixed in our computations. In the next step, we want to find all partial spreads of length 2, as given above, which are compatible with spaces L_v, i.e. which have trivial intersection with the L_v's. So, a subspace $\begin{bmatrix} C & I & 0 & 0 \\ X & 0 & C & I \end{bmatrix}$ is compatible if and only if $vC\neq 0,v$, for all $v\in\mathbb{F}_2^2$, which is equivalent to $C^2+C+I=0$.

A subspace $\begin{bmatrix} B & 0 & I & 0 \\ Y & B & 0 & I \end{bmatrix}$ is compatible if and only if $vB\neq 0,v$, for all $v\in\mathbb{F}_2^2$, which is equivalent to $B^2+B+I=0$.

A subspace $\begin{bmatrix} C & I & I & 0 \\ Z & C+I & 0 & I \end{bmatrix}$ is compatible if and only if $vC\neq 0,v$, for all $v\in\mathbb{F}_2^2$, which is equivalent to $C^2+C+I=0$, and there exist exactly 2 matrices $M\in GL_2(2)$ with $M^2+M+I=0$.

From Reamrk 3.2, it follows that we have the following compatible candidates:

$S_X=\begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}$,

$\begin{bmatrix} M+I & I & 0 & 0 \\ X & 0 & M+I & I \end{bmatrix}$

$T_Y=\begin{bmatrix} M & 0 & I & 0 \\ Y & M & 0 & I \end{bmatrix}$,

$\begin{bmatrix} M+I & 0 & I & 0 \\ Y & M+I & 0 & I \end{bmatrix}$,

$ST_Z=\begin{bmatrix} M & I & I & 0 \\ Z & M+I & 0 & I \end{bmatrix}$.

These subspaces must be also compatible among themselves and hence we prove Lemma 3.1.

Lemma 3.1. *(i) S_X and T_Y are compatible if and only if $X+Y$ is non-singular.*

(ii) T_Y and ST_Z are compatible if and only if $Y+Z$ is non-singular.

(iii) S_X and ST_Z are compatible if and only if $X+Z+I$ is non-singular.

Proof. We only do the proof of part (i). S_X and T_Y are compatible if and only if the matrix

$$\begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \\ M & 0 & I & 0 \\ Y & M & 0 & I \end{bmatrix}$$

is non-singular which is equivalent to

$$\begin{bmatrix} M & I & 0 \\ M & 0 & I \\ X+Y & 0 & 0 \end{bmatrix}$$

is non-singular. It follows that S_X and T_Y are compatible if $X+Y$ is non-singular.

Note 3.1. *Let $X+Y=g\in GL_2(2)$, $Y+Z=h\in GL_2(2)$. Hence $X=Y+g, Z=Y+h$ and $X+Z+I=g+h+I$ or equivalently we need to classify all pairs $(g,h)\in GL_2(2)\times GL_2(2)$ such that $g+h+I\in GL_2(2)$. Hence we prove Proposition 3.3.*

Proposition 3.3. *The admissible triples (X,Y,Z) for which $(g,h)\in GL_2(2)\times GL_2(2)$ and $g+h+I\in GL_2(2)$ are: For a 2×2 matrix Y, we have*

(i) $(Y+g, Y, Y+g)$, $g\in GL_2(2)$, *and there are* 96 *triples of this type.*

(ii) $(Y+I, Y, Y+h)$, $I\neq h\in GL_2(2)$, *and there are* 80 *triples of this type.*

(iii) $(Y+g, Y, Y+I)$, $I\neq g\in GL_2(2)$, *and there are* 80 *triples of this type.*

Proof. For the pairs (g,h), we have three types

Type 1: $g=h$, this implies that we have 6 pairs (g,g) with $g\in GL_2(2)$.

Type 2: $g=I$, $h\neq I$, this implies that there are 5 pairs (I,h), with $h\in GL_2(2)$.

Type 3: $g\neq I$, $h=I$, this gives 5 pairs of this type.

We claim that all solutions have been found. To prove this, let m be the set of all 2×2 matrices over $\mathbb{F}_2$, m_0 and m_1 are as follows:

$$m_0=\left\{\begin{bmatrix} a & b\\ a+b & a\end{bmatrix} \mid a,b\in\mathbb{F}_2\right\},\text{ and}$$

$$m_1=\left\{\begin{bmatrix} a & b\\ b & a+b\end{bmatrix} \mid a,b\in\mathbb{F}_2\right\}.$$

From this it follows that $|m_i|=4$, $i=1,2$, $m=m_0\oplus m_1$, m_0,m_1 are closed under addition, m_1 is closed under multiplication, $m_1\cong\mathbb{F}_4$ and the 3 non-zero elements in m_0 are precisely the involutions in $GL_2(2)$.

Assume that we have a further solution (g,h) with $I\neq g\neq h\neq I$ such that $g+h+I\in GL_2(2)$, it is clear that $g\in m_0\cup m_1$, $h\in m_0\cup m_1$, and if $0\neq M_0\in m_0, 0\neq M_1\in m_1$, then M_0+M_1 is singular.

- Case 1: If $I\neq g\in m_1$, then $0\neq g+I\in m_1$, hence h must be in m_1 as it is proved above. Thus, $I\neq g\neq h\neq I$ are contained in m_1. This implies that $h=g^2$ and $g+h+I=g+g^2+I=0$, a contradiction. Likewise if $h\in m_1$, we get a contradiction.

- Case 2: If $g,h\in m_0$, then $0\neq g+h\in m_0$. Hence, $g+h$ is an involution in $GL_2(2)$ and $g+h+I$ is singular because $g+h\in m_0$, $I\in M_1$. Hence a contradiction.

So we have a complete classification (up to conjugation in $C_{GL(V)}(G)$) of partial spreads of length 9 which are invariant under G where G has 3 fixed points and 3 orbits of length 2. □

The above observations can now be summarized in the following theorem.

Theorem 3.1. *Let $G=\langle\sigma,\tau\rangle\leq GL(V)$ with $V=\mathbb{F}_2^8$, where*

$$\sigma=\left[\begin{array}{c|c} I_4 & 0\\ \hline I_4 & I_4\end{array}\right],\ \tau=\left[\begin{array}{cc|c} I_2 & 0 & \\ I_2 & I_2 & 0\\ \hline & & I_2\quad 0\\ 0 & & I_2\quad I_2\end{array}\right]$$

such that $(V,G,\mathscr{K})_{\mathbb{F}_2}$ is a Baer triple of type 2, then

1. *Up to conjugation in $N_{GL(V)}(G)$, the three fixed components in $\mathscr{K}$ are the subspace*

   $$L_v=\{(\lambda_1 v,\lambda_2 v,\lambda_3 v,\lambda_4 v)\mid \lambda_i\in\mathbb{F}_2\},$$

   $0\neq v\in\mathbb{F}_2^2$.

2. *$\mathscr{K}$ contains* 3 *partial spreads of length* 2*, U_σ^G with $N_G(U_\sigma)=\langle\sigma\rangle$, U_τ^G with $N_G(U_\tau)=\langle\tau\rangle$ and $U_{\sigma\tau}^G$ with $N_G(U_{\sigma\tau})=\langle\sigma\tau\rangle$.*

3. *If $M\in GL_2(2)$ with $M^2+M+I=0$, i.e. $M\in GL_2(2)$ of order* 3*, then there exist $X,Y,Z\in Mat(2\times 2,\mathbb{F}_2)$ such that*

- $U^G_\sigma = U^G_\sigma(X) = \begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}$ *and* $\begin{bmatrix} \overline{M} & I & 0 & 0 \\ X & 0 & \overline{M} & I \end{bmatrix}$, *where* $\overline{M} = M^2 = M+I$,
- $U^G_\tau = U^G_\tau(Y) = \begin{bmatrix} M & 0 & I & 0 \\ Y & M & 0 & I \end{bmatrix}$ *and* $\begin{bmatrix} \overline{M} & 0 & I & 0 \\ Y & \overline{M} & 0 & I \end{bmatrix}$,
- $U^G_{\sigma\tau} = U^G_{\sigma\tau}(Z) = \begin{bmatrix} \overline{M} & I & I & 0 \\ Z & M & 0 & I \end{bmatrix}$ *and* $\begin{bmatrix} M & I & I & 0 \\ Z & \overline{M} & 0 & I \end{bmatrix}$

and the partial spreads $U^G_\sigma(X)$, $U^G_\tau(Y)$ *and* $U^G_{\sigma\tau}(Z)$ *are compatible (i.e. distinct elements have trivial intersection) if and only if* $X+Y$, $Z+Y$, *and* $X+Z+I$ *are non-singular.*

4. *The triples* X,Y,Z *satisfying these conditions are precisely the triples:*

$$\begin{array}{llll} X & Y & Z, & \\ Y+g & Y & Y+g, & g\in GL_2(2), \\ Y+I & Y & Y+h, & I\neq h\in GL_2(2), \\ Y+g & Y & Y+Z, & I\neq g\in GL_2(2). \end{array}$$

This gives a total of 256 *admissible partial spreads invariant under* G *having the prescribed orbit structure.*

Note 3.2. *Let* H *be the stabilizer of* $\{L_v\,|\,0\neq v\}$ *in* $N_{GL(V)}(G)$. *Then* H *acts on these partial spreads constructed above, i.e. if* $\sum = \{L_1,L_2,\cdots,L_g\}$ *is a partial spread, as above, and* $h\in H$, *then* $\sum^h = \{L_1^h,L_2^h,\cdots,L_g^h\}$ *is also a partial spread as bove.*

The group H can be described explicitly as follows: If H_0 denotes the stabilizer of $\{L_v\,|\,v\neq 0\}$ in $C_{GL(V)}(G)$, then $H_0 = M:D$ where M consists of all matrices

$$X_{abc} = \left[\begin{array}{cc|cc} I & 0 & \multicolumn{2}{c}{0} \\ aI & I & & \\ \hline bI & 0 & I & 0 \\ cI & bI & aI & I \end{array}\right], \text{for } a,b,c\in\mathbb{F}_2,$$

D consists of all matrices $\hat{g} = \begin{bmatrix} g & & & \\ & g & & \\ & & g & \\ & & & g \end{bmatrix}, g\in GL_2(2)$, where $L_v^{\hat{g}} = L_{vg}$, and X_{abc} fixes all components L_v. Hence $H = H_0 : K$ where K consists of all matrices

$$\left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ (a+b+1)I & aI & bI & 0 \\ \hline (c+d+1)I & cI & dI & 0 \\ 0 & 0 & 0 & I \end{array}\right],$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(2)$.

In particular, $|H| = 288$, $G \leq X = \{X_{abc}\,|\,a,b,c\in\mathbb{F}_2\}$ of index 2. To see the action of H on the triples (X,Y,Z) we prove Proposition 3.4.

Proposition 3.4. *The action of* H *on the triples* (X,Y,Z) *is generated by the permutations:*

$$\begin{array}{lcl} (X,Y,Z) & \to & (X+I,Y+I,Z+I), \\ (X,Y,Z) & \to & (X^g,Y^g,Z^g), g\in GL_2(2), \\ (X,Y,Z) & \to & (X,Y,Z+I), \\ (X,Y,Z) & \to & (Z,X+I,Y+I), \\ (X,Y,Z) & \to & (Z,Y,X), \\ (X,Y,Z) & \to & (Z+I,X,Y), \\ (X,Y,Z) & \to & (Y+I,X+I,Z). \end{array}$$

Proof. The action of $t = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & I \end{bmatrix}$ is as follows:

$U_\sigma(X)^t = U_\sigma(X+I)$, $U_\tau(Y)^t = U_\tau(Y+I)$, $U_{\sigma\tau}(Z)^t = U_{\sigma\tau}(Z+I)$ and $\hat{g}$ acts as $U_\sigma(X)^{\hat{g}} = U_\sigma(X)^g$, $U_\tau(Y)^{\hat{g}} = U_\tau(Y)^g$, $U_{\sigma\tau}(Z)^{\hat{g}} = U_{\sigma\tau}(Z)^g$. The matrix $J = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ acts as $U_\sigma(X)^J = U_\tau(X)$, $U_\tau(Y)^J = U_\sigma(Y)$, $U_{\sigma\tau}(Z)^J = U_{\sigma\tau}(Z+I)$ and $K = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ I & I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ acts as $U_\sigma(X)^K = U_\tau(X+I)$, $U_\tau(Y)^K = U_{\sigma\tau}(Y+I)$, $U_{\sigma\tau}(Z)^K = U_\sigma(Z)$. From these actions the action of H follows.

Note 3.3. *In order to complete the partial spreads of length 9 to full spreads of length 17, we have to find a G-invariant set of 8 subspaces which form a partial spread and decomposes in 2 G-orbits of length 4, and must be compatible with one of the partial spreads of length 9 from above.*

We can denote the partial spread above by $\mathscr{K}(X,Y,Z) = \{L_v \mid v \neq 0\} \cup U_\sigma(X)^G \cup U_\tau(Y)^G \cup U_{\sigma\tau}(Z)^G$.

In order to complete the spread $\mathscr{K}(X,Y,Z)$ to a full spread $\mathscr{K}$ of length 17, we have to find subspaces R_1, R_2 such that $|R_1^G| = |R_2^G| = 4$ and $\mathscr{K} = \mathscr{K}(X,Y,Z) \cup R_1^G \cup R_2^G$ is a spread. In particular, R_i^G, $i = 1,2$, must be a partial spread of length 4. So, we prove Observation 3.1.

Observation 3.1. *For a subspace R of V, if $1 \neq x \in G$, $R \cap R^x \neq \{0\}$ if and only if $R \cap C_V(x) \neq \{0\}$.*

Proof. If $R \cap C_V(x) \neq \{0\}$, then there exists $0 \neq v \in R$ with $v^x = v$ and hence $v = v^x \in R \cap R^x$ so that $R \cap R^x \neq \{0\}$. Assume $D = R \cap R^x \neq \{0\}$. As $x^2 = 1$, the space D is x-invariant, and x fixes some non-zero $v \in D$. Hence $\{0\} \neq D \cap C_V(x) \leq R \cap C_V(x)$.

From Observation 3.1, one has: If $R < V$ such that $\dim(R) = \frac{1}{2}\dim(V)$, then R^G is a partial spread if and only if $R \cap R^x = \{0\}$ for all $1 \neq x \in G$ if and only if $R \cap C_V(x) = \{0\}$ for all $1 \neq x \in G$.

Remark 3.3. *For the subspaces R_i, $i = 1,2$, it holds that $R_i \cap C_V(x) = \{0\}$ for all $1 \neq x \in G$ if and only if $V = R_i \oplus C_V(x)$. Hence $C_V(\sigma) = \{(x,y,0,0) \mid x,y \in \mathbb{F}_2^2\}$ and $R \cap C_V(\sigma) = \{0\}$ if and only if R is of type $\begin{bmatrix} A & B & I & 0 \\ C & D & 0 & I \end{bmatrix}$. $C_V(\tau) = \{(x,0,y,0) \mid x,y \in \mathbb{F}_2^2\}$ and $R \cap C_V(\tau) = \{0\}$ if and only if $vB \neq 0$ for all $0 \neq v \in \mathbb{F}_2^2$ and $C_V(\sigma\tau) = \{(x,y,y,0) \mid x,y \in \mathbb{F}_2^2\}$ and $R \cap C_V(\sigma\tau) = \{0\}$ if and only if $vB \neq v$ for all $0 \neq v \in \mathbb{F}_2^2$.*

So, the subspaces R such that R^G is a partial spread of length $|G|$ are exactly the subspaces $\begin{bmatrix} A & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ with $M^2 + M + I = 0$. There are $2(16)^3$ such subspaces and R^G consists of the subspaces

$$\begin{bmatrix} A & M & I & 0 \\ C & D & 0 & I \end{bmatrix}, \quad \begin{bmatrix} A+I & M & I & 0 \\ C & D+I & 0 & I \end{bmatrix},$$

$$\begin{bmatrix} A+M & M & I & 0 \\ A+C+D+M & D+M & 0 & I \end{bmatrix}, \quad \text{and}$$

$$\begin{bmatrix} A+M+I & M & I & 0 \\ A+C+D+M & D+M+I & 0 & I \end{bmatrix}.$$

Remark 3.4. *Let $m = m_0 \oplus m_1$ where $m_0 = \left\{ \begin{bmatrix} a & b \\ a+b & a \end{bmatrix} \mid a,b \in \mathbb{F}_2 \right\}$, and $m_1 = \left\{ \begin{bmatrix} a & b \\ a & a+b \end{bmatrix} \mid a,b \in \mathbb{F}_2 \right\}$, then R^G contains a subspace $\begin{bmatrix} A_0 & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ with $A_0 \in m_0$.*

Proof. It is clear that m_1 is the field $\{0, I, M, \overline{M}\}$. If $A \in m$, then $A = M_0 + M_1$ where $M_0 \in m_0$ and $M_1 \in m_1$. Then $A + M_1 = M_0 \in m_0$. So R^G contains a subspace $\begin{bmatrix} A_0 & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ with $A_0 \in m_0$. Without loss of generality we can assume that $A \in m_0$, and we have $2,048$ such subspaces.

Proposition 3.5. *All the vectors of shape $(x_1,x_2,x_3,0)$ contained in a component of R_1^G, where $R_1^G = \begin{bmatrix} A & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$, if and only if $x_2 = x_3M$ and $x_1 = x_3A$, where none of the vectors $(y_1,y_2,y_3,0)$ with $y_2 = y_3\overline{M} = y_3M^2$ lies in a component of R_1^G.*

Proof. Let $X = (x_1,x_2,x_3,0)$ be a nonzero vector. Then $X \in L_v$ for some $v \neq 0$ if and only if $\dim(\langle x_1,x_2,x_3 \rangle) = 1$.
$X \in U_\sigma^G$ if and only if $x_3 = 0$ and $x_1 = x_2M$,
$X \in U_\tau^G$ if and only if $x_1 = x_3M$ and $x_2 = 0$,
$X \in U_{\sigma\tau}^G$ if and only if $x_2 = x_3$ and $x_1 = x_2\overline{M}$.

In particular, if $\dim\langle x_2,x_3 \rangle = 2$, then $(x_1,x_2,x_3,0)$ is not contained in any component of $\mathscr{K}(X,Y,Z)$, hence $X \in R_1$ if and only if $x_2 = x_3M$ and $x_1 = x_3A$.

Corollary 3.1. *Let $R_1 = \begin{bmatrix} A_0 & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ and let $X = (0, v\overline{M}, v, 0)$, where $v \neq 0$, then X is not contained in any component of R_1^G where the spaces R_1^G are of the form $U = \begin{bmatrix} A' & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$, and the vectors $(y_1,y_2,y_3,0)$*

in U are of shape $(ZA', ZM, Z, 0)$ for some $Z \in \mathbb{F}_2^2$. In particular, if $(x_1, x_2, x_3, 0)$ is contained in such space then $x_2 = x_3 M$. If $x_3 \neq 0$, then $x_3 M \neq x_3\overline{M} = x_3 M + x_3$. From this, we get vectors $X = (0, v\overline{M}, v, 0)$ for $0 \neq v \in \mathbb{F}_2^2$, are not contained in components of R_1^G, and they are also not contained in a component of $\mathcal{K}(X,Y,Z)$. Hence X must be contained in a component of R_2^G. This implies that $R_2 = \begin{bmatrix} A' & \overline{M} & I & 0 \\ C' & D' & 0 & I \end{bmatrix}$.

3.2. Orbits of length 4

In this section, we investigate the orbits R_1^G and R_2^G of length 4.

As R_1^G is of shape $\begin{bmatrix} A & M & I & 0 \\ C & D & 0 & M \end{bmatrix}$ and R_2^G is of shape $\begin{bmatrix} A' & \overline{M} & I & 0 \\ C' & D' & 0 & M \end{bmatrix}$ for 2×2 matrices A, C, D, A', C', D' then R_1^G consists of the following spaces:

$$R_1 = \begin{bmatrix} A & M & I & 0 \\ C & D & 0 & M \end{bmatrix},$$

$$R_1^\sigma = \begin{bmatrix} A+I & M & I & 0 \\ C & D+I & 0 & M \end{bmatrix},$$

$$R_1^\tau = \begin{bmatrix} A+M & M & I & 0 \\ A+C+D+M & D+M & 0 & I \end{bmatrix}, \text{ and}$$

$$R_1^{\sigma\tau} = \begin{bmatrix} A+M+I & M & I & 0 \\ A+C+D+M & D+M+I & 0 & I \end{bmatrix}.$$

Let $m = Mat(2\times 2, \mathbb{F}_2)$, then $m = m_0 + m_1$, where

$$m_0 = \left\{ \begin{bmatrix} a & b \\ a+b & a \end{bmatrix} : a, b \in \mathbb{F}_2 \right\} = \{0\} \cup \{x \in GL_2(2) \,|\, x^2 = 1\}, \text{ and}$$

$$m_1 = \left\{ \begin{bmatrix} a & b \\ b & a+b \end{bmatrix} : a, b \in \mathbb{F}_2 \right\} = \{0, I, M, \overline{M}\} \cong \mathbb{F}_4$$

Spaces in R_1^G can be written in a form $\begin{bmatrix} A+f & M & I & 0 \\ C_f & D+f & 0 & I \end{bmatrix}$ where $f \in m_1$ and

$$C_f = \begin{cases} C & \text{if } f = 0, I, \\ A+C+D+M & \text{if } f = M, \overline{M}. \end{cases}$$

R_1^G contains a unique $\begin{bmatrix} A & M & I & 0 \\ C & D & 0 & M \end{bmatrix}$ with $D = D_0 + D_1$ where $D_0 \in m_0$ and $D_1 \in m_1$, take $f = D_1$, then $D + f = D_0 \in m_0$.

Now we find the conditions such that $R_1^G \cap L_v = \{0\}$ for all $g \in G, v \neq 0$.

Note 3.4. *If $R_1 \cap L_v = \{0\}$ for all $v \neq 0$, then $R_1^g \cap L_v = \{0\}$ for all $v \neq 0$, as $R_1^g \cap L_v = R_1^g \cap L_v^g = (R_1 \cap L_v)^g$ for all $g \in G$.*

So we have to investigate only $R_1 = \begin{bmatrix} A & M & I & 0 \\ C & D & 0 & M \end{bmatrix}$ with $D \in m_0$.

Proposition 3.6. *1. Let*

$$R_1 = \begin{bmatrix} A & M & I & 0 \\ C & D & 0 & M \end{bmatrix}$$

with $D \in m_0$. If $D = 0$, then R_1 is compatible with L_v if and only if $C = M$ or $C = \overline{M}$ and $R_1 = \begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$ or $R_1 = \begin{bmatrix} A & M & I & 0 \\ \overline{M} & 0 & 0 & I \end{bmatrix}$.

2. If $D = j \in GL_2(2)$ is an involution and $R_1 = \begin{bmatrix} A & M & I & 0 \\ C & j & 0 & I \end{bmatrix}$, then $R_1 \cap L_v = \{0\}$ for all $v \neq 0$ if and only if

(α) $v \neq 0$ and $v_j = v$ implies $vC \neq 0, v$,

(β) $v \neq 0$ and $v(M+j) \in \{0, v\}$ implies $v(A+C) \notin \{0, v\}$.

Proof. 1. R_1 contains all vectors $(vC, vD, 0, v) = [C\ D\ 0\ I]$. As $D = 0$, then R_1 contains all vectors $(vC, 0, 0, v)$ for all $v \neq 0$. As $R_1 \cap L_v = \{0\}$ and $(0,0,0,v) \in L_v$, $(v,0,0,0) \in L_v$, it follows that $vC \neq 0, v$ for all $v \neq 0$. Hence $C = M$ or $\overline{M}$.

Conversely if $vC \neq 0, v$ for all $v \neq 0$, then $\begin{bmatrix} A & M & I & 0 \\ C & 0 & 0 & I \end{bmatrix} \cap L_v = \{0\}$ for all $v \neq \{0\}$ implies that vectors $(xA + yC, xM, x, y)$ in R_1 are in L_v if $x = 0$, which implies that $(yC, 0, 0, y) \in L_v$, a contradiction as $yC \neq 0, y$.

2. Elements in R_1 are of the form $(xA + yC, xM + yj, x, y)$. If such a vector is in L_v, then $y \neq 0$ and we have only to check vectors $(yC, yj, 0, y)$ and $(x(A + C), x(M + j), x, x)$.

   The involution j has a unique fixed vector $0 \neq u \in \mathbb{F}_2^2$ such that

$$\begin{aligned} u(M+j) &= uM + u = u\overline{M}, \\ uM(M+j) &= uM^2 + uMj = u\overline{M} \\ &= uj\overline{M} = 0, \\ u\overline{M}(M+j) &= u\overline{M}M + u\overline{M}j \\ &= u + ujM = u + uM = u\overline{M}. \end{aligned}$$

   Hence the conditions (α) and (β) are equivalent to

   (α') $uC \neq 0, u$,

   (β') $uM(A+C) \neq 0, uM$ or $uM(A+C) \in \{u, u\overline{M}\}$
   $u\overline{M}(A+C) \neq 0, u\overline{M}$ or $u\overline{M}(A+C) \in \{u, uM\}$

   As $uM, u\overline{M}$ are independent, there exist 4 matrices $A+C$ which satisfy (β'), these matrices are $M, \overline{M}, j, j+I$. Hence if $D = j$, then $\begin{bmatrix} A & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ is compatible with L_v, $v \neq 0$ if and only if $uC \neq 0, u$ for $u \neq 0$, $uD = u$ and $A + C \in \{M, \overline{M}, D, D+I\}$.

For a given involution D there exist 8 matrices C with $uC \neq 0, u$ and for such C there exist exactly $4 \cdot 8 = 32$ compatible spaces $R_1 = \begin{bmatrix} A & M & I & 0 \\ C & D & 0 & I \end{bmatrix}$ for a given involution D, and likewise for $\overline{M}$.
The number of G-transitive partial spreads of length 4 which are compatible with L_v is then 2^8

Lemma 3.2. *Let* $L = \begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & 0 \end{bmatrix}$, $M^2 + M + I = 0$ *be a representative of a partial spread of length* 4*, then*

1. *L is compatible with u_σ and $\overline{u}_\sigma$ if and only if $X + MA + \overline{M}$ and $X + \overline{M}A + I$ are non-singular.*

2. *L is compatible with u_τ and $\overline{u}_\tau$ if and only if $Y + A$ and $Y + MA + \overline{M}$ are non-singular.*

3. *L is compatible with $u_{\sigma\tau}$ and $\overline{u}_{\sigma\tau}$ if and only if $Z + A$ and $Z + \overline{M}A$ are non-singular.*

Proof. L is compatible with L_v's (by Proposition 3.6).

1. L is compatible with $u_\tau = \begin{bmatrix} M & 0 & I & 0 \\ Y & M & 0 & I \end{bmatrix}$ and with $\overline{u}_\tau = \begin{bmatrix} \overline{M} & 0 & I & 0 \\ Y & \overline{M} & 0 & I \end{bmatrix}$ if and only if $\begin{bmatrix} A+M & M \\ M+Y & M \end{bmatrix}$ and $\begin{bmatrix} A+\overline{M} & M \\ M+Y & \overline{M} \end{bmatrix}$ are both non-singular which is equivalent to $A + Y$ and $M + Y + M(A + \overline{M}) = MA + Y + \overline{M}$ are both non-singular, respectively.

2. L is compatible with $u_\sigma = \begin{bmatrix} M & I & 0 & 0 \\ X & 0 & M & I \end{bmatrix}$ and with $\overline{u}_\sigma = \begin{bmatrix} \overline{M} & I & 0 & 0 \\ X & 0 & \overline{M} & I \end{bmatrix}$ if and only if $X + MA + \overline{M}$ and $X + \overline{M}A + I$ are non-singular respectively, for: Let $(vM + wX, v, wM, w) \in u_\sigma$, then $(vM + wX, v, wM, w) \in L$ if and only if $(vM + wX, v, wM, w) = (wMA + wM, wM^2, wM, w)$ if and only if $v = wM^2$ and $wM + wX = wMA + wM$. Hence L is compatible with u_σ if and only if $X + MA + \overline{M}$ is non-singular. Also $(v\overline{M} + wX, v, w\overline{M}, w) \in \overline{u}_\sigma \cap L$ if and only if $(v\overline{M} + wX, v, w\overline{M}, w) = (w\overline{M}A + wM, w\overline{M}M, w\overline{M}, w)$ if and only if $v = w$ and $w\overline{M} + wX = w\overline{M}A + wM$. From this it follows that L is compatible with $\overline{u}_\sigma$ if and only if $X + \overline{M}A + I$ is non-singular.

3. L is compatible with $u_{\sigma\tau} = \begin{bmatrix} M & I & I & 0 \\ Z & M & 0 & I \end{bmatrix}$ and with $\overline{u}_{\sigma\tau} =$

$\begin{bmatrix} \overline{M} & I & I & 0 \\ Z & M & 0 & I \end{bmatrix}$ if and only if $\begin{bmatrix} A+M & \overline{M} \\ M+Z & \overline{M} \end{bmatrix}$ and $\begin{bmatrix} A+\overline{M} & \overline{M} \\ M+Z & M \end{bmatrix}$ are non-singular which is equivalent to $A+Z$ $A+\overline{M}+M(M+Z)=A+MZ$ are non-singular, and $A+MZ$ is non-singular if and only if $\overline{M}A+Z$ is non-singular. Also from Lemma 3.1, we have $X+Y, Y+Z$ and $X+Z+I$ are non-singular.

Lemma 3.3. *$L=\begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$ is compatible with $u_{\sigma\tau}=\begin{bmatrix} M & I & I & 0 \\ Z & \overline{M} & 0 & I \end{bmatrix}$ and with $\overline{u}_{\sigma\tau}=\begin{bmatrix} \overline{M} & I & I & 0 \\ Z & M & 0 & I \end{bmatrix}$ if and only if*

1. *$A,Z\in m_0$ and $Z\neq A,\overline{M}A$,*
2. *$A,Z\in m_1$ and $Z\neq A,\overline{M}A$,*
3. *$A=A_0+A_1$ with $0\neq A_i\in m_i$ for $i=0,1$ and $Z=A_0+\overline{M}A_1$ or $Z=\overline{M}A_0+A_1$.*

Proof. As L is compatible with $u_{\sigma\tau}$ and $\overline{u}_{\sigma\tau}$, then $A+Z$ and $Z+\overline{M}A$ are non-singular by Lemma 3.2. Let $Z+A=(Z_0+A_0)+(Z_1+A_1)$ and $Z+\overline{M}A=(Z_0+\overline{M}A_0)+(Z_1+\overline{M}A_1)$ where $Z_0,A_0\in m_0$ and $Z_1,A_1\in m_1$, hence we have the following cases

- Case 1: $Z_0+A_0=Z_0+\overline{M}A_0=0$ implies $Z_0=A_0=0$, then $Z,A\in m_1$ and $Z\neq A,\overline{M}A$.
- Case 2: $Z_1+A_1=Z_1+\overline{M}A_1=0$ implies $Z_1=A_1=0$, then $Z,A\in m_0$ and $Z\neq A,\overline{M}A$.
- Case 3: $Z_0=A_0\neq 0$ implies $Z_1+A_1\neq 0$, $Z_1+\overline{M}A_1=0$, $A_1\neq Z_1$ if and only if $A_1\neq 0$, $Z_1=\overline{M}A_1\neq 0$. Thus, $A=A_0+A_1$ with $A_i\in m_i$ for $i=0,1$ implies that $Z=\overline{M}A_0+A_1$.
- Case 4: $Z_0\neq A_0, Z_1=A_1, Z_0+\overline{M}A_0=0$, and $Z_1+\overline{M}A_1\neq 0$. Hence $A_0\neq Z_0=\overline{M}A_0$, $Z_1=A_1\neq\overline{M}A_1$. From this it follows that $Z=\overline{M}A_0+A_1$ with $A_i\neq 0$ for $i=0,1$.

Proposition 3.7. *Let $S=\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and*

$$t=\left[\begin{array}{cc|cc} I & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & I & & \\ \hline I & I & I & 0 \\ 0 & 0 & 0 & I \end{array}\right].$$

Then,

(i) *$S,t\in N(G)$,*

(ii) *$\sigma^S=\tau$, $\tau^S=\sigma$, and $(\sigma\tau)^S=\sigma\tau$,*

(iii) *$u_\sigma(X)^S=u_\tau(X)$, $u_\tau(Y)^S=u_\sigma(Y)$, and $u_{\sigma\tau}(Z)^S=u_{\sigma\tau}(Z+I)$,*

(iv) *$L^S=\begin{bmatrix} \overline{M}A & \overline{M} & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$ where $L=\begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$,*

(v) *$\sigma^t=\sigma$, $\tau^t=\sigma\tau$, and $(\sigma\tau)^t=\tau$,*

(vi) *$u_\sigma(X)^t=u_\sigma(X+I)$,*

(vii) *$u_\tau(Y)^t=\overline{u_{\sigma\tau}(Y)}$,*

(viii) *$u_{\sigma\tau}(Z)^t=\overline{u_\tau(Z)}$,*

(ix) *$L^t=\begin{bmatrix} A+I & \overline{M} & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$.*

Proposition 3.8. *Let $d=St$ where S and t are defined in Proposition 3.7. Then,*

(i) *$u_\sigma(X)^d=\overline{u_{\sigma\tau}(X)}$,*

(ii) *$u_\tau(Y)^d=u_\sigma(Y+I)$,*

(iii) *$\left(\overline{u_\sigma(X)}\right)^d=u_{\sigma\tau}(X)$,*

(iv) *$\left(\overline{u_\tau(Y)}\right)^d=\overline{u_\sigma(Y+I)}$,*

(v) *$\left(\overline{u_{\sigma\tau}(Z)}\right)^d=u_\tau(Z+I)$,*

(vi) *$L^d=\begin{bmatrix} \overline{M}A+I & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$ where $L=\begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$.*

Remark 3.5. *L is compatible with $u_{\sigma\tau}$ if and only if L^d is compatible with $u^d_{\sigma\tau}$.*

Observation 3.2. *If $x = x_0 + x_1$, $x_i \in m_i$ for $i = 0, 1$ is non-singular, then $x_0 = 0$ or $x_1 = 0$.*

Corollary 3.2. *Let $x \in m = m_0 \cup m_1$. Assume that there exists $i \in \{0,1\}$ and $a, b \in m_i$, $a \neq b$ such that $x+a$ and $x+b$ are non-singular, then $x \in m_i \backslash \{a,b\}$.*

Proof. Let $x = x_i + x_j$, $x_i \in m_i$, $x_j \in m_j$ where $\{i, j\} = \{0,1\}$, then it follows:

$$x+a = (x_i+a)+x_j, \quad \text{and} \quad x+b = (x_i+b)+x_j.$$

As $a \neq 0$ this implies that $x_i + a \neq 0$ or $x_i + b \neq 0$. Hence $x_j = 0$. From this it follows that $x_i \in m_i \backslash \{a,b\}$ for $i = 0, 1$.

We summarize the results above by the following theorem.

Theorem 3.2. *$\mathscr{K}(X,Y,Z) = \{L_v \mid v \neq 0\} \cup \{u_\sigma(X), \overline{u_\sigma(X)}, u_\tau(Y), \overline{u_\tau(Y)}, u_{\sigma\tau}(Z), \overline{u_{\sigma\tau}(Z)}\}$ form a partial spread if and only if $X+Y$, $Y+Z$, $X+Z+I$ are non-singular and there are exactly 256 suitable triples (X,Y,Z).*

Next, we have to find subspaces L, such that L^G is a partial spread of length 4, which is compatible with $\mathscr{K}(X,Y,Z)$.

We have shown that candidates for such L may be chosen as $\begin{bmatrix} A & M & I & 0 \\ K & 0 & 0 & I \end{bmatrix}$ where $M^2 + M + I = K^2 + K + I = 0$ or as $\begin{bmatrix} A & M & I & 0 \\ C & J & 0 & I \end{bmatrix}$ where $M^2 + M + I = 0$, $J \in GL_2(2)$ is an involution, and $A + C \in \{M+I, M, J, J+I\}$ and $vC \neq 0, v$ for all $v \neq 0$ with $vJ = v$. For a fixed M there are in both cases 32 possibilites for such L. These spaces are compatible with the fixed lines L_v, $v \neq 0$. We first solve for $L = \begin{bmatrix} A & M & I & 0 \\ M & 0 & 0 & I \end{bmatrix}$.

The case $L = \begin{bmatrix} A & M & I & 0 \\ \overline{M} & 0 & 0 & I \end{bmatrix}$ is an immediate consequence of the first case by applying an element in $N(G)$.

The last step is to find the admissible A's, X's, Y's, and Z's for which the following necessary and sufficient conditions hold

1. $X + MA + \overline{M}$ and $X + \overline{M}A + I$ are non-singular.
2. $Y + A$ and $Y + MA + \overline{M}$ are non-singular.
3. $Z + A$ and $Z + \overline{M}A$ are non-singular.

Also we have $X_1 + Y_1$, $Y_1 + Z_1$, $X_1 + Z_1 + I$ are non-singular, where $X_1 = X + A$, $Y_1 = Y + A$ and $Z_1 = Z + A$ or

(1') $X_1 + \overline{M}(A+I)$ and $X_1 + MA + I$ are non-singular.

(2') Y_1 and $Y_1 + \overline{M}(A+I)$ are non-singular

(3') Z_1 and $Z_1 + MA$ are non-singular

Theorem 3.3. *1 If $A = 0$, then the admissible X's, Y's, and Z's are*

X	Y	Z
0	I	$M, \overline{M}$
0	M	$\overline{M}$
M	I	M

2 If A is non-singular and $A \in m_0$, then the admissible X's, Y's, and Z's are

X	Y	Z
A	$\overline{M}A$	A
$\overline{M}A + I$	$\overline{M}A$	A

Proof. 1. As Y and $Y + \overline{M}$ are non-singular, this implies that $Y \in m_1 \backslash \{\overline{M}\}$, and as Z, Y, and $Y + Z$ are non-singular, then by Corollary 3.2 $Y \in m_1$ and $Z \in m_1$.

$X + I$ and $X + M + I$ are non-singular implies that $X + I \in m_1$ and $X \in m_1$. Hence $X, Y, Z \in m_1$ where $X \neq I, \overline{M}$, $Y \neq 0, \overline{M}$, $Z \neq 0$, $X \neq Y$, $Y \neq Z$, $Z \neq X + I$, $Z \neq 0, Y, X + I$. Hence the claim.

2. As Z and $Z + MA$ are non-singular, it follows that $Z \in m_0$. As Y and $Y + Z$ are non-singular, it follows that $Y \in m_0$ and $Y + \overline{M}(A+I) = (Y + \overline{M}A) + \overline{M}$ is non-singular which implies that $Y = \overline{M}A \in m_0$ because $Y + \overline{M}A \in m_0$ and $\overline{M} \in m_1$. As $Y = \overline{M}A$, $Z \in m_0$, then $Z = kA$ where $k \in m_1 \backslash \{0, \overline{M}\}$.

Thus $Z+MA=(k+M)A$ implies that $k \neq M$. Hence $k \neq 0, M, \overline{M}$ and thus $k=I$, $Z=A$, $Y=\overline{M}A$, and

$$X+Y=X+\overline{M}A=(x_0+\overline{M}A)+x_1$$
$$X+Z+I=X+A+I$$
$$=(x_0+A)+x_1+I.$$

If $x_1=0$, then $x_0=A$ and $X=A$. If $x_1 \neq 0$, then $x_0=\overline{M}A \neq A$ which implies $x_1=I$ and thus $X=\overline{M}A+I$. From this

it follows that we have the following admissible X's, Y's, and Z's

X	Y	Z
A	$\overline{M}A$	A
$\overline{M}A+I$	$\overline{M}A$	A

Corollary 3.3. *Theorems 3.1, 3.2, and 3.3 implies the full spread of length 17.*

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حول وجود الثلاثي $(V، G، K)F_2$ Baer Triples من النوع 2

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الملخص

هدف هذا البحث هو بناء غطاء كامل K طوله 17 بحيث أن الثلاثي $(V، G، K)F_2$ هو ثلاثي Baer من النوع 2 و G زمره إبدالية متشاكله مع الزمره 2^2.