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Three iterative methods for solving Jeffery-Hamel flow problem

Majeed A. AL-Jawary*, AL-Zahraa J. Abdul Nabi

Dept. of Mathematics, College of Education for Pure Science (Ibn AL-Haytham), University of Baghdad, Baghdad, Iraq

**Corresponding author: majeed.a.w@ihcoedu.uobaghdad.edu.iq*

Abstract

In this article, the nonlinear problem of Jeffery-Hamel flow has been solved analytically and numerically by using reliable iterative and numerical methods. The approximate solutions obtained by using the Daftardar-Jafari method (DJM), Temimi-Ansari method (TAM) and Banach contraction method (BCM). The obtained solutions are discussed numerically, in comparison with other numerical solutions obtained from the fourth order Runge-Kutta (RK4), Euler and previous analytic methods available in the literature. In addition, the convergence of the proposed methods is given based on the Banach fixed point theorem. The results reveal that the presented methods are reliable, effective and applicable to solve other nonlinear problems. The computational work to evaluate the terms in the iterative processeswascarried out using the computer algebra system MATHEMATICA®10.

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1. Introduction

The theory of viscous fluid flow through convergent-divergent channels has many applications in aerospace, chemical, civil, mechanical, biomechanical and environmental engineering. It also plays a role in understanding rivers and canals and in human anatomy in how capillaries and arteries are linked to each other. The mathematical formulations of the concerned problem were carried out in 1915 by George Barker Jeffery and in 1916 by Georg Karl Hamel (Jeffery, 1915; Hamel,1916). The theory has been extensively discussed and studied by several authors. Many articles have been published on the topics of analytical structure of a solution and the properties of overall flow structure, including velocity field, flow reversal control, and bifurcations (Fraenkel, 1962; Hamadiche *et al.*, 1994; McAlpine & Drazin, 1998; Makinde & Mhone, 2006; Moghimi *et al.*, 2011). In particular, if the Navier-Stokes equations (of two-dimensional flow through a channel with inclined walls) are simplified, one can finally obtain the Jeffery-Hamel equation (Rosenhead, 1940; Schlichting, 1955; Batchelor, 1967; Sobey & Drazin, 1986).

Most of the scientific problems are primarily nonlinear, such as Jeffery-Hamel flow and other fluid mechanical problems. Except for a restricted number of such for most cases of them. This kind of problems can be analytically solved by using iterative methods and then comparing the solutions numerically by using other numerical solutions. For example, solving mechanical or heat problems by iterative and numerical methods (Bayat *et al.*, 2015; Noroozi *et al.*, 2017) are some solutions.

Recently, these types of problems have been resolved with known semi-analytical or numerical methods. These problems include: the Lattice-Boltzmann method (Házi & Farkas, 2003), He's variational iteration method (VIM) with the HPM (Ganji *et al.*, 2009), the homotopy analysis method (HAM) with the homotopy perturbation method (HPM) and the differential transformation method (DTM) (Joneidi *et al.*, 2010), the Adomian decomposition method (ADM) (Esmaili *et al.*, 2008; Sheikholeslami *et al.*, 2012), the modified Adomian decomposition method (MADM) (Lu *et al.*, 2016; Bougoffa *et al.*, 2016; Patel & Meher, 2016), and others (Marinca & Herişanu, 2011; Sushila *et al.*, 2014).

In this study, three iterative methods will be used to obtain the approximate solutions of the Jeffery-Hamel flow problem. Varsha Daftardar-Gejji and Hossein Jafari presented the Daftardar-Jafari method (DJM) in 2006. This is the first iterative method used. It has been implemented for solving different problems in many areas (Bhalekar & Daftardar-Gejji, 2008; Daftardar-Gejji & Bhalekar, 2010; Yaseen*et al.*, 2013; Al-Jawary & Al-

Qaissy, 2015; Al-Jawary & Abd-Al-Razaq, 2016b). The second iterative method that we will use is known as the Temimi-Ansari method (TAM) (Temimi& Ansari, 2011). It has been employed for finding the solutions for various problems (Ehsani*et al.*, 2013; Al-Jawary&Hatif, 2017; Al-Jawary& Al-Razaq, 2016a; Al-Jawary & Raham, 2017; Al-Jawary, 2017; Al-Jawary*et al.*, 2017). The third iterative method for solving the presented nonlinear problem was suggested by the Banach contraction method (BCM), and which was presented by Varsha and Sachin Bhalekar in 2009 (Daftardar-Gejji&Bhalekar, 2009). This method is based on using what is known by the Banach contraction principle.

This paper has been organized as follows: Section 2 shows the mathematical formulation of the Jeffery-Hamel equation. Section 3 presents the basic concepts of the three iterative methods. In section 4, the convergence of the proposed methods is presented. Section 5 presents the approximate solution of the problem by using the proposed methods. The numerical simulations and error analyses of the approximate solutions are shown in section 6. Finally, the conclusions are given in section 7.

2. The mathematical formulation of the governing equation

Consider the two-dimensional flow for some incompressible conducting viscous fluid flowing from some source or sink at the intersection between two plane walls. Let us assume the flow happens in a system of cylindrical polar coordinates (r, θ, z), where the walls are intersecting in the axis of z as in Figure 1. Assume that the velocity is only in the radial direction and that it depends on r and θ, and bear in mind that there is no magnetic field in the z-direction.

The continuity, Navier-Stokes, and Maxwell's equations are in the following reduced forms (Schlichting, 2000):

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(rv(r,\theta)\right) = 0, \tag{1}$$

$$v(r,\theta)\frac{\partial v(r,\theta)}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + w\left[\frac{\partial^2 v(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2} - \frac{v(r,\theta)}{r^2}\right] - \frac{\sigma\beta_0^2}{\rho r^2}v(r,\theta), \tag{2}$$

$$-\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{2w}{r^2}\frac{\partial v(r,\theta)}{\partial \theta} = 0, \tag{3}$$

where β, σ, $v(r,\ \theta)$, P, w and ρ represent the electromagnetic induction, the fluid conductivity, the velocity along a radial direction, the pressure of the fluid, the kinematic viscosity parameter and the density of the fluid, respectively. By using the dimensionless parameters in equation (1) we have:

$$u(\theta) = rv(r,\theta), \tag{4}$$

$$u(x) = \frac{u(\theta)}{u_{max}}, \qquad x = \frac{\theta}{\alpha}. \tag{5}$$

When putting (5) in (2) and (3) and then eliminating P, the following ODE for the normalized function profile$u(x)$ can be obtained (Ganji, 2006):

$$u'''(x) + 2\alpha Re u(x)u'(x) + 4\alpha^2 u'(x) = 0, \tag{6}$$

with the boundary conditions:

$$u(0) = 1,\quad u'(0) = 0,\quad u(1) = 0. \tag{7}$$

The following parameter represents the Reynolds number:

$$Re = \frac{u_{max}\alpha}{w} = \frac{V_{max}r\alpha}{w}\begin{pmatrix}\text{divergent-channel: } \alpha>0, u_{max}>0\\ \text{convergent-channel:} \alpha<0, u_{max}<0\end{pmatrix}. \tag{8}$$

The corresponding Jeffery-Hamel equation above in Eq. (6) will be solved by implementing the three methods with the boundary conditions (7) and with the imposition of the following condition:

$$u''(0) = a, \tag{9}$$

where a will be evaluated later.

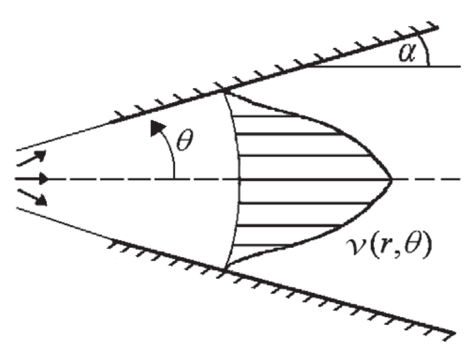

Fig. 1. The geometry of the problem.

3. The fundamentals of the three iterative methods

The basic concepts of the three proposed iterative methods are introduced in this section.

3.1. The basic idea of the DJM

Let us begin by considering the general form of a functional equation (Daftardar-Gejji & Jafari, 2006)

$$u = f + L(u) + N(u), \tag{10}$$

where L is the linear operator, N represents the nonlinear operator, f denotes a known function and u represents the unknown function that need to be evaluated and which is the solution for Eq. (10). It can be decomposed in the following series:

$$u = \sum_{i=0}^{\infty} u_i. \tag{11}$$

Now, let us define the following forms:

$$\mathrm{G}_0 = N(u_0), \tag{12}$$

$$\mathrm{G}_m = N\left(\sum_{i=0}^{m} u_i\right) - N\left(\sum_{i=0}^{m-1} u_i\right). \tag{13}$$

The $N(u)$ equals to the following series:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = \underbrace{N(u_0)}_{\mathrm{G}_0} + \underbrace{[N(u_0+u_1) - N(u_0)]}_{\mathrm{G}_1} + \underbrace{[N(u_0+u_1+u_2) - N(u_0+u_1)]}_{\mathrm{G}_2} + \underbrace{[N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)]}_{\mathrm{G}_3} + \cdots. \tag{14}$$

Also, the relation will be defined in the recurrent procedure

$$v_0 = f, \tag{15}$$

$$v_1 = L(u_0) + \mathrm{G}_0, \tag{16}$$

$$v_2 = L(u_1) + \mathrm{G}_1, \tag{17}$$

$$\vdots$$

$$v_m = L(u_{m-1}) + \mathrm{G}_{m-1}, \qquad m = 1,2,\dots. \tag{18}$$

Since the linear operator L satisfies $\sum_{i=0}^{m} L(u_i) = L(\sum_{i=0}^{m} u_i)$, we have

$$\sum_{i=1}^{m} v_i = \sum_{i=0}^{m-1} L(u_i) + N\left(\sum_{i=0}^{m-1} u_i\right)$$

$$= L\left(\sum_{i=0}^{m-1} u_i\right) + N\left(\sum_{i=0}^{m-1} u_i\right), m = 1,2,\dots. \tag{19}$$

Therefore,

$$\sum_{i=0}^{\infty} v_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{20}$$

Clearly the approximate solution of Eq. (10) is represented in the n-term series approximate solution $v_n = \sum_{i=0}^{n-1} u_i$.

3.2. The basic idea of the TAM

For verifying the standard steps of the TAM (Temimi & Ansari, 2011), let us re-consider Eq. (10) with the boundary conditions:

$$B\left(u, \frac{d^i u}{dx^i}\right) = 0, \qquad i \geq 0, \tag{21}$$

where B denotes the boundary operator.

Primarily, the first step in the TAM is finding the initial approximation which is the function $u_0(x)$. This function represents the solution for the following initial problem

$$L\big(u_0(x)\big) - f(x) = 0, \tag{22}$$

with

$$B\left(u_0, \frac{d^i u_0}{dx^i}\right) = 0, \qquad i \geq 0. \tag{23}$$

The second step is to obtain the next iterative functions. Hence, to find $u_1(x)$, the next problem should be solved

$$L\big(u_1(x)\big) + N\big(u_0(x)\big) = 0, \tag{24}$$

with

$$B\left(u_1, \frac{d^i u_1}{dx^i}\right) = 0, \qquad i \geq 0. \tag{25}$$

In general, the following problem will be used for finding the other iterations for $n \in \mathbb{N}$

$$L\big(u_{n+1}(x)\big) + N\big(u_n(x)\big) = 0, \tag{26}$$

with

$$B\left(u_{n+1}, \frac{d^i u_{n+1}}{dx^i}\right) = 0, \qquad i \geq 0. \tag{27}$$

Continuing in this iterative procedure, one can get the suitable function, which is the exact solution for Eq. (10). It is clear that each iterative function $u_n(x)$ is alone considered some approximate solution for Eq. (10). By increasing the rank of $u_n(x)$, it will be convergent to the exact solution.

3.3. The basic idea of the BCM:

Let us first reconsider Eq. (10) (Daftardar-Gejji & Bhalekar, 2009), now we have to define some successive approximations as follows:

$$\begin{aligned} u_0 &= f, \\ u_1 &= u_0 + L[u_0] + N[u_0], \\ u_2 &= u_0 + L[u_1] + N[u_1], \\ &\vdots \end{aligned} \tag{28}$$

$$u_n = u_0 + L[u_{n-1}] + N[u_{n-1}],\ n \in \mathbb{N} \tag{29}$$

If $(L+N)^k$ is a contraction mapping for some positive integer k, so $(L+N)^k$ has a unique fixed point and thus the nth sequence that is defined by (29) is convergent, see (Daftardar-Gejji & Bhalekar, 2009). Thus the obtained solution of Eq. (10) will be given by:

$$u = \lim_{n\to\infty} u_n. \tag{30}$$

4. The convergence of the proposed techniques

The basic concepts and the fundamental theorem of the convergence for the proposed iterative techniques will be presented in this section.

In the following, the convergence for the DJM, TAM and BCM will be introduced. The iterations occurred from the DJM can be directly used to prove the convergence of this technique. But, to prove the convergence of the TAM or BCM, we should use the following approach, where it can be applied for handling Eq. (6) with the boundary conditions (7). We have the following terms

$$\begin{aligned} v_0 &= u_0(x), \\ v_1 &= F[v_0], \\ v_2 &= F[v_0 + v_1], \\ &\vdots \\ v_{n+1} &= F[v_0 + v_1 + \cdots + v_n], \end{aligned} \tag{31}$$

where F is the operator that can be defined as

$$F[v_k] = S_k - \sum_{i=0}^{k-1} v_i(x),\ k \in \mathbb{N}. \tag{32}$$

The term S_k represents the solution for one of the following problems:

For the TAM:

$$L(v_k(x)) + g(x) + N\left(\sum_{i=0}^{k-1} v_i(x)\right) = 0. \tag{33}$$

For the BCM:

$$v_k = v_0 + N\left[\sum_{i=0}^{k-1} v_i(x)\right]. \tag{34}$$

The same given conditions with the used iterative method will be employed. Thus, in this way, we have $u(x) = \lim_{n\to\infty} u_n(x) = \sum_{n=0}^{\infty} v_n$. So, by using (31) and (32), we can get the following solution in a series form:

$$u(x) = \sum_{i=0}^{\infty} v_i(x). \tag{35}$$

According to the recursive algorithm of the DJM, TAM and BCM, the sufficient conditions for convergence of these techniques will be presented in the following theorems.

Theorem 4.1 Let F defined in (32), be an operator from a Hilbert space H to H. The series solution $u_n(x) = \sum_{i=0}^{n} v_i(x)$ converges if $\exists\ 0 < \gamma < 1$ such that $\|F[v_0 + v_1 + \cdots + v_{i+1}]\| \leq \gamma \|F[v_0 + v_1 + \cdots + v_i]\|$ (such that $\|v_{i+1}\| \leq \gamma \|v_i\|$) $\forall i = 0,1,2,\ldots$.

This theorem is just a special case of Banach's fixed point theorem, which is a sufficient condition to study the convergence of our proposed iterative techniques. Moreover, the norm used in Theorem 4.1 is that induced by the inner product of H.

Proof: See (Odibat, 2010).

Theorem 4.2 If the series solution $u(x) = \sum_{i=0}^{\infty} v_i(x)$ is convergent, then this series will represent the exact solution of the current nonlinear problem.

Proof: See (Odibat, 2010).

Theorem 4.3 Suppose that the series solution $\sum_{i=0}^{\infty} v_i(x)$ (which is defined in (35))is convergent to the solution $u(x)$. If the truncated series $\sum_{i=0}^{n} v_i(x)$ is used as an approximation to the solution of the current problem, then the maximum error $E_n(x)$ is estimated by

$$E_n(x) \leq \frac{1}{1-\gamma}\gamma^{n+1}\|v_0\|. \tag{36}$$

Proof: See (Odibat, 2010).

Theorems **4.1** and **4.2** state that the solutions obtained by the DJM given in (18), the TAM given in (26) or (31), or the solution of the BCM given in (29) or (31) for the nonlinear equation (6), converge to the exact solution under the condition $\exists\, 0 < \gamma < 1$ such that $\|F[v_0 + v_1 + \cdots + v_{i+1}]\| \leq \gamma \|F[v_0 + v_1 + \cdots + v_i]\|$ (that is $\|v_{i+1}\| \leq \gamma\|v_i\|$) $\forall i = 0,1,2,\ldots$. Also, for each i, if we define the parameters:

$$\beta_i = \begin{cases} \frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0 \\ 0, & \|v_i\| = 0 \end{cases}, \tag{37}$$

then the series solution $\sum_{i=0}^{\infty} v_i(x)$ of Eq. (6) converges to the exact solution $u(x)$, when $0 \leq \beta_i < 1, \forall i = 0,1,2,\ldots$. In addition, as in Theorem **4.3**, the maximum truncation error is estimated to be $\|u - \sum_{i=0}^{n} v_i\| \leq \frac{1}{1-\beta}\beta^{n+1}\|v_0\|$, where $\beta = \max\{\beta_i, i = 0,1,\ldots,n\}$.

5. Solving the Jeffery-Hamel flow problem

In this section, the approximate solution for the nonlinear Jeffery-Hamel flow problem given in Eqs. (6) and (7) is evaluated by the three previously presented iterative methods. The convergence condition is also examined for the obtained solution of these proposed methods.

5.1. Solving the Jeffery-Hamel equation by the DJM

Consider the Jeffery-Hamel problem given by (6) and (7) by the DJM. The following integral form will be obtained as in the following steps:

Let's rewrite (6) as the following:

$$u'''(x) = N(u(x)), \tag{38}$$

where $N\big(u(x)\big) = -2\alpha Re u(x)u'(x) - 4\alpha^2 u'(x)$.

Integrating (38) three times from 0 to x, we get:

$$u(x) = 1 + \frac{a}{2}x^2 + \int_0^x \int_0^x \int_0^x N(u(t))\,dt\,dt\,dt, \tag{39}$$

where $a = u''(0)$ as in (9). For simplicity, according to the rule of reducing multiple integrals (Wazwaz, 2011), the integral form given in Eq. (39) will be reduced to the following Volterra integral equation:

$$u(x) = 1 + \frac{a}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 N(u(t))\,dt. \tag{40}$$

So, according to the basic steps of the DJM, we have:

$$v_0(x) = 1 + \frac{a}{2}x^2,$$

$$v_1(x) = \tfrac{1}{2}\int_0^x (x-t)^2 N(u_0(t))\,dt.$$

Then, we get

$$v_1(x) = -\frac{1}{6}ax^4\alpha^2 - \frac{1}{12}aRex^4\alpha - \frac{1}{120}a^2Rex^6\alpha\,.$$

In general, for each $n \in \mathbb{N}$, we have

$$v_{n+1}(x) = \frac{1}{2}\int_0^x (x-t)^2 N\Big(\sum_{i=0}^{n} u_i(t)\Big)\,dt - \frac{1}{2}\int_0^x (x-t)^2 N\Big(\sum_{i=0}^{n-1} u_i(t)\Big)\,dt. \tag{41}$$

Hence,

$$\begin{aligned} v_2(x) = {} & \frac{1}{180}aRe^2x^6\alpha^2 + \frac{1}{560}a^2Re^2x^8\alpha^2 + \frac{a^3Re^2x^{10}\alpha^2}{10800} + \frac{1}{45}aRex^6\alpha^3 + \frac{1}{280}a^2Rex^8\alpha^3 \\ & - \frac{a^2Re^3x^{10}\alpha^3}{12960} - \frac{a^3Re^3x^{12}\alpha^3}{95040} - \frac{a^4Re^3x^{14}\alpha^3}{2620800} + \frac{1}{45}ax^6\alpha^4 \\ & - \frac{a^2Re^2x^{10}\alpha^4}{3240} - \frac{a^3Re^2x^{12}\alpha^4}{47520} - \frac{a^2Rex^{10}\alpha^5}{3240}. \end{aligned}$$

⋮

and so on. Proceeding in this way, the components v_3, v_4 and v_5 were also calculated but for brevity, they are not listed here.

In order to determine the value of a, we solved $v_5(1) = 0$, which provides better accuracy than other roots computed from others iterations (Table 1) with $Re = 50$ and $\alpha = 5°$. Then we get the root $a = -3.539416833235919$. By substituting a with α and Re in the obtained iterations of the DJM, we compute the values of β_i by using the second norm for the Jeffery-Hamel problem as in (38), we get

$$\beta_0 = \frac{\|v_1\|}{\|v_0\|} = 0.499947 < 1$$

$$\beta_1 = \frac{\|v_2\|}{\|v_1\|} = 0.0946053 < 1$$

$$\beta_2 = \frac{\|v_3\|}{\|v_2\|} = 0.101353 < 1 \quad (42)$$

$$\beta_3 = \frac{\|v_4\|}{\|v_3\|} = 0.0498859 < 1$$

$$\beta_4 = \frac{\|v_5\|}{\|v_4\|} = 0.0346925 < 1,$$

where for $i \geq 0$ the β_i's, the values are less than one for all $0 \leq x \leq 1$. Hence, this DJM approach is convergent according to the convergence condition.

5.2. Solving the Jeffery-Hamelequation by the TAM

To solve the current problem presented in Eqs. (6) and (7) by the TAM according to the presented steps in the subsection **3.2**, let us begin by solving the following initial problem:

$$u_0'''(x) = 0, \quad u_0(0) = 1,$$
$$u_0'(0) = 0 \text{ and } u_0''(0) = a, \quad (43)$$

we get,

$$u_0(x) = 1 + \frac{ax^2}{2}.$$

The second step is solving the following problem:

$$u_1'''(x) = N\big(u_0(x)\big), u_1(0) = 1,$$
$$u_1'(0) = 0, \quad u_1''(0) = a, \quad (44)$$

This produces:

$$u_1(x) = 1 + \frac{ax^2}{2} - \frac{1}{6}ax^4\alpha^2 - \frac{1}{12}aRex^4\alpha - \frac{1}{120}a^2Rex^6\alpha. \quad (45)$$

The same step for finding $u_2(x)$, will be used, which means solving the following problem:

$$u_2'''(x) = N\big(u_1(x)\big), u_2(0) = 1,$$
$$u_2'(0) = 0, \quad u_2''(0) = a. \quad (46)$$

We get

$$u_2(x) = 1 + \frac{ax^2}{2} - \frac{1}{12}aRex^4\alpha - \frac{1}{120}a^2Rex^6\alpha - \frac{1}{6}ax^4\alpha^2 + \frac{1}{180}aRe^2x^6\alpha^2 + \frac{1}{560}a^2Re^2x^8\alpha^2 + \frac{a^3Re^2x^{10}\alpha^2}{10800} + \frac{1}{45}aRex^6\alpha^3 + \frac{1}{280}a^2Rex^8\alpha^3 - \frac{a^2Re^3x^{10}\alpha^3}{12960} - \frac{a^3Re^3x^{12}\alpha^3}{95040} - \frac{a^4Re^3x^{14}\alpha^3}{2620800} + \frac{1}{45}ax^6\alpha^4 - \frac{a^2Re^2x^{10}\alpha^4}{3240} - \frac{a^3Re^2x^{12}\alpha^4}{47520} - \frac{a^2Rex^{10}\alpha^5}{3240} \quad (47)$$

⋮

and so on. Proceeding in this way, the components u_3, u_4 and u_5 were also calculated but for brevity are not listed here.

To prove the convergence analysis for the TAM, we have applied the approach given in (31). We get the following results

$$v_0 = u_0(x) = 1 + \frac{ax^2}{2},$$

$$v_1 = -\frac{1}{12}aRex^4\alpha - \frac{1}{120}a^2Rex^6\alpha - \frac{1}{6}ax^4\alpha^2,$$

$$v_2 = \frac{1}{180}aRe^2x^6\alpha^2 + \frac{1}{560}a^2Re^2x^8\alpha^2 + \frac{a^3Re^2x^{10}\alpha^2}{10800} + \frac{1}{45}aRex^6\alpha^3 + \frac{1}{280}a^2Rex^8\alpha^3 - \frac{a^2Re^3x^{10}\alpha^3}{12960} - \frac{a^3Re^3x^{12}\alpha^3}{95040} - \frac{a^4Re^3x^{14}\alpha^3}{2620800} + \frac{1}{45}ax^6\alpha^4 - \frac{a^2Re^2x^{10}\alpha^4}{3240} - \frac{a^3Re^2x^{12}\alpha^4}{47520} - \frac{a^2Rex^{10}\alpha^5}{3240},$$

⋮

and so on. Clearly these iterations are the same iterations obtained by the DJM. In the same way, we use the v_i iterations to prove the convergence condition, we must find a by solving $u_5(1)=0$ with $Re=50$ and $\alpha=5°$; therefore, we get $a=-3.539416833235919$. By substituting a with α and Re in the iterations above of the TAM and by computing the values of β_i for the problem as in (38), we get the same β_i's of the DJM as in (42). Hence, the condition of the convergence for the TAM has been carried out.

5.3. Solving the Jeffery-Hamel equation by the BCM

In this subsection, the BCM will be used to solve the nonlinear Jeffery-Hamel problem given in Eqs. (6) and (7). Let us start by the same integration operations in section 4 which got us the integral form (40). According to the BCM steps in 3.3.2, we have

$$u_0(x)=1+\frac{a}{2}x^2,$$

$$u_1(x)=u_0(x)+\frac{1}{2}\int_0^x (x-t)^2N(u_0(t))\,dt.$$

We obtain,

$$u_1(x)=1+\frac{ax^2}{2}-\frac{1}{6}ax^4\alpha^2-\frac{1}{12}aRex^4\alpha-\frac{1}{120}a^2Rex^6\alpha.$$

In general, we have,

$$u_{n+1}(x)=u_0(x)+\frac{1}{2}\int_0^x (x-t)^2N(u_n(t))\,dt, n\in\mathbb{N}. \quad (48)$$

Hence,

$$u_2(x)=1+\frac{ax^2}{2}-\frac{1}{12}aRex^4\alpha-\frac{1}{120}a^2Rex^6\alpha-\frac{1}{6}ax^4\alpha^2+\frac{1}{180}aRe^2x^6\alpha^2+\frac{1}{560}a^2Re^2x^8\alpha^2+\frac{a^3Re^2x^{10}\alpha^2}{10800}+\frac{1}{45}aRex^6\alpha^3+\frac{1}{280}a^2Rex^8\alpha^3-\frac{a^2Re^3x^{10}\alpha^3}{12960}-\frac{a^3Re^3x^{12}\alpha^3}{95040}-\frac{a^4Re^3x^{14}\alpha^3}{2620800}+\frac{1}{45}ax^6\alpha^4-\frac{a^2Re^2x^{10}\alpha^4}{3240}-\frac{a^3Re^2x^{12}\alpha^4}{47520}-\frac{a^2Rex^{10}\alpha^5}{3240},$$

⋮

and so on. Proceeding in this way, the components u_3, u_4 and u_5 were also calculated but for brevity, they are not listed here.

To prove the convergence for the BCM, by applying the approch given in Eq. (31), we obtain the same iterations obtained by the TAM. Moreover, the same value of a is obtained for the same parameters. Thus the same β_i values of the DJM and TAM as in (42) are reached. Therefore, the condition of convergence is completed for the BCM.

6. The numerical discussion

In order to assess the accuracy of the proposed methods the DJM, TAM and BCM for the approximate solution of the Jeffery-Hamel problem, one must examine the effect of the methods for the function $u(x)$. Thus, when selecting the values of α and Re in our approximate solution, we can get several approximate appropriate solutions. According to the numerical investigations, we observed that the numerical results obtained by the three proposed iterative methods are similar to each other. To determine the value of a, we solved for the given boundary condition $u_5(1)=0$. This provides better accuracy than other roots obtained by other iterations with $Re=50$ and $\alpha=5°$ and we get the root $a=-3.539416833235919$. The other values for the root a_i were evaluated for the same values of Re and α by solving the obtained iterations at the given boundary condition, i.e. $u_i(1)=0,\ i=0,\dots,5$, and then solving these relations at each iterate. Table 1 provides the best obtained real roots a_i among the other complex ones.

Table 1. The obtained real values for solving $u_i''(0)=a$

u_i	a_i
u_1	-3.7056175166082888
u_2	-3.514652445627693
u_3	-3.540697091424481
u_4	-3.539369250766523
u_5	-3.539416833235919

For the uniqueness of the solutions for the obtained condition; we refer to (Lasota & Yorke, 1972; Henderson & Jackson 1983) which stated at least one real root exists for such types of nonlinear BVP.

Table 2 shows the values of α for different values of Re and α. Figures 2 and 3 show the velocity function obtained by the proposed methods for several values of Re and α. Figures 4 and 5 show the root mean square function (RMS) for our obtained velocity function $u(x)$. The RMS function for any obtained iterations u_i can be defined by:

$$\mathrm{RMS}(u_i) = \sqrt{\frac{\sum_{j=0}^{1}\left(u_i(x_j) - u_{\mathrm{RKM}}(x_j)\right)^2}{\sum_{j=0}^{1}\left(u_{\mathrm{RKM}}(x_j)\right)^2}}, \tag{49}$$

where $u_i(x_j)$ is the obtained approximate solution. The function $u_{\mathrm{RKM}}(x_j)$ represents the numerical solution obtained by the classical fourth order Runge-Kutta method (RKM). The values for x_j are increased as $0, 0.1, 0.2, \dots, 1$. Since the exact solution is unknown, $u_{\mathrm{RKM}}(x_j)$ has been used as a benchmark to access the performance of the approximate solution. Furthermore, to estimate the accuracy of the obtained approximate solution, the error remainder function has been evaluated in the following form:

$$ER_n(x) = u_n'''(x) + 2\alpha Re u_n(x)u_n'(x) + 4\alpha^2 u_n'(x), \tag{50}$$

with the following maximal error remainder parameter

$$MER_n = \max_{0\le x\le 1}|ER_n(x)|. \tag{51}$$

For $\alpha = 2.5°$ and $Re = 50$, the maximal error remainder MER_n values for the numerical solutions obtained by our proposed methods are plotted in Figure 6. It is important to note that we wrote the programs for the ADM (Esmaili *et al.*, 2008) and the VIM (Ganji *et al.*, 2009). One can clearly see that the MER_n value for the proposed methods is less than the value for the ADM. This confirms that the proposed methods converge faster than the ADM. Figure 7 shows that the numerical solutions obtained by our proposed methods are in a good agreement with the results obtained by the ADM (Esmaili *et al.*, 2008) and the VIM (Ganji *et al.*, 2009), RK4 and Euler methods.

Table 2. The values of $u_5''(0) = a$ for the approximate solution by using 5-iterations.

α	Re	a_5
5°	50	−3.539416833235919
5°	100	−5.869196490018455
5°	118	−6.880245457633106
5°	160	−9.478590329181
5°	209	−12.74250234452334
−5°	50	−1.1219899962255044
−5°	100	−0.640207744659544
−5°	118	−0.5269683706213608
−5°	160	−0.34011384167250835
−5°	209	−0.20995974152883548

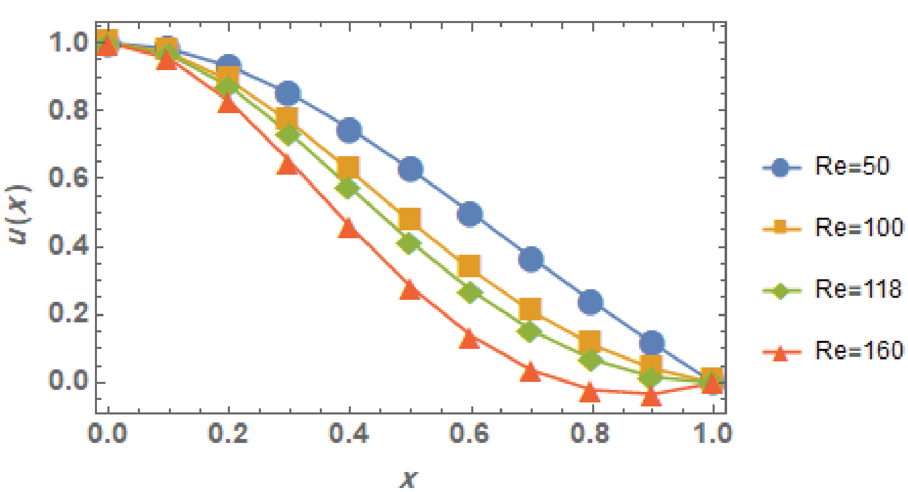

Fig. 2. The velocity solution obtained by proposed methods for various values of the Reynolds number and $\alpha = 5°$.

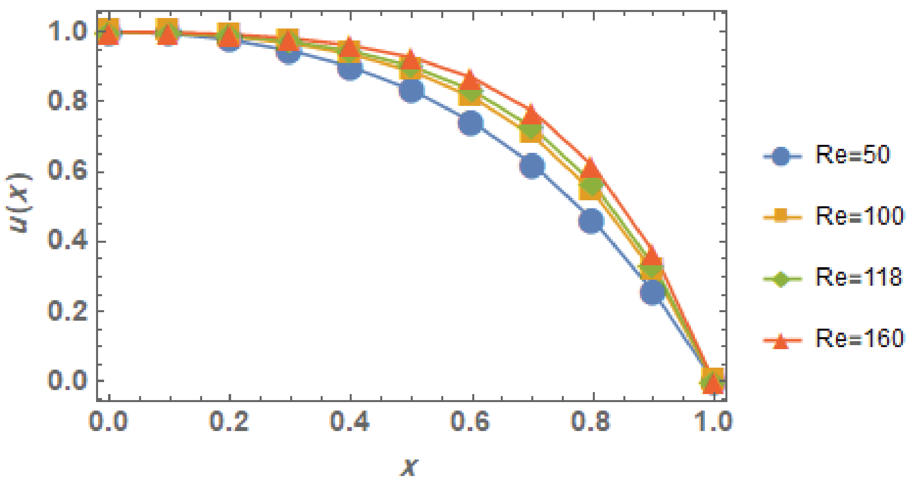

Fig. 3. The velocity solution obtained by proposed methods for various values of the Reynolds number and $\alpha = -5°$.

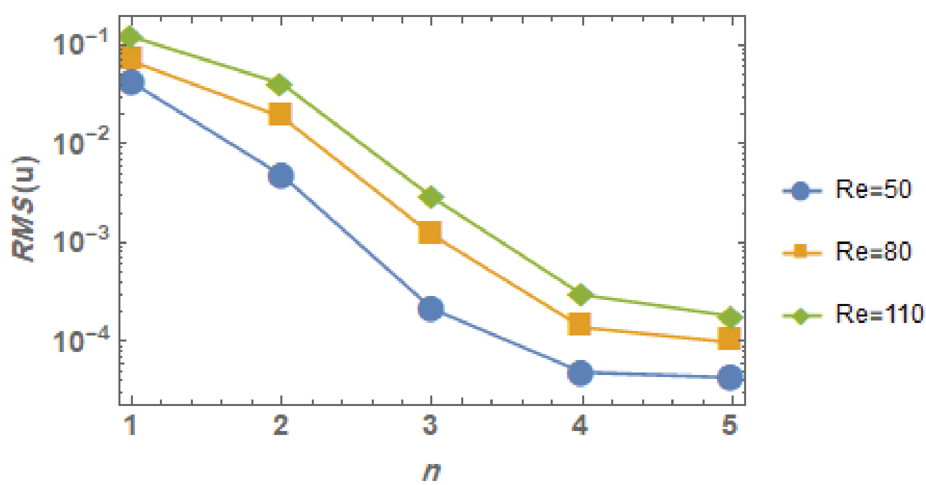

Fig. 4. Logarithmic plots of the RMS function for different values of Re when $\alpha = 5°$.

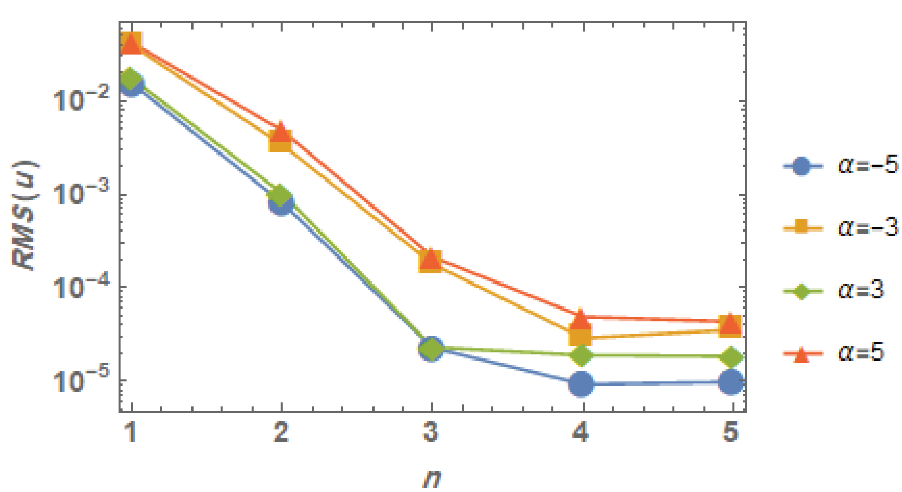

Fig. 5. Logarithmic plots of the RMS function for different values of α when $Re = 50$.

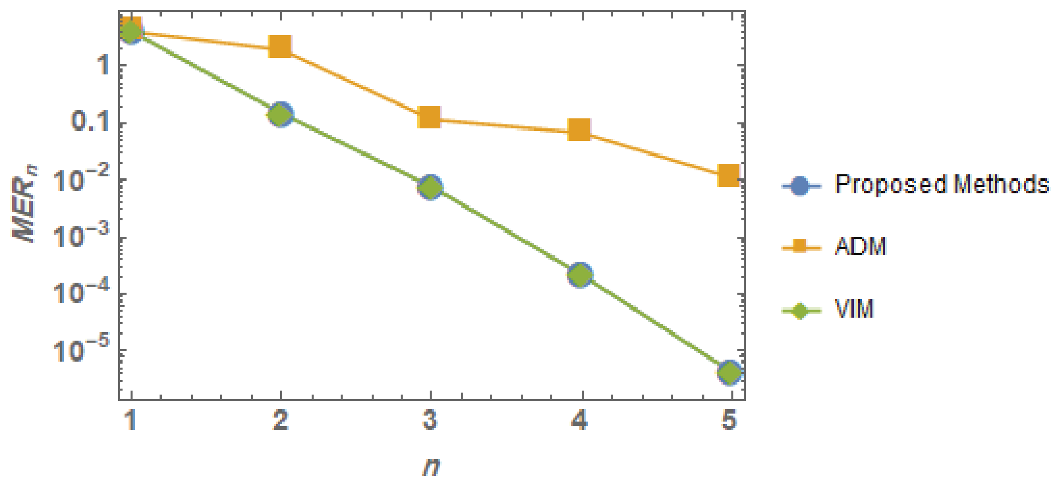

Fig. 6. The MER_n values obtained by proposed methods, the ADM, and the VIM when $\alpha = 2.5°$ and $Re = 50$.

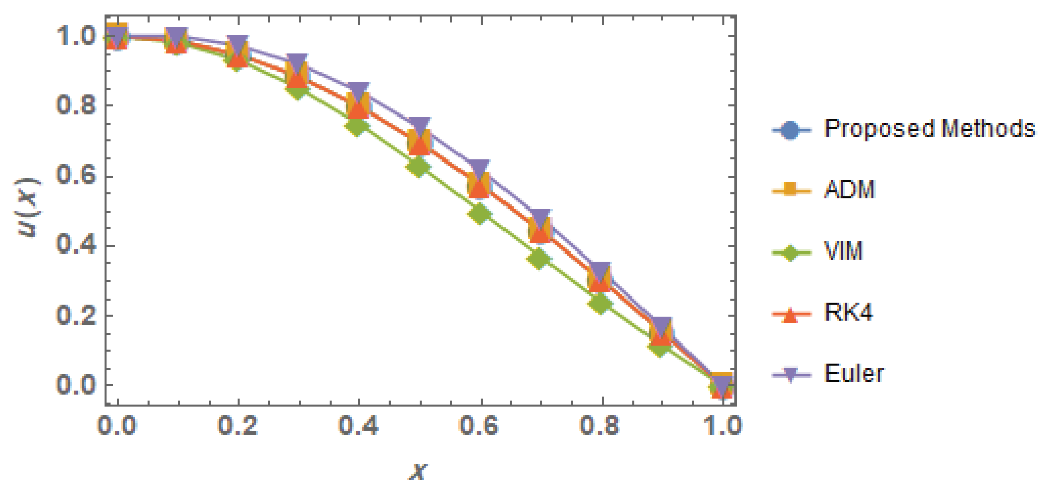

Fig. 7. The numerical solutions obtained by our proposed methods, the ADM and the VIM at $\alpha = 2.5°$ and $Re = 50$.

When comparing the proposed methods to other existing analytic and numerical methods, the main features of the proposed methods are that there is no need for linearization or discretization step-size for determining a sequence of subintervals over a large interval. Furthermore, the large computational work, the additional parameters, and the round-off errors are avoided without using any restricted assumptions for the nonlinear terms. We also overcome the difficulty that arises when calculating Adomian polynomials to handle the nonlinear terms in ADM. Our methods do not require calculating Lagrange multipliers as in VIM.

However, the limitation of our iterative methods is that by increasing the values of α or the Reynolds number Re, the accuracy and convergence decreases.

7. Conclusion

In this paper, the approximate solutions for the Jeffery-Hamel flow problem are obtained by using the three iterative methods (DJM, TAM and BCM). The obtained approximate solutions are presented in a convergent series without any restrictive assumptions in order to deal with the nonlinear terms. In a numerical simulation, we have shown that the values of the maximum error remainder decrease when the number of the iterations increases. Furthermore, the numerical results of the proposed methods were compared with those obtained by the Runge-Kutta method (RKM). This was carried out by evaluating the root mean square norm. In conclusion, the proposed methods are very accurate since they find reliable results.

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ثلاث طرائق تكرارية لحل مسألة تدفق جيفري-هامل

مجيد أحمد الجواري، الزهراء جابر عبد النبي

قسم الرياضيات، كلية التربية للعلوم الصرفة (ابن الهيثم)، جامعة بغداد، بغداد، العراق

الملخص

في هذه المقالة، تم حل مسألة تدفق جيفري-هامل غير الخطية تحليلياً وعددياً باستخدام طرائق تكرارية وعددية موثوقة. تم استنتاج الحلول التقريبية باستخدام طريقة دافتاردار-جعفري، طريقة التميمي-الأنصاري، وطريقة تقلص بناخ. تمت مناقشة النتائج المستنتجة عددياً، بالمقارنة مع حلول عددية أخرى تم الحصول عليها عددياً من طرائق رانج-كوتا ذات الرتبة الرابعة، أويلر وطرائق تحليلية سابقة متاحة. بالإضافة إلى ذلك، تم اعطاء التقارب للطرائق المقترحة استناداً إلى نظرية بناخ للنقطة الثابتة. تشير النتائج إلى أن الطرائق المُقدّمَة موثوقة، وذات فعالية وقابلة للتطبيق لحل مسائل غير خطية أخرى. تمت أعمالنا الحاسوبية باستخدام برنامج ماثيماتيكا لحُسبان الحدود في العمليات التكرارية.