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Abundant rational solutions and conservation laws of the generalized BKP-Boussinesq equation and its dimensionally reduced equations

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Abstract

By virtue of a variable transformation and a quadratic function approach, we generate rational analytical solutions to the (3+1)-dimensional gBKP-Boussinesq equation and lump solutions to its three types of dimensionally reduced equations. In addition, we derive quite a few conservation laws for those equations by employing the first homotopy method.

Keywords: (3+1)-dimensional gBKP-Boussinesq equation; conservation laws; dimensionally reduced equations; lump solutions; rational solutions

1. Introduction

It is well known that soliton solutions to nonlinear partial differential equations (NLPDEs), describing significant wave phenomena, have been a popular research topic in the field of nonlinear and modern mathematical physics. Rational solutions (Zhang & Ma, 2015a; Zhang & Ma, 2015b; Shi *et al*., 2015) to NLPDEs have especially attracted a great deal of attention in recent years. Lump solutions, as a particular type of rational solutions, are rationally localized in all directions in space. By taking advantage of the variable transformations and quadratic functions, a powerful novel approach has been presented that seeks lump solutions, since a study (Ma, 2015a) by one of the authors (Ma). By means of this method, a sequence of NLPDEs has been investigated and shown to possess lump solutions. These include the following equations: the reduced p-gKP and p-gBKP equations (Ma, Qin & Lü, 2016), the potential-YTSF equation (Lü & Chen, 2015), the BKP equation (Yang & Ma, 2016), the generalized B-type KP equation (Wu *et al*., 2017), the generalized shallow water-like equation (Zhang *et al*., 2017), the Jimbo-Miwa equation (Yang & Ma, 2017), the asymmetrical Nizhnik-Novikov-Veselov equation (Zhao *et al*., 2017), the Fokas equation (Cheng & Zhang, 2017), the generalized KP-Boussinesq equation (Lü *et al*., 2016), and so forth.

On the other hand, conservation laws play a significant role in the study of integrability of NLPDEs, for example, integrable reductions, well-possedness of Cauchy problems, and various numerical techniques. Conservation laws, closely related constants of motion, are also used in attempting linearization problems to explore integrability.

The so-called (3+1)-dimensional B-type of the KP equation (Ma & Fan, 2011; Ma & Zhu, 2012) takes the form

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0, \tag{1}$$

where u is a function of the variables x, y, z and t. Recently, via adding one extra term of second order in time t to the B-type of the KP Equation (1), a new (3+1)-dimensional gBKP-Boussinesq equation (Wazwaz & El-Tantawy, 2017)

$$u_{ty}-u_{xxxy}-3(u_xu_y)_x+u_{tt}+3u_{xz}=0 \tag{2}$$

was proposed. This equation is used as a mathematical model to describe both right and left-going waves as the Boussinesq equation. By use of the simplified Hirota's approach, the general phase shift and one- and two-soliton solutions for Equation (2) were derived.

The purpose of this article is twofold. First, the focus is on the construction of rational solutions to the (3+1)-dimensional gBKP-Boussinesq Equation (2) and lump solutions to three types of dimensionally reduced equations utilizing the quadratic function method. Second, the focus is on deriving some conservation laws of Equation (2) and its dimensionally reduced equations using the first homotopy method.

2. Rational solutions to the gBKP-Boussinesq equation

This section is devoted to seeking rational analytical solutions to Equation (2). Under the relation between f and u

$$u=2(\ln f)_x, \tag{3}$$

where $f=f(x,y,z,t)$ is a real unknown function, the gBKP-Boussinesq Equation (2) can be cast into the following form:

$$\begin{aligned}&-2f_xf_t^2-f_{xxxx}f_yf-4f_{xxxy}f_xf\\&+2f_{xxx}f_{xy}f+6f_{xxy}f_x^2+2f_{xt}f_tf\\&+f_xf_{tt}f+3f_{xx}f_zf+6f_xf_{xz}f\\&-6f_x^2f_z+f_xf_{yt}f+f_{xt}f_yf\\&+f_{xy}f_tf+f_{xxxxy}f^2-f_{xtt}f^2\\&-3f_{xxz}f^2-f_{xyt}f^2-2f_xf_tf_y\\&+2f_{xxx}f_xf_y-6f_{xx}f_xf_{xy}=0.\end{aligned} \tag{4}$$

Thus, it is obvious that if f solves Equation (4), then $u=2(ln\ f)_x$ presents a solution to Equation (2). For the derivation of the quadratic function solutions to Equation (4), we assume that

$$\begin{cases}f=g^2+h^2+\delta,\\g=\alpha x+\beta y+\mu z+\nu t,\\h=ax+by+cz+dt,\end{cases} \tag{5}$$

where $\alpha,\beta,\mu,\nu,a,b,c,d$ and δ are some real undetermined parameters. Inserting (5) into Equation (4) directly gives rise to a set of algebraic equations for the parameters, from which we obtain

$$a=-\frac{d(d^2+\nu^2)}{3\mu\nu},b=-d,c=0,$$

$$\alpha=-\frac{d^2+\nu^2}{3\mu},\beta=\frac{d^2}{\nu}.$$

Here μ,ν,d,δ are arbitrary parameters and $\mu\nu\neq 0$. Therefore, we arrive at the positive quadratic function solutions for Equation (4)

$$\begin{aligned}f&=(-\frac{d^2+\nu^2}{3\mu}x+\frac{d^2}{\nu}y+\mu z+\nu t)^2\\&+(-\frac{d(d^2+\nu^2)}{3\mu\nu}x-dy+dt)^2+\delta,\end{aligned} \tag{6}$$

provided that the parameter δ is positive. The resulting class of quadratic function solutions, in turn, leads to a class of rational solutions to the (3+1)-dimensional gBKP-Boussinesq equation (2)

$$\begin{aligned}u&=4[\alpha(-\frac{d^2+\nu^2}{3\mu}x+\frac{d^2}{\nu}y+\mu z+\nu t)\\&+a(-\frac{d(d^2+\nu^2)}{3\mu\nu}x-dy+dt)]\Big/[(\frac{d^2+\nu^2}{3\mu}x\\&-\frac{d^2}{\nu}y-\mu z-\nu t)^2+(\frac{d(d^2+\nu^2)}{3\mu\nu}x+dy-dt)^2+\delta]\end{aligned} \tag{7}$$

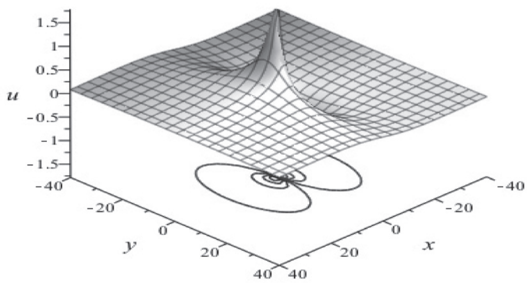
which is a class of rational solutions, involving four arbitrary parameters, for Equation (2). The solutions are similar to the results (Yan *et al*. 2018). With regard to (7), through taking particular values of the parameters as

$$d=1.5,\mu=2,\nu=4,\delta=10,$$

the following specific solution is

$$\begin{aligned}u&=\frac{18.25}{144}[(333.0625x-192z\\&-438t)]\Big/[(\frac{18.25}{6}x-\frac{2.25}{4}y-2z-4t)^2\\&+(\frac{9.125}{8}x+1.5y-1.5t)^2+10]\end{aligned} \tag{8}$$

According to Solution (8), some plots are depicted (Figures *1* and 2) to show localized characteristics and dynamic behaviors. Figure *1* displays the localized structures of Solution (8) in the *(x,y)*-, *(y,z)*- and *(x,z)*-planes respectively. It is clear that the wave possesses two adjacent humps in opposite directions: one is above the plane and the other is below. From Figure 2, It is evidently seen that the wave in *(x,y)*-plane propagates towards the positive direction of the x-axis as time evolves, and it retains its shape during the propagation.



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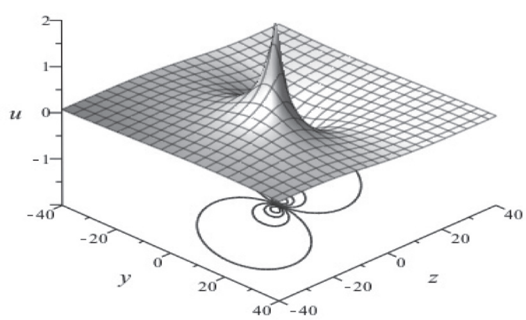

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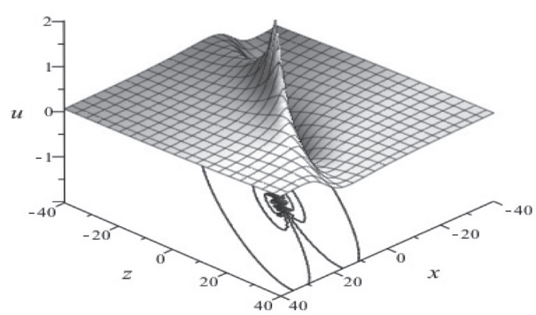
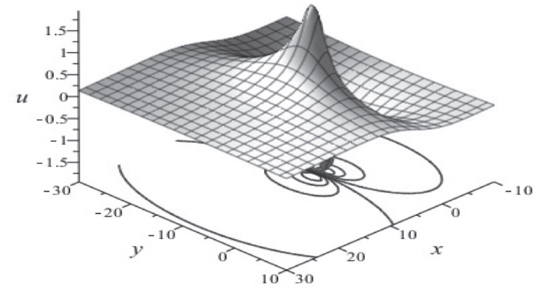


(c)

Fig. 1. Graphs of rational solution (8). (a) $z = t = 0$; (b) $x = t = 0$; (c) $y = t = 0$. Curved lines drawn in the horizontal planes are contour lines.



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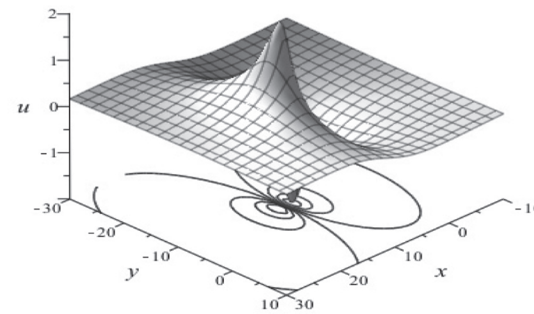

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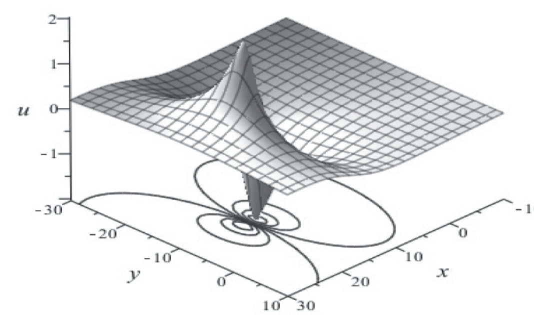

(c)

Fig. 2. Graphs of rational solution (8) when $z = 20$ with different times. (a) $t = -5$; (b) $t = 0$; (c) $t = 5$. Curved lines drawn in the horizontal plane are contour lines.

3. Lump solutions to the dimensionally reduced gBKP-Boussinesq equations

3.1 The reduction of z=x

In this subsection, we consider the case of $z=x$, then a dimensionally reduced form of Equation (2) is

$$u_{ty}-u_{xxxy}-3(u_xu_y)_x+u_{tt}+3u_{xx}=0. \tag{9}$$

A direct substitution of the relation (3) and this assumption

$$\begin{cases} f=g^2+h^2+\delta, \\ g=\alpha x+\beta y+\nu t, \\ h=ax+by+dt, \end{cases} \tag{10}$$

into Equation (9) generates the constraining equations

$$\beta=\frac{3a^2\nu-6a\alpha d-3\alpha^2\nu-d^2\nu-\nu^3}{d^2+\nu^2},$$
$$b=-\frac{3a^2d+6a\alpha\nu-3\alpha^2d+d^3+d\nu^2}{d^2+\nu^2},$$
$$\delta=-\frac{1}{(a\nu-\alpha d)^2}(3a^5d+3a^4\alpha\nu+6a^3\alpha^2d+a^3d^3+a^3d\nu^2+6a^2\alpha^3\nu+a^2\alpha d^2\nu+a^2\alpha\nu^3+3a\alpha^4d+a\alpha^2d^3+a\alpha^2d\nu^2+3\alpha^5\nu+\alpha^3d^2\nu+\alpha^3\nu^3),$$

where α,a,ν,d are arbitrary parameters and $d^2+\nu^2\neq 0$. By imposing $a\nu-\alpha d\neq 0$ and the positivity condition $\delta>0$, it follows that

$$f=(\alpha x+\frac{1}{d^2+\nu^2}(3a^2\nu-6a\alpha d-3\alpha^2\nu-d^2\nu-\nu^3)y+\nu t)^2+(ax-\frac{1}{d^2+\nu^2}(3a^2d+6a\alpha\nu-3\alpha^2d+d^3+d\nu^2)y+dt)^2-\frac{1}{(a\nu-\alpha d)^2}(3a^5d+3a^4\alpha\nu+6a^3\alpha^2d+a^3d^3+a^3d\nu^2+6a^2\alpha^3\nu+a^2\alpha d^2\nu+a^2\alpha\nu^3+3a\alpha^4d+a\alpha^2d^3+a\alpha^2d\nu^2+3\alpha^5\nu+\alpha^3d^2\nu+\alpha^3\nu^3), \tag{11}$$

which gives rise to a class of lump solutions to Equation (9)

$$u=\frac{4(\alpha g+ah)}{f},$$

where f is expressed by (11) and

$$\begin{cases} g=\alpha x+\frac{1}{d^2+\nu^2}(3a^2\nu-6a\alpha d-3\alpha^2\nu-d^2\nu-\nu^3)y+\nu t, \\ h=ax-\frac{1}{d^2+\nu^2}(3a^2d+6a\alpha\nu-3\alpha^2d+d^3+d\nu^2)y+dt. \end{cases}$$

3.2 The reduction of z=y

For the reduction with $z=y$, Equation (2) is changed into

$$u_{ty}-u_{xxxy}-3(u_xu_y)_x+u_{tt}+3u_{xy}=0. \tag{12}$$

Carrying (3) and (10) into Equation (12), we derive the constraining equations

$$\alpha=\frac{1}{3d\beta}(3a\beta^2+\beta^2d+d^3),$$
$$\nu=-\frac{\beta}{d}(3a+d),\quad b=\frac{\beta^2}{d^2}(3a+d),$$
$$\delta=\frac{1}{9d^2\beta^2}(9a^2\beta^4+9a^2\beta^2d^2+6a\beta^4d+6a\beta^2d^3+\beta^4d^2+2\beta^2d^4+d^6),$$

where β,a,d are free parameters, $d\beta\neq 0$ and $\delta>0$. We can therefore write a class of lump solutions to Equation (12) as follows

$$u=\frac{4(\alpha g+ah)}{f},$$

in which

$$f=(\frac{3a\beta^2+\beta^2d+d^3}{3d\beta}x+\beta y-\frac{\beta(3a+d)}{d}t)^2+(ax+\frac{\beta^2}{d^2}(3a+d)y+dt)^2+\frac{1}{9d^2\beta^2}(9a^2\beta^4+9a^2\beta^2d^2+6a\beta^4d+6a\beta^2d^3+\beta^4d^2+2\beta^2d^4+d^6),$$

and

$$\begin{cases} g=\frac{3a\beta^2+\beta^2d+d^3}{3d\beta}x+\beta y-\frac{\beta(3a+d)}{d}t, \\ h=ax+\frac{\beta^2(3a+d)}{d^2}y+dt. \end{cases}$$

3.3 The reduction of z=t

If we let $z=t$, then Equation (2) is turned into

$$u_{ty}-u_{xxxy}-3(u_xu_y)_x+u_{tt}+3u_{xt}=0. \tag{13}$$

The substitution of (3) and (10) into Equation (13) yields the constraining equations

$$\beta = -3\alpha - \nu, b = \frac{\alpha}{a}(3\alpha + \nu),$$

$$d = -\frac{1}{a}(3a^2 + 3\alpha^2 + \alpha\nu),$$

where α, ν are free parameters, $a \neq 0$ and $\delta > 0$. Hence, we obtain a class of lump solutions to equation (13)

$$u = \frac{4(\alpha g + ah)}{f},$$

where

$$f = (\alpha x - (3\alpha + \nu)y + \nu t)^2 + (ax + \frac{\alpha}{a}(3\alpha + \nu)y - \frac{1}{a}(3a^2 + 3\alpha^2 + \alpha\nu)t)^2 + \delta,$$

and

$$\begin{cases} g = \alpha x - (3\alpha + \nu)y + \nu t, \\ h = ax + \dfrac{\alpha(3\alpha + \nu)}{a} y - \dfrac{3a^2 + 3\alpha^2 + \alpha\nu}{a} t. \end{cases}$$

4. Conservation laws for the gBKP-Boussinesq equation

In this section, we find the conservation laws for the gBKP-Boussinesq Equation (2). Until now, there are a variety of systematic approaches for obtaining conservation laws of NLPDEs, including the Noether theorem (Olver, 1993), the symmetry and adjoint symmetry method (Ma, 2015b; Ma, 2018), the multiplier method (Moleleki *et al*., 2017), the first homotopy method (Cheviakov, 2010; Volterra, 1913; Flanders, 1963), and so on. The method used here for computing conservation laws is the first homotopy method. We first give a brief review on the main steps of the first homotopy method. For more details, we recommend that readers refer to (Cheviakov, 2010).

Without loss of generality, we consider a given nonlinear differential equation with independent variables

$$x = (x_1 = t, x_2, \cdots, x_n)$$

and dependent variables

$$u(x) = (u^1, u^2, \cdots, u^m)(x):$$

$$\Phi(x, u, \partial u, \cdots) = 0, \tag{14}$$

where the symbol ∂u denotes the set of the first-order derivatives of u with respect to the independent variables t and x_i, $2 \leq i \leq n$. A conservation law for Equation (14) is a divergence expression

$$\mathrm{Div}T[U] = \mathrm{D}_t T^t[U] + \sum_{i=2}^{n} \mathrm{D}_i T^{x_i}[U] = 0, \tag{15}$$

which vanishes for all solutions of Equation (14). Here $T^t[U]$ is referred to as a conserved density, and $T^{x_i}[U]$ as spatial fluxes of the conservation law (15).

The main steps of the first homotopy method can be summarized as follows:

Step 1. Computation of the conservation law multipliers

One first needs to solve multiplier determining equations

$$\mathrm{E}_U(\Lambda\Phi(x, U, \partial U, \cdots, \partial^k U)) = 0, \tag{16}$$

with

$$\Lambda = \Lambda(x, U, \partial U, \cdots, \partial^l U)$$

holding for arbitrary function *U(x)*, which yields the set of linear determining equations to obtain all sets of conservation law multipliers of Equation (14).

Step 2. Calculation of the n-dimensional Euler operator

After finding multipliers for Equation (14), one can calculate the *n*-dimensional Euler operator through the formula given by

$$\mathrm{E}_U^{(s_1,\cdots,s_n)} = \sum_{k_1=s_1}^{\infty} \cdots \sum_{k_n=s_n}^{\infty} \binom{k_1}{s_1} \cdots \binom{k_n}{s_n} \mathrm{D}_1^{k_1-s_1} \cdots \mathrm{D}_n^{k_n-s_n} \frac{\partial}{\partial U^{(k_1+\cdots+k_n)}},$$

where $U^{(k_1+\cdots+k_n)} = \dfrac{\partial^{k_1+\cdots+k_n} U}{\partial^{k_1} x_1 \cdots \partial^{k_n} x_n}$.

Step 3. Determination of the n-dimensional homotopy operator

The *n*-dimensional homotopy operator associated with the respective independent variable x_i can be derived via the formula

$$\mathrm{H}^{(x_i)}(f[U]) = \int_0^1 \sum_{j=1}^{m} \mathrm{I}_j^{(x_i)}(f[\tilde{U}])\Big|_{\tilde{U}=\lambda U} \frac{\mathrm{d}\lambda}{\lambda},$$

where for j=1,…,m,

$$\mathrm{I}_j^{(x_i)}(f[\tilde{U}]) = \sum_{s_1=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} \left(\frac{1+s_i}{1+s_1+\cdots+s_n}\right) \mathrm{D}_1^{s_1} \cdots \mathrm{D}_n^{s_n} (\tilde{U}^j \mathrm{E}_{\tilde{U}^j}^{(s_1,\cdots,s_{i-1},1+s_i,s_{i+1},\cdots,s_n)}(f[\tilde{U}])).$$

Based on the above steps, one can write a conserved density $T^t[U]$ and fluxes $T^{x_i}[U]$ as

$$T^t[U]=\mathrm{H}^{(t)}(f[U]),T^{x_i}[U]=\mathrm{H}^{(x_i)}(f[U]).$$

Therefore, one can completely determine the conserved density and fluxes in a conservation law on the solutions $U(x) = u(x)$ of the considered equation.

In order to derive conservation laws for Equation (2), we first compute multipliers, which are also adjoint symmetries (Ma, 2015b; Ma, 2018). Here we search for conservation laws arising from multipliers of the form

$$\Lambda=\Lambda(t,x,y,z,u,u_x)$$

Then the determining equations about Λ can be generated from (16) as

$$\Lambda_{yt}=-\Lambda_{tt},$$

$$\Lambda_{yu_r}=\Lambda_{zu_r}=\Lambda_{tu_r}=\Lambda_{u_ru_r}=\Lambda_x=\Lambda_u=0.$$

Now, a direct calculation leads to

$$\Lambda=cu_x+F_1(z,y)+F_2(z,t-y),$$

where c is an arbitrary constant and $F_1(z, y)$ denotes an arbitrary function of the variables z and y, while $F_2(z, t-y)$ depends on z and $t-y$ freely. It is noted that in the following statement, the conserved density T^t denotes the density of the medium, whereas the associated fluxes T^x, T^y, T^z represent the spatial fluxes. It turns out that we can find the multiplier $\Lambda=u_x$ and its corresponding conserved density and fluxes

$$T^t=-\frac{1}{2}uu_{xt}+\frac{1}{2}u_tu_x+\frac{1}{2}u_xu_y,$$

$$T^x=-uu_xu_{xy}-2u_x^2u_y+\frac{3}{2}u_xu_z-\frac{1}{2}u_{xxx}u_y$$

$$+\frac{1}{2}u_{xx}u_{xy}-\frac{1}{2}u_xu_{xxy}+\frac{1}{2}uu_{tt}+\frac{1}{2}uu_{yt}$$

$$-\frac{1}{2}uu_{xxxy}+\frac{3}{2}uu_{xz},$$

$$T^y=uu_xu_{xx}-\frac{1}{2}uu_{xt}+\frac{1}{2}uu_{xxxx},$$

$$T^z=-\frac{3}{2}uu_{xx}.$$

5. Conservation laws for dimensionally reduced gBKP-Boussinesq equations

Here we apply the first homotopy method to three dimensionally reduced equations and present multipliers and their corresponding conserved density and fluxes. We suppose that the multipliers take the form $\Lambda=\Lambda(t, x, y, u, u_x)$.

5.1 The reduction of z=x

As to Equation (9), we can determine the multipliers via direct computation as follows

$$\Lambda=cu_x+F_1(y)+F_2(t-y),$$

where c is an arbitrary constant, while $F_1(y)$ and $F_2(t-y)$ represent arbitrary functions of y and $t-y$, respectively.

Case 1. Corresponding to the multiplier $\Lambda = u_x$, we have

$$T_1^t=-\frac{1}{2}uu_{xt}+\frac{1}{2}u_tu_x+\frac{1}{2}u_xu_y,$$

$$T_1^x=-uu_xu_{xy}-2u_x^2u_y+\frac{3}{2}u_x^2-\frac{1}{2}u_{xxx}u_y$$

$$+\frac{1}{2}u_{xx}u_{xy}-\frac{1}{2}u_xu_{xxy}+\frac{1}{2}uu_{tt}+\frac{1}{2}uu_{yt}$$

$$-\frac{1}{2}uu_{xxxy},$$

$$T_1^y=uu_xu_{xx}-\frac{1}{2}uu_{xt}+\frac{1}{2}uu_{xxxx}.$$

Case 2. Corresponding to the multiplier $\Lambda=F_2(t-y)$, we have

$$T_{F_2}^t=-uF_2'(t-y)+u_tF_2(t-y)+u_yF_2(t-y),$$

$$T_{F_2}^x=-3u_xu_yF_2(t-y)+3u_xF_2(t-y)$$

$$-u_{xxy}F_2(t-y),$$

$$T_{F_2}^y=-uF_2'(t-y).$$

5.2 The reduction of z=y

As to Equation (12), the multipliers we are looking for are

$$\Lambda=cu_x+F_1(y)+F_2(t-y),$$

where c is an arbitrary constant, while $F_1(y)$ and $F_2(t-y)$ represent arbitrary functions of y and $t-y$, respectively.

Case 1. Corresponding to the multiplier $\Lambda = u_x$, we find

$$T_1^t = -\frac{1}{2}uu_{xt} + \frac{1}{2}u_t u_x + \frac{1}{2}u_x u_y,$$

$$T_1^x = -uu_x u_{xy} - 2u_x^2 u_y + \frac{3}{2}u_x u_y - \frac{1}{2}u_{xxx}u_y$$

$$+\frac{1}{2}u_{xx}u_{xy} - \frac{1}{2}u_x u_{xxy} + \frac{1}{2}uu_{tt} + \frac{1}{2}uu_{yt}$$

$$-\frac{1}{2}uu_{xxxy} + \frac{3}{2}uu_{xy},$$

$$T_1^y = uu_x u_{xx} - \frac{1}{2}uu_{xt} - \frac{3}{2}uu_{xx} + \frac{1}{2}uu_{xxxx}.$$

Case 2. Corresponding to the multiplier $\Lambda = F_2(t-y)$, we find

$$T_{F_2}^t = -uF_2'(t-y) + u_t F_2(t-y) + u_y F_2(t-y),$$

$$T_{F_2}^x = -3u_x u_y F_2(t-y) + 3u_y F_2(t-y)$$

$$-u_{xxy}F_2(t-y),$$

$$T_{F_2}^y = -uF_2'(t-y).$$

5.3 The reduction of z=t

As to Equation (13), the multipliers have the following expression

$$\Lambda = cu_x + F_1(y) + F_2(t-y),$$

where c is an arbitrary constant, while $F_1(y)$ and $F_2(t-y)$ represent arbitrary functions of y and $t-y$, respectively.

Case 1. Corresponding to the multiplier $\Lambda = u_x$, we have

$$T_1^t = -\frac{1}{2}uu_{xt} + \frac{1}{2}u_t u_x + \frac{1}{2}u_x u_y,$$

$$T_1^x = -uu_x u_{xy} - 2u_x^2 u_y + \frac{3}{2}u_x^2 - \frac{1}{2}u_{xxx}u_y$$

$$+\frac{1}{2}u_{xx}u_{xy} - \frac{1}{2}u_x u_{xxy} + \frac{1}{2}uu_{tt} + \frac{1}{2}uu_{yt}$$

$$-\frac{1}{2}uu_{xxxy}$$

$$T_1^y = uu_x u_{xx} - \frac{1}{2}uu_{xt} + \frac{1}{2}uu_{xxxx}.$$

Case 2. Corresponding to the multiplier $\Lambda = F_2(t-y)$, we have

$$T_{F_2}^t = -uF_2'(t-y) + u_t F_2(t-y) + 3u_x F_2(t-y)$$

$$+u_y F_2(t-y),$$

$$T_{F_2}^x = -3u_x u_y F_2(t-y) - 3uF_2'(t-y)$$

$$-u_{xxy}F_2(t-y),$$

$$T_{F_2}^y = -uF_2'(t-y).$$

6. Concluding remarks

In summary, based on a variable transform and a quadratic function method, we first examined the (3+1)-dimensional gBKP-Boussinesq equation and derived a class of rational solutions to this equation. Some graphs were made to show the localized characteristics and dynamic behaviors of the obtained solutions. Similarly, three classes of lump solutions were also presented for the dimensionally reduced forms of the (3+1)-dimensional gBKP-Boussinesq equation with $z = x$, $z = y$ and $z = t$, respectively. These lump solutions contain a set of free parameters. If different values are selected for these parameters, the spatial structures of the solutions will changed accordingly. Furthermore, upon applying the first homotopy method, we constructed quite a few conservation laws for both the (3+1)-dimensional gBKP-Boussinesq equation and its dimensionally reduced equations.

There are also some systematic studies on lump solutions in any dimensions (Ma, Zhou & Dougherty, 2016; Ma & Zhou, 2018). Diversity of interaction solutions between lumps and solitons (or periodic waves) would be a very interesting problem (Ma *et al*., 2017; Zhao & Ma, 2017; Zhang & Ma, 2017).

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حلول منطقية وفيرة وقوانين الحفظ لمعادلة BKP-Boussinesq المُعممة ومعادلاتها المختزلة بشكل بُعدي

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الملخص

بفضل التحويل المتغير ونهج الدالة التربيعية، فإننا ننتج حلولاً تحليلية منطقية جديدة لمعادلة (1 + 3 BKP-Boussinesq) البعدية ومجموعة حلول لأنواعها الثلاثة من المعادلات المختزلة بشكل بُعدي. علاوة على ذلك، فإننا نشتق عدداً قليلاً من قوانين الحفظ لتلك المعادلات من خلال استخدام الطريقة التوفيقية الأولى.