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The characteristic polynomial of some anti-tridiagonal 2-Hankel matrices of even order

João Lita da Silva*
Dept. of Mathematics and GeoBioTec
Faculty of Sciences and Technology, NOVA University of Lisbon
Quinta da Torre, 2829-516 Caparica, Portugal
**Corresponding author: jfls@fct.unl.pt*

Abstract

In this paper, we derive the characteristic polynomial for a family of anti-tridiagonal 2-Hankel matrices of even order in terms of Chebyshev polynomials, giving also a representation of its eigenvectors. An orthogonal diagonalization for these type of matrices having null northeast-to-southwest diagonal is also provided using prescribed eigenvalues.

Keywords: Anti-tridiagonal 2-Hankel matrix; Chebyshev polynomials; eigenvalue; eigenvector
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1. Introduction

The concept of an r-Toeplitz matrix was introduced by Gover and Barnett in the eighties (Gover & Barnett, 1985), which also established many of its properties (Gover & Barnett, 1985; Gover, 1989). They defined an r-Toeplitz matrix as an $n \times n$ matrix $\mathbf{A}_n$, such that $[\mathbf{A}_n]_{k+r,\ell+r} = [\mathbf{A}_n]_{k,\ell}$ for all $k,\ell = 1,2,\ldots,n-r$. Following this idea, we say that an $n \times n$ matrix $\mathbf{B}_n$ is an *r-Hankel matrix* if $[\mathbf{B}_n]_{k+r,\ell-r} = [\mathbf{B}_n]_{k,\ell}$ for every $k = 1,2,\ldots,n-r$ and $\ell = r+1,\ldots,n$. Note that when $r = 1$, the matrix $\mathbf{B}_n$ becomes a Hankel matrix.

Let us point out that Hankel matrices appear not only in engineering problems of system and control theory (Olshevsky & Stewart, 2001 and the references therein), but also in computational mathematics (Bultheel & Van Barel, 1997).

In this note, we shall consider a particular type of anti-tridiagonal 2-Hankel matrices of even order, concretely, real $2n \times 2n$ matrices of the form

$$\mathbf{H}_{2n} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & b_1 & c \\ \vdots & & & \iddots & a_2 & d & a_1 \\ \vdots & & \iddots & \iddots & c & b_2 & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a_2 & d & \iddots & \iddots & & \vdots \\ b_1 & c & b_2 & \iddots & & & \vdots \\ d & a_1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{1}$$

with $cd = 0$. It is our goal to obtain an explicit expression for the characteristic polynomial of $\mathbf{H}_{2n}$ as well as a representation of its eigenvectors for eigenvalues given *a priori*. As a consequence, sufficient conditions are announced to get an orthogonal diagonalization of anti-tridiagonal 2-Hankel matrices of even order having null northeast-to-southwest diagonal. We emphasize that, in general, $\mathbf{H}_{2n}$ is not a persymmetric matrix, which makes some recent approaches concerning this issue unfeasible (Akbulak, da Fonseca & Yilmaz, 2013; Wu, 2010). Therefore, our results emerge as a complement for these and other papers about spectral properties of anti-tridiagonal matrices.

2. Main results

For any integer $p \geqslant -1$, we shall denote by $U_p(x)$ the pth degree Chebyshev polynomial of the second kind

$$U_p(x) = \frac{\sin[(p+1)\arccos x]}{\sin(\arccos x)}, \quad -1 < x < 1,$$

with $U_p(\pm 1) = (\pm 1)^p(p+1)$ (Mason & Handscomb, 2003). This expression as a sum of powers of x can, of course, be evaluated for any x. The symbols $\lfloor x \rfloor$ and $\otimes$ will be used to indicate the largest integer not greater than x and the Kronecker product, respectively. The Euclidean norm will be denoted by $\| \cdot \|$.

Let ξ, b_1, b_2 be real numbers such that $b_1 b_2 \neq 0$. Throughout, we shall consider the sequence of polynomials $\{Q_k(x,\xi)\}_{k \geqslant 0}$ defined by

$$Q_k(x,\xi) := \begin{cases} x(b_1b_2)^{\frac{k-1}{2}} U_{\frac{k-1}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right), & k \text{ odd} \\[2ex] (b_1b_2)^{\frac{k}{2}} U_{\frac{k}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) + \\ \quad \xi^2 (b_1b_2)^{\frac{k}{2}-1} U_{\frac{k}{2}-1}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right), & k \text{ even} \end{cases} \tag{2}$$

as well as the $n \times n$ matrix $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ whose (k,ℓ)-entry is

$$\begin{cases} -b_{\frac{3-(-1)^k}{2}}^{\left\lfloor \frac{\ell-k}{2} \right\rfloor} b_{\frac{3+(-1)^k}{2}}^{\left\lfloor \frac{\ell-k+1}{2} \right\rfloor} \dfrac{Q_{k-1}(\lambda,b_2) Q_{n-\ell}\left[\lambda, b_{\frac{3+(-1)^n}{2}}\right]}{Q_n(\lambda,b_2)}, & k \leqslant \ell \\[4ex] -b_{\frac{3-(-1)^\ell}{2}}^{\left\lfloor \frac{k-\ell}{2} \right\rfloor} b_{\frac{3+(-1)^\ell}{2}}^{\left\lfloor \frac{k-\ell+1}{2} \right\rfloor} \dfrac{Q_{\ell-1}(\lambda,b_2) Q_{n-k}\left[\lambda, b_{\frac{3+(-1)^n}{2}}\right]}{Q_n(\lambda,b_2)}, & k > \ell \end{cases} \tag{3}$$

and the $n \times n$ matrix $\mathbf{S}_n\left[x, b_{\frac{3+(-1)^n}{2}}, b_2\right]$ given by

$$\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right] - \frac{b_{\frac{3+(-1)^n}{2}} Q_n(x,b_2)}{Q_n(x,b_2) - b_{\frac{3+(-1)^n}{2}} Q_{n-1}(x,b_2)} \cdot \mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right] \mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]^\top \tag{4}$$

with $\mathbf{q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ the last column of $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$. Further, we shall suppose the $n \times n$ matrix $\mathbf{T}_n(x,y)$ defined by

$$\begin{cases} \begin{bmatrix} 0 & x & 0 & \dots & \dots & \dots & 0 \\ x & 0 & y & 0 & & & \vdots \\ 0 & y & 0 & x & \ddots & & \vdots \\ \vdots & 0 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & x \\ 0 & \dots & \dots & \dots & 0 & x & y \end{bmatrix}, & n \text{ even} \\ \begin{bmatrix} 0 & x & 0 & \dots & \dots & \dots & 0 \\ x & 0 & y & 0 & & & \vdots \\ 0 & y & 0 & x & \ddots & & \vdots \\ \vdots & 0 & x & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 & y \\ 0 & \dots & \dots & \dots & 0 & y & x \end{bmatrix}, & n \text{ odd}. \end{cases} \tag{5}$$

Set

$$\mathbf{J}_n := \left[\delta_{k+\ell,n+1}\right]_{k,\ell},$$

$$\mathbf{E}_n := \left[\tfrac{1+(-1)^k}{2}\delta_{k,\ell}\right]_{k,\ell},$$

$$\mathbf{K}_n := \left[\tfrac{1-(-1)^k}{2}\delta_{k,\ell}\right]_{k,\ell}$$

where δ is the Kronecker delta. For $ab \neq 0$, let $\mathbf{u}_n(x,a,b)$ be the n-dimensional vector whose the kth component is

$$\begin{cases} U_{\frac{k-1}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) + \frac{b}{a} U_{\frac{k-3}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ odd} \\ \frac{x}{a} U_{\frac{k}{2}-1}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ even} \end{cases} \tag{6}$$

In what follows, we shall assume the anti-tridiagonal 2-Hankel matrix $\mathbf{H}_{2n}$ defined in (1) with $d = 0$. Notwithstanding, similar results hold for any real number d and $c = 0$, *mutatis mutandis*.

Theorem 1 *Let n be a positive integer, c a real number, $\{Q_k(x,\xi)\}_{k\geqslant 0}$ the sequence of polynomials* (2) *and $\mathbf{T}_n(a_1,a_2)$, $\mathbf{T}_n(b_1,b_2)$ the matrices defined by* (5) *for nonzero reals a_1,a_2,b_1,b_2.*

(a) *If n is even, then the eigenvalues of $\mathbf{H}_{2n}$ in* (1) *are precisely the zeros of*

$$f(x) = (a_1a_2b_1b_2)^{\frac{n}{2}} \cdot \left[U_{\frac{n}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right) + \frac{a_2-x}{a_1} U_{\frac{n}{2}-1}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right)\right] \cdot \left[U_{\frac{n}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) + \frac{b_2-x}{b_1} U_{\frac{n}{2}-1}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right)\right] \tag{7}$$

Moreover, if λ is an eigenvalue of $\mathbf{T}_n(a_1,a_2)$, μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$, $Q_n(\lambda,b_2) \neq b_2 Q_{n-1}(\lambda,b_2)$ and $\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$, then

$$\mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{u}_n(\lambda,a_1,a_2) \\ -c\,\mathbf{S}_n(\lambda,b_2,b_2)\mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix} \tag{8}$$

and

$$\mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_n(\mu,b_1,b_2) \end{bmatrix} \tag{9}$$

are eigenvectors of $\mathbf{H}_{2n}$ associated to λ and μ, respectively, where $\mathbf{P}_{2n}$ is the $2n \times 2n$ permutation matrix

$$\mathbf{P}_{2n} := \left[\begin{array}{c|c} \mathbf{E}_n & \mathbf{J}_n\mathbf{E}_n \\ \hline \mathbf{K}_n & \mathbf{J}_n\mathbf{K}_n \end{array}\right] \tag{10}$$

$\mathbf{u}_n(\lambda,a_1,a_2)$, $\mathbf{u}_n(\mu,b_1,b_2)$ are the n-dimensional vectors defined by (6) *and $\mathbf{S}_n(\lambda,b_2,b_2)$ is the $n \times n$ matrix given in* (4).

(b) *If n is odd, then the eigenvalues of $\mathbf{H}_{2n}$* in (1) *are precisely the zeros of*

$$f(x) = (a_1a_2b_1b_2)^{\frac{n-1}{2}} \cdot \left[(x-a_1)U_{\frac{n-1}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right) - a_2 U_{\frac{n-3}{2}}\left(\frac{x^2-a_1^2-a_2^2}{2a_1a_2}\right)\right] \cdot \left[(x-b_1)U_{\frac{n-1}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right) - b_2 U_{\frac{n-3}{2}}\left(\frac{x^2-b_1^2-b_2^2}{2b_1b_2}\right)\right] \tag{11}$$

Furthermore, if λ is an eigenvalue of $\mathbf{T}_n(a_1,a_2)$, μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$, $Q_n(\lambda,b_2) \neq b_1 Q_{n-1}(\lambda,b_2)$ and $\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$, then

$$\mathbf{P}_{2n}^\top \begin{bmatrix} -c\,\mathbf{S}_n(\lambda,b_1,b_2)\mathbf{u}_n(\lambda,a_1,a_2) \\ \mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix} \tag{12}$$

and

$$\mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{u}_n(\mu,b_1,b_2) \\ \mathbf{0} \end{bmatrix} \tag{13}$$

are eigenvectors of $\mathbf{H}_{2n}$ associated to λ and μ, respectively, where $\mathbf{P}_{2n}$ is the $2n \times 2n$ permutation matrix

$$\mathbf{P}_{2n} := \left[\begin{array}{c|c} \mathbf{K}_n & \mathbf{E}_n\mathbf{J}_n \\ \hline \mathbf{E}_n & \mathbf{K}_n\mathbf{J}_n \end{array}\right] \tag{14}$$

$\mathbf{u}_n(\lambda,a_1,a_2)$, $\mathbf{u}_n(\mu,b_1,b_2)$ are the n-dimensional vectors defined by (6) *and $\mathbf{S}_n(\lambda,b_1,b_2)$ is the $n \times n$ matrix given in* (4).

Remark It is worthwhile to note that by taking $c = 0$ and $a_2 = b_1$, $a_1 = b_2$ in (7) or (11), we recover the expressions obtained in section 4 of da Fonseca

(2018) for the matrices of even order analysed therein.

The previous result leads us to an orthogonal diagonalization for anti-tridiagonal 2-Hankel matrices (1) with null northeast-to-southwest diagonal, i.e. for matrices of the form

$$\mathbf{H}^*_{2n} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & b_1 & 0 \\ \vdots & & & \iddots & a_2 & 0 & a_1 \\ \vdots & & \iddots & \iddots & 0 & b_2 & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & a_2 & 0 & \iddots & \iddots & & \vdots \\ b_1 & 0 & b_2 & \iddots & & & \vdots \\ 0 & a_1 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \tag{15}$$

Put

$$\begin{aligned} \mathbf{V}_n &:= \left[\frac{\mathbf{u}_n(\lambda_1,a_1,a_2)}{\|\mathbf{u}_n(\lambda_1,a_1,a_2)\|} \cdots \frac{\mathbf{u}_n(\lambda_n,a_1,a_2)}{\|\mathbf{u}_n(\lambda_n,a_1,a_2)\|} \right] \\ \mathbf{W}_n &:= \left[\frac{\mathbf{u}_n(\mu_1,b_1,b_2)}{\|\mathbf{u}_n(\mu_1,b_1,b_2)\|} \cdots \frac{\mathbf{u}_n(\mu_n,b_1,b_2)}{\|\mathbf{u}_n(\mu_n,b_1,b_2)\|} \right] \end{aligned} \tag{16}$$

where $\mathbf{u}_n(\lambda_k,a_1,a_2)$ and $\mathbf{u}_n(\mu_k,b_1,b_2)$ are the n-dimensional vectors whose kth components are defined by (6).

Corollary 1 *Let n be a positive integer, a_1,a_2,b_1,b_2 nonzero real numbers, $\mathbf{H}^*_{2n}$ the $2n \times 2n$ matrix* (15), $\mathbf{T}_n(a_1,a_2)$ and $\mathbf{T}_n(b_1,b_2)$ *matrices defined by* (5) *having eigenvalues $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$, respectively. Suppose that* $\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$ *and the sequence of polynomials* $\{Q_k(x,\xi)\}_{k\geqslant 0}$ *given by* (2) *satisfies* $Q_n(\lambda_k,b_2) \neq b_{\frac{3+(-1)^n}{2}} Q_{n-1}(\lambda_k,b_2)$. *for each $k = 1,\ldots,n$.*

(a) *If n is even, then*

$$\mathbf{H}^*_{2n} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)\mathbf{U}_{2n}^\top, \tag{17}$$

where

$$\mathbf{U}_{2n} = \mathbf{P}_{2n}^\top \left[\begin{array}{c|c} \mathbf{V}_n & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{W}_n \end{array} \right], \tag{18}$$

$\mathbf{P}_{2n}$ *is the permutation matrix* (10) *and* $\mathbf{V}_n,\mathbf{W}_n$ *are the $n\times n$ matrices in* (16).

(b) *If n is odd, then*

$$\mathbf{H}^*_{2n} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)\mathbf{U}_{2n}^\top \tag{19}$$

where

$$\mathbf{U}_{2n} = \mathbf{P}_{2n}^\top \left[\begin{array}{c|c} \mathbf{O} & \mathbf{W}_n \\ \hline \mathbf{V}_n & \mathbf{O} \end{array} \right],$$

$\mathbf{P}_{2n}$ *is the permutation matrix* (14) *and* $\mathbf{V}_n,\mathbf{W}_n$ *are the $n \times n$ matrices in* (16).

Remark More generally, Theorem 1 also leads to an eigendecomposition for $\mathbf{H}_{2n}$ in (1) with $d = 0$, taking eigenvector matrices formed by the column vectors (8), (9) or (12), (13) according to whether n is even or odd, respectively.

3. Lemmata and proofs

In order to prove Theorem 1, we will need some auxiliary results. The first one is well-known in the literature (Akbulak, da Fonseca & Yilmaz, 2013) and locates the eigenvalues of tridiagonal matrices having the form (5). Indeed, the characteristic polynomial of $\mathbf{T}_n(a,b)$ is

$$(ab)^{\frac{n}{2}} \left[U_{\frac{n}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) + \frac{b-x}{a} U_{\frac{n}{2}-1}\left(\frac{x^2-a^2-b^2}{2ab}\right) \right],$$

when n is even and

$$(ab)^{\frac{n-1}{2}} \left[(x-a) U_{\frac{n-1}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) - b U_{\frac{n-3}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) \right]$$

whenever n is odd. Next, we shall provide a representation of its eigenvectors.

Lemma 1 *Let n be a positive integer and $\mathbf{T}_n(a,b)$ the $n\times n$ matrix* (5) *with a,b nonzero reals. If λ is an eigenvalue of $\mathbf{T}_n(a,b)$, then $\mathbf{u}_n(\lambda,a,b)$ given in* (6) *is an eigenvector* of $\mathbf{T}_n(a,b)$ *associated to λ* .

Proof. Suppose a positive integer n and reals a,b such that $a \neq 0, b \neq 0$. Consider the three-term recurrence relation,

$$\begin{cases} P_{-1}(x) \equiv 0, \\ P_0(x) \equiv 1, \\ P_k(x) = \frac{x-\beta_k}{\alpha_k} P_{k-1}(x) - \frac{\gamma_{k-1}}{\alpha_k} P_{k-2}(x), \quad 1 \leqslant k \leqslant n \end{cases}$$

with $\gamma_0 = \alpha_n = 1$,

$$\alpha_k = \gamma_k = \begin{cases} a, & k \text{ odd} \\ b, & k \text{ even} \end{cases}$$

and

$$\beta_k = \begin{cases} 0, & k < n \\ b, & k = n \text{ and } n \text{ even} \\ a, & k = n \text{ and } n \text{ odd.} \end{cases}$$

Hence, $P_k(x)$ is expressed by

$$\begin{cases} U_{\frac{k}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right) + \frac{b}{a} U_{\frac{k}{2}-1}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ even} \\ \frac{x}{a} U_{\frac{k-1}{2}}\left(\frac{x^2-a^2-b^2}{2ab}\right), & k \text{ odd} \end{cases}$$

for each $0 \leqslant k \leqslant n-1$ and $[P_0(\lambda),P_1(\lambda),\ldots,P_{n-1}(\lambda)]^\top$ is an eigenvector of $\mathbf{T}_n(a,b)$ associated to the eigenvalue λ (da Fonseca, 2005). The thesis is established. □

The following auxiliary statement is an explicit formula for the inverse of sort of slightly perturbed tridiagonal 2-Toeplitz matrices.

Lemma 2 *Let n be a positive integer, λ a real number,* $\{Q_k(x,\xi)\}_{k\geqslant 0}$ *the sequence of polynomials defined by* (2) and $\mathbf{T}_n(b_1,b_2)$ *the $n \times n$ matrix defined by* (5) *with nonzero reals b_1,b_2. If* $Q_n(\lambda,b_2) \neq b_{\frac{3+(-1)^n}{2}} Q_{n-1}(\lambda,b_2)$, *then*

$$[\mathbf{T}_n(b_1,b_2) - \lambda \mathbf{I}_n]^{-1} = \mathbf{S}_n\left[\lambda, b_{\frac{3+(-1)^n}{2}}, b_2\right] \tag{20}$$

where $\mathbf{S}_n\left[\lambda, b_{\frac{3+(-1)^n}{2}}, b_2\right]$ *is the* $n\times n$ *matrix given by* (4).

Proof. Suppose a positive integer n and real numbers λ, b_1, b_2 such that $b_1 \neq 0$, $b_2 \neq 0$. Employing the Second Principle of Mathematical Induction on the variable n, we can state that $\det[\mathbf{T}_n(b_1,b_2)] = (-1)^{\lfloor \frac{n}{2} \rfloor} b_1^n$, which ensure the nonsingularity of $\mathbf{T}_n(b_1,b_2)$. Denoting $\mathbf{e}_n$ the n-dimensional vector (0,...,0,1), the inverse of $\mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n - b_{\frac{3+(-1)^n}{2}}\mathbf{e}_n$ is the matrix $\mathbf{Q}_n\left[b_{\frac{3+(-1)^n}{2}}\right]$ in (3) (see Theorem 4.1 of da Fonseca & Petronilho, 2001), and the thesis is a direct consequence of the well-known Sherman-Morrison-Woodbury formula. □

Proof of Theorem 1. Since both assertions can be proven in the same way, we only prove (a). Let n be an even positive integer. It is straightforward to see that

$$\mathbf{P}_{2n}\mathbf{H}_{2n}\mathbf{P}_{2n}^\top = \left[\begin{array}{c|c} \mathbf{T}_n(a_1,a_2) & \mathbf{O} \\ \hline c\mathbf{I}_n & \mathbf{T}_n(b_1,b_2) \end{array}\right], \tag{21}$$

where $\mathbf{P}_{2n}$ is the permutation matrix (10). Thus,

$$\det(t\mathbf{I}_{2n} - \mathbf{H}_{2n}) = \det[t\mathbf{I}_n - \mathbf{T}_n(a_1,a_2)]\det[t\mathbf{I}_n - \mathbf{T}_n(b_1,b_2)]$$

and we obtain (7). Let λ be an eigenvalue of $\mathbf{T}_n(a_1,a_2)$. According to (21) we can rewrite the relation $(\mathbf{H}_{2n} - \lambda\mathbf{I}_{2n})\,x = 0$ as

$$\left[\begin{array}{c|c} \mathbf{T}_n(a_1,a_2) - \lambda\mathbf{I}_n & \mathbf{O} \\ \hline c\mathbf{I}_n & \mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n \end{array}\right]\mathbf{P}_{2n}\mathbf{x} = \mathbf{0},$$

that is,

$$\begin{aligned} &[\mathbf{T}_n(a_1,a_2) - \lambda\mathbf{I}_n]\,\mathbf{y}^{(1)} = \mathbf{0}, \\ &c\mathbf{y}^{(1)} + [\mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n]\,\mathbf{y}^{(2)} = \mathbf{0}, \\ &\begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \mathbf{P}_{2n}\mathbf{x}. \end{aligned} \tag{22}$$

Since $\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0$, the matrices $\mathbf{T}_n(a_1,a_2)$ and $\mathbf{T}_n(b_1,b_2)$ have no eigenvalues in common (see Laub, 2005, page 145) which implies $\det[\mathbf{T}_n(b_1,b_2) - \lambda\,\mathbf{I}_n] \neq 0$ and Lemma 1 ensures that the solution of (22) is

$$\mathbf{x} = \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{u}_n(\lambda,a_1,a_2) \\ -c\,[\mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n]^{-1}\mathbf{u}_n(\lambda,a_1,a_2) \end{bmatrix},$$

where $\mathbf{u}_n(\lambda, a_1, a_2)$ is given by (6). From Lemma 2,

$$[\mathbf{T}_n(b_1,b_2) - \lambda\mathbf{I}_n]^{-1} = \mathbf{S}_n(\lambda, b_2, b_2),$$

and (8) is an eigenvector of $\mathbf{H}_{2n}$ associated to the eigenvalue λ. On the other hand, suppose that μ is an eigenvalue of $\mathbf{T}_n(b_1,b_2)$. Since $\mathbf{H}_{2n}\mathbf{x} = \mu\,\mathbf{x}$ is equivalent to

$$\begin{aligned} &[\mathbf{T}_n(a_1,a_2) - \mu\mathbf{I}_n]\,\mathbf{y}^{(1)} = \mathbf{0}, \\ &c\mathbf{y}^{(1)} + [\mathbf{T}_n(b_1,b_2) - \mu\mathbf{I}_n]\,\mathbf{y}^{(2)} = \mathbf{0}, \\ &\begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix} = \mathbf{P}_{2n}\mathbf{x}, \end{aligned}$$

and $\det[\mathbf{T}_n(a_1,a_2) - \mu\,\mathbf{I}_n] \neq 0$, we obtain

$$\mathbf{x} = \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_n(\mu, b_1, b_2) \end{bmatrix},$$

where $\mathbf{u}_n(\mu, b_1, b_2)$ is defined in (6). Therefore, (9) is an eigenvector of $\mathbf{H}_{2n}$ associated to the eigenvalue μ. □

Proof of Corollary 1. Consider an even positive integer n. From

$$\det[\mathbf{I}_n \otimes \mathbf{T}_n(a_1,a_2) - \mathbf{T}_n(b_1,b_2) \otimes \mathbf{I}_n] \neq 0,$$

we can guarantee that all eigenvalues of $\mathbf{H}^*_{2n}$ are distinct. Setting

$$\mathbf{v}_n(\lambda_k) := \mathbf{u}_n(\lambda_k, a_1, a_2),$$

$$\mathbf{w}_n(\mu_k) := \mathbf{u}_n(\mu_k, b_1, b_2)$$

and

$$\widehat{\mathbf{v}}_n(\lambda_k) := \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{v}_n(\lambda_k) \\ \mathbf{0} \end{bmatrix},$$

$$\widehat{\mathbf{w}}_n(\mu_k) := \mathbf{P}_{2n}^\top \begin{bmatrix} \mathbf{0} \\ \mathbf{w}_n(\mu_k) \end{bmatrix}$$

it follows that

$$\left\{ \frac{\widehat{\mathbf{v}}_n(\lambda_1)}{\|\widehat{\mathbf{v}}_n(\lambda_1)\|}, \ldots, \frac{\widehat{\mathbf{v}}_n(\lambda_n)}{\|\widehat{\mathbf{v}}_n(\lambda_n)\|}, \frac{\widehat{\mathbf{w}}_n(\mu_1)}{\|\widehat{\mathbf{w}}_n(\mu_1)\|}, \ldots, \frac{\widehat{\mathbf{w}}_n(\mu_n)}{\|\widehat{\mathbf{w}}_n(\mu_n)\|} \right\} \tag{23}$$

is a complete set of orthogonal eigenvectors according to Theorem 1. Hence,

$$\mathbf{H}^*_{2n} = \mathbf{U}_{2n}\mathrm{diag}(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\mathbf{U}_{2n}^{-1},$$

where

$$\mathbf{U}_{2n} = \begin{bmatrix} \frac{\widehat{\mathbf{v}}_n(\lambda_1)}{\|\widehat{\mathbf{v}}_n(\lambda_1)\|} & \cdots & \frac{\widehat{\mathbf{v}}_n(\lambda_n)}{\|\widehat{\mathbf{v}}_n(\lambda_n)\|} & \frac{\widehat{\mathbf{w}}_n(\mu_1)}{\|\widehat{\mathbf{w}}_n(\mu_1)\|} & \cdots & \frac{\widehat{\mathbf{w}}_n(\mu_n)}{\|\widehat{\mathbf{w}}_n(\mu_n)\|} \end{bmatrix}$$

$$= \mathbf{P}_{2n}^\top \begin{bmatrix} \frac{\mathbf{v}_n(\lambda_1)}{\|\mathbf{v}_n(\lambda_1)\|} & \cdots & \frac{\mathbf{v}_n(\lambda_n)}{\|\mathbf{v}_n(\lambda_n)\|} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \frac{\mathbf{w}_n(\mu_1)}{\|\mathbf{w}_n(\mu_1)\|} & \cdots & \frac{\mathbf{w}_n(\mu_n)}{\|\mathbf{w}_n(\mu_n)\|} \end{bmatrix}$$

provided that $\mathbf{P}_{2n}^\top$ is an orthogonal matrix. Since (23) is an orthonormal set, $\mathbf{U}_{2n}$ is an orthogonal matrix and (17) is established. The proof of (b) is analogous and so will be omitted. □

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كثيرات الحدود المميزة لبعض المصفوفات ثلاثية القطرية
ثنائية هانكل العكسية للترتيب المتساوي

جو أو ليتا دا سيلفا
قسم الرياضيات و تكنولوجيا البيولوجيا الجيولوجية
كلية العلوم والتكنولوجيا، جامعة نوفا في لشبونة، كوينتا دا توري، البرتغال

الملخص

في هذا البحث، نستنتج كثيرات الحدود المميزة لعائلة من المصفوفات ثلاثية القطرية ثنائية هانكل العكسية لترتيب متساوي من حيث كثيرات حدود Chebyshev والتي تقدم أيضا تمثيلاً لمتجهاتها الذاتية. كما تم توفير قطرية متعامدة لهذا النوع من المصفوفات التي لها قطر صفري من الشمال الشرقي إلى الجنوب الغربي باستخدام القيم الذاتية المحددة.