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K-modal *BL*-algebras

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Abstract

This article will introduce *K*-modal *BL*-algebra and investigate some properties of this new algebra. Consequently, *K*-modal filters and □-tautology filters as filters of *K*-modal *BL*-algebras will be dealt with. We will prove that the class of all *K*-modal *BL*-algebras is a variety of algebra. Our final goal in this paper is to prove that a *K*-modal *BL*-algebra is a sub-algebra of direct product of a system of linearly ordered *K*-modal *BL*-algebras under special conditions.

Keywords: Fuzzy logic; Fuzzy modal logic; *K*-modal *BL*-algebra; *K*-modal filter; □-tautology filter.

1. Introduction

Modal logic is an important branch of logic developed firstly in the category of non-classical logics (Fitting, 1991; Fitting, 1992; Fitting & Richard, 1998) and has now been widely used as a formalism for knowledge representation in artificial intelligence and analysis tool in computer science (Abramsky *et al.*, 1992; Gabbay *et al.*, 1994; Gabbay *et al.*, 2003). The fuzzy modal logic $S5(C)$, which was constructed by Hajek, used a schematic extension of *BL*-algebras in order to establish the fuzzy modal logic of $S5$ (Hajek, 2010).The algebraic view of *BL*-logics has been studied and investigated by some authors (Abbasloo & Borumand Saeid, 2014; Ma *et al.,*2009; Tayebi Khorami & Borumand Saeid, 2014; Zhan *et al.,*2014; Zhan *et al.,*2009). In order to answer the question, "what is an algebraic counterpart of a fuzzy modal logic in Hajek's sense?", we must firstly construct the algebraic counterpart of fuzzy minimal modal logic *K*, as the minimal modal logic is that of modal logic that satisfies only the axiom $K\colon \Box(\phi \Rightarrow \varphi) \Rightarrow (\Box\phi \Rightarrow \Box\varphi)$ among modal axioms. Moreover, every other modal logic can be obtained by extending this system through a (possibly infinite) set of extra axioms (Gabbay *et al.,* 2003).The above idea motivated us to introduce an algebraic structure satisfying only the algebraic property of modal principle *K*. Therefore, we enrich *BL*-algebras by modal operators to get algebras named *K*-modal *BL*-algebras, which is the algebraic counterpart of fuzzy minimal modal logic. Our

K-modal *BL*-algebra may have numerous applications in linguistics (Moss & Tiede, 2007) and computer programming (Pratt Vaughan, 1980). It is also used as effective formalisms for arguments on time, space, knowledge, belief, actions, obligations, provability, etc (Fitting, 1998). This paper is organized as follows: in section 2 we give some preliminaries. In section 3 we give definition of *K*-modal *BL*-algebra and several examples of it. We show that axioms $\Box 1 - \Box 3$ are independent of each other. In section 4 we investigate some properties of this algebra. Finally in section 5, the notions *K*-modal filter, $\Box$-tautology filter and *K*-modal prime filter are defined and the theorem which states that the *K*-modal *BL*-algebra is a sub-algebra of direct product of a system of linearly ordered *K*-modal *BL*-algebras under special conditions is proved.

2. Preliminaries

In this section, we give some definitions and theorems that we need in the sequel.

Definition 2.1. (Hajek, 1998) A residuated lattice is an algebra $\mathcal{A} = (A, \cup, \cap, \rightarrow, *, 0, 1)$ of type (2,2,2,2,0,0) such that:

(i) $(A, \cup, \cap, 0, 1)$ is a bounded lattice;

(ii) $(A, *, 1)$ is a commutative monoid and

(iii) the operation $*$ and $\rightarrow$ form an adjoint pair, i.e. $x * y \leq z$ if and only if $x \leq y \rightarrow z$ for all $x, y, z \in A$.

Definition 2.2.(Hajek, 1998) A residuated lattice $\mathcal{A} = (A, \cup, \cap, \rightarrow, *, 0, 1)$ is a *BL*-algebra if and only if the following two identities hold, for all $x, y \in A$:

(iv) $x \cap y = x * (x \rightarrow y)$ (divisibility);

(v) $(x \rightarrow y) \cap (y \rightarrow x) = 1$ (prelinearity).

Theorem 2.3. (Hajek, 1998; Piciu, 2007) In any residuated lattice $\mathcal{A} = (A, \cup, \cap, \rightarrow, *, 0,1)$ the following properties hold for all $x, y, z \in A$:

(1) $x * y \leq x, y$; hence $x * y \leq x \cap y$;

(2) $x * (x \rightarrow y) \leq x \cap y \leq x, y$;

(3) $x \leq y$ if and only if $x \rightarrow y = 1$;

(4) $x \leq y$ implies $x * z \leq y * z$ and $z * x \leq z * y$;

(5) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$;

(6) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$;

(7) $x * (y \rightarrow z) \leq y \rightarrow (x * z) \leq (x * y) \rightarrow (x * z)$;

(8) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$;

(9) $x * (y \cap z) \leq (x * y) \cap (x * z)$;

(10) $x * (\cup_i y_i) = \cup_i (x * y_i)$.

Theorem 2.4. (Piciu, 2007) In any BL-algebra $\mathcal{A} = (A, \cup, \cap, \rightarrow, *, 0, 1)$ the following additional properties hold for all $x, y, z \in A$:

(11) $((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x) \leq x \cup y$;

(12) $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$;

(13) $(x \rightarrow y)^n \cup (y \rightarrow x)^n = 1$.

Definition 2.5. (Hajek, 1998) A non-empty set F of a BL-algebra $\mathcal{A}$ is called a filter if and only if

F1) If $x, y \in F$, then $x * y \in F$;

F2) If $x \in F$ and $x \leq y$, then $y \in F$.

Definition 2.6. (Piciu, 2007) Let $\mathcal{A} = (A, \cap, \cup, 0, 1)$ be a bounded lattice. An element $a \in A$ is called complemented if there is an element $a' \in A$ such that $a \cup a' = 1$ and $a \cap a' = 0$. If such element a' exists it is called a complement of *a*. Let $B(A)$ be the set of all complemented elements of the lattice $\mathcal{A} = (A, \cap, \cup, 0, 1)$.

Lemma 2.7. (Kowalski & Ono, 2001) If $e \in B(A)$, then $e * x = e \cap x$, for any $x \in A$.

Definition 2.8. (Blackburn *et al.*, 2001) A modal algebra is a pair $\mathcal{M} = (\mathcal{A}, \Box)$ such that $(A, \cap, \cup, 0, 1)$ is a Boolean algebra and $\Box: A \longrightarrow A$ is a unary function on $\Box$ satisfying:

(1) $\Box (a \cap b) = \Box a \cap \Box b$;

(2) $\Box 1 = 1$.

3. K-modal BL-algebra

Consider BL-algebra $\mathcal{A} = (A, \cup, \cap, *, \rightarrow, 0, 1)$, we define a unary operator $\Box$ on A, where $\Box: A \longrightarrow A$ satisfies the following conditions:

($\Box$1) $\Box x * \Box y \leq \Box(x * y)$;

($\Box$2) If $x \leq y$ then $\Box x \leq \Box y$;

($\Box$3) $1 \leq \Box 1$;

where $\leq$ is defined as $x \leq y$ if and only if $x \cap y = x$, for all $x, y \in A$.

Lemma 3.1. Let $\mathcal{M} = (\mathcal{A}, \Box)$ where $\Box: A \longrightarrow A$, satisfies the conditions $\Box 1 - \Box 3$. Then

$\square(x \to y) \leq \square x \to \square y$, for all $x, y \in A$.

Proof. Let $x, y \in A$. Residuation property implies $x * (x \to y) \leq y$. Then $\square(x * (x \to y)) \leq \square y$. Thus $\square x * \square(x \to y) \leq \square(x * (x \to y)) \leq y$ by $\square 1$. Hence $\square(x \to y) \leq \square x \to \square y$.

Remark 3.2. The relation $\square(x \to y) \leq \square x \to \square y$ is the algebraic counterpart of the normal principle $K: \square(\phi \Rightarrow \varphi) \Rightarrow (\square\phi \Rightarrow \square\varphi)$ of modal logics, where φ and ϕ are formulas of the related language. Since the algebra $\mathcal{M} = (\mathcal{A}, \square)$ satisfies the algebraic counterpart of principle *K*, we used the sign K for the name of the algebra $\mathcal{M}$. Now, we have the following definition:

Definition 3.3. The algebra $\mathcal{M} = (\mathcal{A}, \square)$, where $\mathcal{A} = (\mathrm{A}, \cup, \cap, \to, *, 0, 1)$ is a BL-algebra, is called a *K*-modal BL-algebra provided that $\square$ satisfies the conditions $\square 1 - \square 3$. Where $\leq$ is defined as $x \leq y$ if and only if $x \cap y = x$, for all $x, y \in A$. We denote the above *K*-modal BL-algebra by $\mathcal{M} = (\mathcal{A}, \square)$.

Example 3.4. (Iorgulescu, 2008) Consider $\mathcal{A} = (A = \{0, a, b, c, 1\}, \cup, \cap, \to, *, 0, 1)$ with lattice order $0 \leq a \leq b \leq 1$ and $a \leq c \leq 1$.

Table 1. The operators $\to$ and $*$ of Example 3.4

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

This structure together with the operations of Table 1, is a BL-algebra.We define the operator $\square$ as follows:

Table 2. The operator $\square$ of Example 3.4

x	0	a	b	c	1
$\square$	0	c	1	c	1

Then the structure $(\mathcal{A}, \square)$ is a *K*-modal BL-algebra.

Example 3.5. Define on the real unit interval $I = [0,1]$ the binary operations $*$ and $\to$ as follows:

$x * y = \max(0, x + y - 1)$ and $x \to y = \min(1, 1 - x + y)$.

Then $(I,\ \cup,\ \cap,\ *,\ \rightarrow,\ 0,\ 1)$ is a BL-algebra (called Lukasiewicz structure) (Hajek, 1998).

Now, we define an operator $\Box$ on this structure as follows:

$$\Box x = \begin{cases} 1, & if\ x = 1 \\ \frac{1}{2}x, & if\ x \neq 1 \end{cases}$$

If $x, y \neq 1$ then we get $\Box x *\ y = \frac{1}{2}x * \frac{1}{2}y = \max\left(0, \frac{1}{2}x + \frac{1}{2}y - 1\right) = 0 \leq \frac{1}{2}$ $\max(0, x + y - 1) = \frac{1}{2}(x * y) = \Box(x * y)$. This shows that the $\Box 1$ holds.If $x = 1$ or $y = 1$ then clearly the axiom $\Box 1$ holds.We can easily verify that the axioms $\Box 2$ and $\Box 3$ hold. Then the structure $(I,\ \leq,\ *,\ \rightarrow,\ 0,\ 1,\ \Box)$ is a K-modal BL-algebra.

Remark 3.6.

(I) If $\Box 4$: $\Box\ (x * y) = \Box x * \Box y$, then $\Box 4$ implies $\Box 1$ and $\Box 2$. But $\Box 1$ and $\Box 2$ do not imply $\Box 4$ generally. Indeed, if $\Box 4$ holds, then clearly $\Box 4$ implies $\Box 1$.

Let $x \leq y$. Then $x = x \cap y$. Thus $\Box x = \Box\ (x \cap y) = \Box(y \cap x)$

$= \Box(y * (y \rightarrow x))$ by divisibility;

$= \Box y * \Box(y \rightarrow x)$ by $\Box 4$;

$\leq \Box y$ by Theorem 2.3(2).

Hence $\Box x \leq \Box y$, i.e., $\Box 4$ implies $\Box 2$. If in the Example 3.5 above we take $x = \frac{1}{2}$ and $y = \frac{3}{4}$ then $\Box x * \Box y \neq \Box(x * y)$, but $\Box 1$ and $\Box 2$ hold.

(II) If $\mathcal{A} = (A,\ \cap,\ \cup,\ *,\ \rightarrow,\ 0,\ 1)$ is a BL-algebra and $B\ (A)$ is the set of all complemented elements of BL-algebra A, then $e * x = e \cap x$ for each $e \in B(A)$ and $x \in A$. If $x, y\ \in B(A)$ then $\Box 4$: $\Box(x * y) = \Box x * \Box y$ reduces to the condition (1): $\Box(x \cap y) = \Box x \cap \Box y$ of the Definition 2.8. Remark 3.6 (II) leads us to a generalization of Definition 2.8 as in the following definition:

Definition 3.7. The algebra $\mathcal{M} = (\mathcal{A},\ \Box)$ where $\mathcal{A} = (A,\ \cap,\ \cup,\ *,\ \rightarrow,\ 0,\ 1)$ is called a modal BL-algebra provided that:

(BL) $\mathcal{A} = (A,\ \cap,\ \cup,\ *,\ \rightarrow,\ 0,\ 1)$ is a BL-algebra;

$(\Box 4)$ $\Box(x * y) = \Box x * \Box y$;

$(\Box 3)$ $1 \leq \Box 1$.

Proposition 3.8. Every modal BL-algebra contains a modal algebra and every modal BL-algebra contains a K-modal BL-algebra.

Proof. It follows from Remark 3.6 (I) and (II).

Remark 3.9. The condition $\square(x * y) \leq \square x * \square y$ implies $\square 2$, but the converse is not true generally. Consider $A = \{0,\ a,\ b,\ 1\}$. Define $\rightarrow$ and $*$ as Table 3. Then $\mathcal{A} = (A,\ \cap,\ \cup,\ *,\ \rightarrow, 0,\ 1)$ with lattice order $0 < a < b < 1$ is a BL-algebra.

Table 3. The operators $\rightarrow$ and $*$ of Remark 3.9

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	1	a	b	1

Table 4. The operator $\square$ of Remark 3.9

x	0	a	b	1
$\square$	0	a	a	1

We can easily check that $\square 2$ is verified, but the condition $\square(x * y) \leq \square x * \square y$ does not hold. In fact, if $x = a$ and $y = b$, we have $x * y = a, \square(x * y) = a = a, \square x * \square y = a * a = 0$ and $a \nleq 0$.

The idea of introducing modal operators in residuated lattices and other algebraic structures has been adopted by some researchers, for several purposes: Belohlavek & Vychodil (2005) defined a so-called" truth stresser" ν for a residuated lattice $(A,\ \cup,\ \cap,\ *,\ \rightarrow,\ 0,\ 1)$ as a unary operator on A such that:

$$\nu x \leq x;$$

$$\nu 1 = 1;$$

$$\nu(x \rightarrow y) \leq \nu x \rightarrow \nu y.$$

They used it to model the (truth function of) unary connective "very true".

Ono (2005) defined modal residuated lattices as structures $(A, \cup, \cap, *, \rightarrow, \nu, 0, 1)$ in which $(A, \cup, \cap, *, \rightarrow, 0, 1)$ is a residuated lattice and ν is aunary operator on A satisfying:

$$\nu x \leq \nu\nu x;$$

$$\nu 1 = 1;$$

$$\nu(x \cap y) \leq \nu x;$$

$$\nu x * \nu y \leq \nu(x * y).$$

Hajek (1998) used a unary operator Δ on the BL-algebra A to get the algebra BL_Δ such that axioms of BL_Δ are those of BL plus:

$$\Delta\varphi \vee \neg\Delta\varphi;$$

$$\Delta(\varphi \vee \psi) \Rightarrow (\Delta\varphi \vee \Delta\psi);$$

$$\Delta\varphi \Rightarrow \varphi;$$

$$\Delta\varphi \Rightarrow \Delta\Delta\varphi;$$

$$\Delta(\varphi \Rightarrow \psi) \Rightarrow (\Delta\varphi \Rightarrow \Delta\psi).$$

The axioms evidently resemble modal logic with Δ as necessity, but in the axiom on $\Delta(\varphi \vee \psi) \Rightarrow (\Delta\varphi \vee \Delta\psi)$, Δ behaves as possibility rather than necessity (Bazz & Hajek, 1996; Hajek, 1998). Magdalena & Rachunek (2006) defined an unary operator f on an MV-algebra A as follows:

If $\mathcal{A} = (A, \oplus, \neg, 0)$ is an MV-algebra where $x \odot y = \neg(\neg x \oplus \neg y)$, then $f: A \rightarrow A$ is called a modal operator on , if for each $x, y \in A$:

$$x \leq f(x);$$

$$f\big(f(x)\big) = f(x);$$

$$f(x \odot y) = f(x) \odot f(y).$$

In fact, the modal operator f behaves as possibility $\Diamond$ in modal logics. Since it satisfies the dual of algebraic counterpart of *T* and satisfies the algebraic counterpart of $K, 4$ by $x \leq f(x)$ and $f(x \odot y) = f(x) \odot f(y)$, $f\big(f(x)\big) = f(x)$, respectively. But we defined a unary operator $\Box$, necessity, by selecting the conditions $\Box 1 - \Box 3$ on BL-algebra A such that our structure, *K*-modal BL-algebra, satisfies only the algebraic counterpart of modal principle *K*. If we extend the unary operator *f* to BL-algebra A, then *f* does not equal to $\Box$. On the other hands, if we restrict the unary operator $\Box$ to MV-center of A then $\Box$ does not equal to *f*. Since the $\Box$ satisfies only the algebraic counterpart of *K* whenever *f* satisfies the algebraic counterpart of K, 4 and satisfies the dual of algebraic counterpart of *T*. If the $\Box$ is restricted to Boolean center of BL-algebra A as we mentioned in the Remark 3.6, then the $\Box$ does not equal to *f*. Since the $\Box$ and *f* have different essence. Indeed, $\Box$ and *f* are correspond to necessity and possibility, respectively. Chakraborty & Sen (1998) defined a unary operator $c: A \rightarrow A$, closure operator, on BL-algebra $\mathcal{A} = (A, \cap, \cup, *, \rightarrow, 0, 1)$ where c satisfies the following conditions:

$(c1)$ $x \leq c(x)$;

$(c2)$ if $x \leq y$ then $c(x) \leq c(y)$;

$(c3)$ $c(c(x)) = c(x)$;

$(c4)$ $c(x) * c(y) \leq c(x * y)$.

In fact, the closure operator c behaves as necessity $\Box$ in modal logics. Clearly, if c is a closure operator then c satisfies the conditions $\Box 1 - \Box 3$, but the converse is not true generally.

Example 3.10. Consider the BL-algebra A of Example 3.4, we define a unary operator $\Box$ on it as Table 5.

Table 5. The operator $\Box$ of Example 3.10

x	0	a	b	c	1
$\Box$	0	a	a	c	1

The unary operator $\Box$ satisfies the conditions $\Box 1 - \Box 3$. Since $b \nleq b = a$, then the condition $(c1)$ is not hold. Tayebi Khorami & Borumand Saeid (2014) defined a multiplier operator on BL-algebra as follows: The operator $m: A \rightarrow A$ is said to be multiplier if $m(x \rightarrow y) = x \rightarrow m(y)$, for all $x, y \in A$. We compare the multiplier operator m with the modal operator $\Box$ and closure operator $c. m(1) = m(0 \rightarrow x) = 0 \rightarrow m(x) = 1$. Hence $1 = m(1) = m(x \rightarrow x) = x \rightarrow m(x)$, i.e., $x \leq m(x)$, for all $x \in A$. Therefore, m satisfies the condition $(c1)$ of closure operators. Let m be a multiplier operator and $m(x) \leq x$. Then $m(x \rightarrow y) = x \rightarrow m(y) \leq m(x) \rightarrow m(y)$ by Theorem 2.3. On the other words, if $m(x) \leq x$ then the multiplier operator m satisfies the algebraic counterpart of normal principle K of modal logics that we mentioned in Lemma 3.1. Furthermore, suppose that $x \leq y$ then we get $1 = m(1) = m(x \rightarrow y) = x \rightarrow m(y)$. With assumption $m(x) \leq x$ we get $m(x) \leq x \leq m(y)$. Now, we ask: when does the modal operator $\Box$ behave as multiplieroperator m? Let $\Box$ be a modal operator satisfying $\Box 1 - \Box 3$ and assume $x \leq \Box x$. Then $\Box(x \rightarrow y) \leq \Box x \rightarrow \Box y \leq x \rightarrow \Box y$ by Theorem 2.3. This implies that the modal operator $\Box$ is a multiplier operator provided that $x \leq \Box x$. Therefore, multiplier operator m is equal to modal operator $\Box$ provided that it is identity operator, i.e., $m(x) = x$.

In the following we give some examples to show that the axioms $\Box 1 - \Box 3$ are independent.

Example 3.11. Consider the structure $\mathcal{A}$ of example 3.4.

Case1. Define the unary operation □ on A as Table 6.

Table 6. The operator □ of Case 1 of Example 3.11

x	0	a	b	c	1
□	0	0	a	a	1

Then the structure $\mathcal{A} = (\{0, a, b, c, 1\}, \cap, \cup, *, \rightarrow, 0, 1, \Box)$, i.e. $(\mathcal{A}, \Box)$ is not a K-modal BL-algebra. We can easily check that 2 and 3 are verified, but □1 does not hold. In fact, if $x = b$ and $y = c$, we have $x * y = b * c = a, \Box(x * y) = \Box a = 0, \Box x * \Box y = \Box b * \Box c = a * a = a$ and $a \nleq 0$. This shows that the axiom □1 is independent of the other axioms.

Case2. Define the unary operator □ on A as Table 7.

Table 7. The operator □ of Case 2 of Example 3.11

x	0	a	b	c	1
□	0	0	0	0	0

The axioms BL, □1, □2 hold, but the axiom □3 does not hold, i.e., this case shows that the axiom □3 is independent of the other axioms.

Case 3. If the unary operator □ on A is defined as Table 8.

Table 8. The operator □ of Case 3 of Example 3.11

x	0	a	b	c	1
□	0	b	a	b	1

Then the axioms BL, □1, □3 hold, but the axiom □2 does not hold for $x = a$ and $y = b$. This case shows that the axiom □2 is independent of the other axioms. Next we show that the inequality in Definition 3.3 can be replaced by some equalities.

Lemma 3.12. The identity $\Box(x \cap y) \cap \Box x = \Box(x \cap y)$ is true in each K-modal BL-algebra and conversely the axiom □2 can be obtained by it.

Proof. We know that $x \cap y \leq x$, then $\Box(x \cap y) \leq \Box x$ by axiom □2. Thus $\Box(x \cap y) \cap x = (x \cap y)$. Now, let $x \leq y$. Then $x = x \cap y$. Thus $\Box x = \Box(x \cap y) = \Box(y \cap x) = \Box(y \cap x) \cap \Box y$. Hence $\Box(y \cap x) \leq \Box y$. Therefore, $\Box x \leq \Box y$.

Theorem 3.13. The class of all K-modal BL-algebras is a variety of algebras.

Proof. The proof is evident from the definition of lattice ordering $\leq$ as follows:

We know that the class of all BL-algebras is a variety of algebras (Hajek, 1998). The axiom $\square 1$ can be replaced by $(\square x * \square y) \cap \square(x * y) = \square x * \square y$. The axiom $\square 2$ can be replaced by the $\square(x \cap y) \cap x = \square(x \cap y)$. The axiom $\square 3$ can be replaced by $1 \cap \square 1 = 1$.

4. Some properties of *K*-modal *BL*-algebras

Lemma 4.1. In each K-modal BL-algebra the following properties hold:

(1) $\square(x \cap y) \leq \square x \cap \square y$;

(2) $\square x \cup \square y \leq \square(x \cup y)$;

(3) $\square(x \to y) * \square(y \to z) \leq \square x \to \square z$;

(4) $\square\big((x \cap y) \to y\big) = 1$;

(5) $\square x \to \square(y \to x) = 1$;

(6) $\square x \to (\square y \to \square x) = 1$;

(7) $\big(\square(x \to y) \cup \square(z \to y)\big) * \square(x \cap z) \leq \square y$;

(8) $\square x * \square(y \cap z) \leq \square(x * y) \cap \square(x * z)$;

(9) $\square\big((x \to y) \to y\big) * \square\big((y \to x) \to x\big) \leq \square(x \cup y)$;

(10) $\square\big((y \to x) \to z\big) \leq \square\big((x \to y) \to z\big) \to \square z$.

Proof.

(1) $x \cap y \leq x, y$ hence $\square(x \cap y) \leq \square x, \square y$. Therefore, $\square(x \cap y) \leq \square x \cap \square y$.

(2) $x, y \leq x \cup y$ then $\square x, \square y \leq \square(x \cup y)$. Hence, $\square x \cup \square y \leq \square(x \cup y)$.

(3) $\square(x \to y) \leq \square x \to \square y$ and $\square(y \to z) \leq \square y \to \square z$; by Lemma 3.1.

Thus, $\square(x \to y) * \square(y \to z) \leq (\square x \to \square y) * (\square y \to \square z) = \square x \to \square z$; by Theorem 2.3.

(4) $x \cap y \to y = 1$, thus $\square\big((x \cap y) \to y\big) = \square 1 = 1$.

(5) $x \leq y \to x$, hence $\square x \leq \square(y \to x)$ by $\square 2$ Then, $\square x \leq \square y \to \square x$, by Lemma 3.1. Finally, $\square x \to \square(y \to x) = 1$.

(6) Similarly as (5) we get $\Box x \leq \Box(y \rightarrow x) \leq \Box y \rightarrow \Box x$, thus $\Box x \rightarrow (\Box y \rightarrow \Box x) = 1$.

(7) $[\Box(x \rightarrow y) \cup \Box(z \rightarrow y)] * \Box(x \cap z) = [\Box(x \rightarrow y) * \Box(x \cap z)] \cup [\Box(z \rightarrow y) * \Box(x \cap z)] \leq [\Box(x \rightarrow y) * (\Box x \cap \Box z)] \cup [\Box(z \rightarrow y) * (\Box x \cap \Box z)] \leq [(\Box x \rightarrow \Box y) * (\Box x \cap \Box z)] \cup [(\Box z \rightarrow \Box y) * (\Box x \cap \Box z)] \leq \left[\left((\Box x \rightarrow \Box y) * \Box x\right) \cap \left((\Box x \rightarrow \Box y) * \Box z\right)\right] \cup \left[\left((\Box z \rightarrow \Box y) * \Box x\right) \cap \left((\Box z \rightarrow \Box y) * \Box z\right)\right] \leq [\Box y \cap ((\Box x \rightarrow \Box y) * \Box z] \cup [\left((\Box z \rightarrow \Box y) * \Box x\right) \cap \Box y] \leq \Box y \cup \Box y = \Box y$.

(8) The $x * (y \cap z) \leq (x * y) \cap (x * z)$ holds by Theorem 2.3. Thus by $\Box 2$ we get :

(4.1) $\Box(x * (y \cap z)) \leq \Box((x * y) \cap (x * z))$. Hence, by (4.1) and $\Box 1$ we conclude that $\Box x * \Box(y \cap z) \leq \Box\left(x * (y \cap z)\right) \leq \Box(x * y) \cap \Box(x * z)$.

(9) The $((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x) \leq (x \cup y)$ holds in each *BL*-algebra by Theorem 2.4. Thus by $\Box 2$ we have

(4.2) $\Box(\left((x \rightarrow y) \rightarrow y\right) \cap \left((y \rightarrow x) \rightarrow x\right)) \leq \Box(x \cup y)$. Hence, by (4.2) and $\Box 1$ we get:

$$\begin{aligned}\Box\left((x \rightarrow y) \rightarrow y\right) * \Box\left((y \rightarrow x) \rightarrow x\right) &\leq \Box(((x \rightarrow y) \rightarrow y) * ((y \rightarrow x) \rightarrow x)))\\ &\leq \Box(((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x)))\\ &\leq \Box(x \cup y).\end{aligned}$$

(10) The inequality $(y \rightarrow x) \rightarrow z \leq ((x \rightarrow y) \rightarrow z) \rightarrow z$ holds in each *BL*-algebra by Theorem 2.4. Now, by $\Box 2$ we have:

(4.3) $\Box((y \rightarrow x) \rightarrow z) \leq \Box((x \rightarrow y) \rightarrow z) \rightarrow z)$.

Hence, by (4.3) and Lemma 3.1 we conclude

$\Box((y \rightarrow x) \rightarrow z) \leq \Box((x \rightarrow y) \rightarrow z) \rightarrow z) \leq \Box((x \rightarrow y) \rightarrow z) \rightarrow \Box z$.

Theorem 4.2. Let $\mathcal{M} = (\mathcal{A}, \Box)$ be a *K*-modal *BL*-algebra and $\Box(A) = \{x \in A: x = \Box x\}$. Then we have the following properties:

(1) $\Box(A) = \{\Box x: x \in A\}$ and it is closed under $\cap$ and $\rightarrow$.

(2) $(\Box(A), \leq_\Box, \cap_\Box, \cup_\Box, *_\Box, \rightarrow_\Box, \Box(0), 1)$, is a *BL*- algebra defined as follows: for each $x, y, z \in \Box(A)$, $x \cap_\Box y = \Box(x \cap y), x \cup_\Box y = \Box(x \cup y), x *_\Box y = (x * y)$, $x \rightarrow_\Box y = (x \rightarrow y)$.

Furthermore, $\cap_\Box = \cap$, $\cup_\Box = \cup$, $\rightarrow_\Box = \rightarrow$.

(3) If A satisfies $x * x = x$ for each $x \in A$, then $\Box(x * y) = \Box(x) * \Box(y)$.

Proof. The modal operator $\Box$ satisfies the conditions $\Box 1 - \Box 3$, plus the condition $\Box x = x$. Hence the modal operator $\Box$ on $\Box A$ satisfies $(c1) - (c4)$ of closure operator. Now, the assertion can be obtained by Theorem 2.3 of Ko & Kim (2004).

5. *K*-modal filters

Let $\mathcal{M} = (\mathcal{A}, \Box)$ be a K-modal BL-algebra. We may consider a non-empty subset F of A as a filter in $\mathcal{M}$ in the same way as it is a filter in BL-algebra $\mathcal{A}$ defined by:

Definition 5.1. A filter F of a K-modal BL-algebra $\mathcal{M} = (\mathcal{A}, \Box)$ is called a K-modal filter if and only if F is closed under $\Box$, i.e., if $x \in F$, then $\Box x \in F$, for all $x \in A$.

Lemma 5.2. If $K_\Box = Ker\Box = \{x \in A: \Box x = 1\}$, then $K_\Box$ is a K-modal filter in $\mathcal{M} = (\mathcal{A}, \Box)$.

Proof. Clearly $K_\Box$ is a filter. If $x \in K_\Box$, then $\Box x = 1$. Hence $\Box(\Box x) = 1$. Thus $\Box x \in K_\Box$.

Definition 5.3. The $K_\Box$ is called the $\Box$- tautology filter in $\mathcal{M} = (\mathcal{A}, \Box)$.

Example 5.4. Consider the BL-algebra $\mathcal{A}$ of Example 3.4, we define a unary operator $\Box$ on it as Table 9.

Table 9. The operator $\Box$

x	0	a	b	c	1
$\Box$	0	c	1	c	1

We can easily verify that the structure $\mathcal{A} = (A, \cup, \cap, *, \to, 0, 1, \Box)$ is a K-modal BL-algebra.The filter $F_1 = \{b, 1\} = K_\Box$ on $\mathcal{A}$ is a $\Box$-tautology filter. Lemma 5.2 verifies that every $\Box$-tautology filter is a K-modal filter but the converse is not true generally. For example, the filter $F_2 = \{a, b, c, 1\}$ is a K-modal filter but is not a $\Box$-tautology filter, since $c = c \neq 1$. Clearly every K-modal filter is a filter, but the converse is not true generally. Consider the BL-algebra $\mathcal{A}$ of Example 3.4, we define a unary operator $\Box$ on it as Table 10.

Table 10. The operator $\Box$

x	0	a	b	c	1
$\Box$	0	a	a	c	1

$F_3 = \{b, 1\}$ is a filter but is not a K-modal filter since $\Box b = a \notin F_3$.

Remark 5.5. (1) we can extend any (type) filter of BL-algebra to K-modal filter.

Indeed, let F be a filter such that F is not a K-modal filter, i.e., there exists an element x in F which is not closed under $\square$. By adding to F all of the elements x which $x \notin F$, we can obtain a K-modal filter such that the K-modal filter contains F as a subset.

(2) Every K-modal filter is an extension of a filter, since every K-modal filter is itself a filter which closed under $\square$.

Lemma 5.6. Let F be a filter and $\square F = \{\square x : x \in F\}$. If $\square x * \square y = \square(x * y)$, then $\square F$ is a filter.

Proof. If $\square x, \square y \in \square F$, then $x * y \in F$, since F is a filter. Hence $\square x * \square y = \square(x * y) \in F$. If $\square x \leq y$ and $\square x \in F$, then $\square y \in F$.

Lemma 5.7. Let $\mathcal{A} = (A, \cup, \cap, *, \rightarrow, 0, 1)$ be a BL-algebra and $\mathfrak{I}(\mathcal{A})$ be a G-algebra. If $a \in \mathfrak{I}(\mathcal{A})$ then the operator $\square_a$ defined as $\square_a(x) = a \rightarrow x$ for every $x \in A$, is a modal operator, i.e., it satisfies the axioms $\square 1 - \square 3$.

Proof.

1. The relations $a * (a \rightarrow x) \leq x$ and $b * (b \rightarrow y) \leq y$ hold by Theorem 2.3(2). Hence

$$(a \rightarrow x) * (b \rightarrow y) * (a * b) = \big(a * (a \rightarrow x)\big) * \big(b * (b \rightarrow y)\big) \leq x * \big(b * (b \rightarrow y)\big) \leq x * y.$$

By residuation property we have:

(5.1) $(a \rightarrow x) * (b \rightarrow y) \leq (a * b) \rightarrow x * y.$

So

$$\square_a(x) * \square_a(y) = (a \rightarrow x) * (a \rightarrow y) \leq (a * a) \rightarrow (x * y) \text{ by } (5.1);$$
$$= a \rightarrow (x * y) = \square_a(x * y).$$

Then axiom $\square 1$ is satisfied.

2. If $x \leq y$, then $a \rightarrow x \leq a \rightarrow y$. Hence $\square_a(x) \leq \square_a(y)$, i.e., the axiom $\square 2$ is satisfied.

3. $\square_a(1) = a \rightarrow 1 = 1$.

Corollary 5.8. Let $\mathcal{A} = (A, \cup, \cap, *, \rightarrow, 0, 1)$ be a BL-algebra and $\mathfrak{I}(\mathcal{A})$ be a G-algebra. If $a \in \mathfrak{I}(\mathcal{A})$ then the interval $[a, 1] = \{x \in A : a \leq x \leq 1\}$ is a $\square$-tautology filter.

Proof. Clearly, $[a, 1]$ is a filter. Let $\square_a$ be as in Lemma 5.7. We show that $[a, 1] = K_a$. If $x \in [a, 1]$, then $a \leq x \leq 1$. Hence $\square_a(a) \leq \square_a(x) \leq \square_a(1)$, i.e., $1 \leq \square_a(x) \leq 1$. Equivalently $\square_a(x) = 1$, i.e., $x \in K_{\square_a}$.

Conversely, let $x \in K_{\square_a}$. Then $\square_a(x) = 1$, i.e., $a \rightarrow x = 1$. The last holds if and only if $a \leq x$. Therefore, $[a, 1]$ is a $\square$-tautology filter.

Now, we give the definition of prime filter in $\mathcal{M} = (\mathcal{A}, \square)$ and definition of K-modal prime filter in $\mathcal{M}$ as:

Definition 5.9. A filter F of a K-modal BL-algebra $\mathcal{M} = (\mathcal{A}, \square)$ is called a prime filter in $\mathcal{M}$ if and only if for each $x, y \in A$, $(x \to y) \in F$ or $(y \to x) \in F$.

Definition 5.10. A filter F of $\mathcal{M} = (\mathcal{A}, \square)$ is called a K-modal prime filter in $\mathcal{M}$ if and only if F is a prime filter in $\mathcal{M} = (\mathcal{A}, \square)$ and F is closed under $\square$.

Example 5.11. In Example 5.4. The filter $F_2 = \{a, c, 1\}$ is a K-modal filter which it is closed under $\square$, i.e., F_2 is a K-modal prime filter. The filter $F_3 = \{a, b, 1\}$ is a prime filter in $\mathcal{M}$, but F_3 is not a K-modal prime filter, since $\square a = c \notin F_3$.

Lemma 5.12. If F is a K-modal prime filter, then $\square x \to \square y \in F$ or $\square y \to \square x \in F$, for each $x, y \in F$.

Proof. Let F be a K-modal prime filter. Then $x \to y \in F$ or $y \to x \in F$. Hence $\square(x \to y) \in F$ or $\square(y \to x) \in F$. By Lemma 3.1 we get $\square(x \to y) \leq \square x \to \square y$ or $\square(y \to x) \leq \square y \to \square x$, i.e., $\square x \to \square y \in F$ or $\square y \to \square x \in F$ since F is a filter.

Theorem 5.13. Let $\mathcal{M} = (\mathcal{A}, \square)$ be a K-modal BL-algebra and F be a K-modal filter of $\mathcal{M}$. Put $x \equiv_F y$ if and only if $(x \to y) \in F$ and $(y \to x) \in F$.

(i) $x \equiv_F y$ is a congruence relation and the corresponding quotient algebra $\mathcal{M}_{/\equiv_F}$ is a K-modal BL-algebra.

(ii) $\mathcal{M}_{/\equiv_F}$ is linearly ordered if and only if F is a K-modal prime filter.

Proof. (i) First we show that $x \equiv_F y$ is an equivalence relation on A. Let $x, y \in A$. We have:

$x \equiv_F y$ if and only if $(x \to y) \in F$ and $(y \to x) \in F$. The reflexivity and symmetry properties are easily verified. To show that $x \equiv_F y$ is transitive, we notice that

(5.2) $(x \to y) * (y \to z) \leq (x \to z)$ holds in any residuated lattice, by Theorem 2.3(6).

Now, let $x \equiv_F y$ and $y \equiv_F z$. Then

$(x \to y) \in F$ and $(y \to x) \in F$;

$(y \to z) \in F$ and $(z \to y) \in F$.

But $(x \to y) * (y \to z) \leq (x \to z)$ holds by (5.2). Thus $(x \to y) \in F$, since $(x \to y) * (y \to z) \in F$ and F is a (K-modal) filter. We get $(z \to x) \in F$, similarly. Hence $x \equiv_F z$. Since F is a filter in the BL-algebra $\mathcal{A}$, the relation $\equiv_F$ is a congruence relation on $\mathcal{A}$ by Lemma 2.3.14 of Hajek (1998). To prove that $\equiv_F$ is a congruence relation on $\mathcal{M}$, it remains only to show that $\equiv_F$ is compatible with $\square$ on A. Let

$x \equiv_F y$. Then $(x \to y) \in F$ and $(y \to x) \in F$. Hence $x \to y \in F$ and $(y \to x) \in F$. Thus $x \to y \in F$ and $y \to x \in F$ by Lemma 3.1 which implies $x \equiv_F y$. If $[x]$ is the congruence class of x, then we form $A_{/\equiv_F} = \{[x]: x \in A\}$. Now, we define the corresponding operations on $A_{/\equiv_F}$ as follows:

$[x]_F * [y]_F = [x * y]_F, [x]_F \to [y]_F = [x \to y]_F, \ \Box[x]_F = [\Box x]_F$, for the other operations similarly. Since $\equiv_F$ is a congruence relation on A, all the above operations are well-defined. Therefore, the system $\mathcal{M}_{/\equiv_F} = (\mathcal{A}_{/\equiv_F}, \ \Box)$ is an algebra called the quotient algebra of $\mathcal{M}$. Now, we claim that $\mathcal{M}_{/\equiv_F}$ is a *K*-modal *BL*-algebra. First, we define the relation $\leq$ on $A_{/\equiv_F}$ as follows: $[x]_F \leq [y]_F$ if and only if $(x \to y) \in F$. It is easily verified that the relation $\leq$ on $\mathcal{M}_{/\equiv_F}$ is an order. To show that $\mathcal{M}_{/\equiv_F}$ is a *K*-modal *BL*-algebra we need only to show that the operation $\Box$ defined on $\mathcal{M}_{/\equiv_F}$ satisfies $\Box 1 - \Box 3$, for all $[x], \ [y] \in \ \mathcal{A}_{/\equiv_F}$.

We know that $\Box x * \Box y \leq \Box(x * y)$. Hence $\Box x * \Box y \to \Box(x * y) = 1 \in F$. Thus $[\Box x * \Box y]_F \leq [(x * y)]_F$. Hence $\Box[x]_F * \Box[y]_F = [\Box x]_F * \ [\Box y]_F = [\Box x * \Box y]_F \leq [\Box(x * y)]_F = [x * y]_F = \Box([x]_F * [y]_F)$. Therefore, $\mathcal{M}_{/\equiv_F}$ satisfies $\Box 1$.

Let $[x]_F \leq [y]_F$. Then $x \to y \in F$. Since F is a *K*-modal filter, $\Box(x \to y) \in F$. Thus $\Box x \to \Box y \in F$ by Lemma 3.1. Therefore, $[\Box x]_F \leq [\Box y]_F$. Equivalently $[x]_F \leq [y]_F$. Therefore, $\mathcal{M}_{/\equiv_F}$ satisfies $\Box 2$. Since $1 \leq \Box 1$ then $1 \to \Box 1 = 1 \in F$, i.e., $[1]_F \leq [\Box 1]_F = \Box[1]_F$. Therefore, $\mathcal{M}_{/\equiv_F}$ satisfies 3. Hence, the $\mathcal{M}_{/\equiv_F}$ is a *K*-modal *BL*-algebra.

To prove (ii), let $\mathcal{M}_{/\equiv_F}$ be linearly ordered, i.e., $[x]_F \leq [y]_F$ or $[y]_F \leq [x]_F$, for every $[x]_F, [y]_F \ \in \mathcal{A}_{/\equiv_F}$. Thus $(x \to y) \in F$ or $(y \to x) \in F$, respectively. Hence *F* is a *K*-modal prime filter. Conversely, let *F* be a *K*-modal prime filter. Then *F* is a prime filter, i.e., we get $(x \to y) \in F$ or $(y \to x) \in F$. Hence $[x]_F \leq [y]_F$ or $[y]_F \leq [x]_F$, i.e., $\mathcal{M}_{/\equiv_F}$ is linearly ordered.

From the above theorem it follows that:

Corollary 5.14. Let *F* be a $\Box$-tautology filter in $\mathcal{M} = (\mathcal{A}, \ \Box)$. Then $\mathcal{M}_{/\equiv_F}$ is linearly ordered if and only if *F* is a $\Box$-tautology prime filter.

Recall that in the modal logics the modal principles *T* and 4 are in the forms $\Box\phi \Rightarrow \ \phi$ and $\Box\phi \Rightarrow \Box\Box \ \phi$, respectively. Clearly, the algebraic counterpart of *T* and 4 are in the forms $\Box 5: \Box x \leq x$ and $\Box 6: \Box x \leq \Box\Box x$, respectively. Below, we show that there are *K*-modal filters containing a given filter and an element under the above conditions.

Lemma 5.15. Let $\mathcal{M} = (\mathcal{A}, \,\Box)$ be a K-modal BL-algebra, F be a K-modal filter on $\mathcal{M}$ and $z \in A$. Then there exists a K-modal filter F' such that F' containing F as a subset and z as an element provided that $\Box$ satisfies two extra conditions:

$\Box 5: x \leq x;$

$\Box 6: \Box x \leq \Box\Box x.$

Proof. Consider F' as follows:

$F' = \{u \in A: \exists v \in F, \exists n \in \mathbb{N} \;\; v * (z)^n \leq u\}$. Where $z^n = z * \ldots * z, n$ times and $z^0 = 1. F' \neq \emptyset$, since $1 \in F'$. We claim that F' is a filter containing F as a subset and z as an element. Let $u_1, u_2 \in F'$. Hence

$v_1 * (\Box z)^{n_1} \leq u_1$ for some $v_1 \in F$ and some $n_1 \in \mathbb{N}$;

$v_2 * (\Box z)^{n_2} \leq u_2$ for some $v_2 \in F$ and some $n_2 \in \mathbb{N}$.

Thus we get:

$(v_1 * v_2) * (\Box z)^{n_1 + n_2} \leq u_1 * u_2$. Therefore $u_1 * u_2 \in F'$, since F is a filter and $(v_1 * v_2) = v_3 \in F$.

Let $u_1 \leq u_2$; $u_1 \in F'$. Thus $v_1 * (\Box z)^{n_1} \leq u_1 \leq u_2$. Then $u_2 \in F'$. Hence F' is a filter. Let $v \in F$. We conclude that $v * (\Box\, z)^0 = v * 1 = v \leq \; v$. Hence $v \in F'$. We claim that F' contains z as an element:

(5.3) $1 * (\Box z)^1 = \Box z \leq \Box z.$

(5.3) implies that $\Box z \in F'$. F' is a filter and $\Box z \leq z$ by $\Box 5$. Thus $z \in F'$. Now, we show that F' is a K-modal filter:

Let $u \in F'$. Thus we have $v * (\Box z)^n \leq u$, for some $v \in F$ and some $n \in \; \mathbb{N}$. Thus we obtain:

(5.4) $\Box v * \; \Box(\Box z\,)^n \leq \Box u.$

So,

$\Box v * (\Box z)^n = \Box v * (\;\Box z * \ldots \; * \Box z)$

$\leq \Box v * \Box\Box z * \ldots * \Box\Box z$ by $\Box 6$;

$\leq \Box v * \Box(\Box z * \ldots * \Box z)$ by $\Box 1$;

$= \;\Box v * \Box(\Box z)^n$

$\leq \Box u$ by (5.4); $v \in F$, since $v \in F$ and F is a K-modal filter. Hence $\Box u \in F'$.

Based on the above lemma, we prove:

Theorem 5.16. Suppose that $\mathcal{M} = (\mathcal{A}, \,\Box)$ is a K-modal BL-algebra and F is a

K-modal filter of $\mathcal{M}$ such that $1 \neq a \notin F$. Then there exists a K-modal prime filter F' on $\mathcal{M}$ containing F and $a \notin F'$, provided that $\square$ satisfies four extra conditions:

$\square 4$: $\square x * \square y = \square(x * y)$;

$\square 5$: $\square x \leq x$;

$\square 6$: $\square x \leq \square\square x$;

$\square 7$: $\square(x \cup y) = \square x \cup \square y$.

Proof. If F is prime, then we are done. If not, for every $(x, y) \in A^2$ with $(x \to y) \notin F$ and $(x \to y) \notin F$, we construct F_1 and F_2 as in Lemma 5.15 in such away that:

$F \subseteq F_1, \quad x \to y \in F_1$;

$F \subseteq F_2, \quad x \to y \in F_2$.

We claim that: $a \notin F_1$ or $a \notin F_2$. Assume on the contrary that: $a \in F_1$ and $a \in F_2$. Then by Lemma 5.15 we get:

$\upsilon_1 * (\square(x \to y))^{n_1} \leq a$ for some $\upsilon_1 \in F$ and some $n_1 \in \mathbb{N}$;

$\upsilon_2 * (\square(y \to x))^{n_2} \leq a$ for some $\upsilon_2 \in F$ and some $n_2 \in \mathbb{N}$.

Let $\upsilon = \upsilon_1 \cap \upsilon_2$ and $n = \max(n_1, n_2)$. Therefore, we obtain:

$\upsilon * (\square(x \to y))^n \leq a$;

$\upsilon * (\square(y \to x))^n \leq a$. Thus

$a \geq (\upsilon * (\square(x \to y))^n \cup \upsilon * (\square(y \to x))^n$

$= \upsilon * ((\square(x \to y))^n \cup (\square(y \to x))^n)$ by 2.3(10);

$= \upsilon * (\square(x \to y)^n \cup \square(y \to x)^n)$ by $\square 4$;

$= \upsilon * \square((x \to y)^n \cup (y \to x)^n)$ by $\square 7$;

$= \upsilon * 1 = \upsilon$;

This implies that $a \in F$, which contradicts our assumption. Corresponding to cardinality of our language of discourse, we can use axiom of choice to enumerate the pairs $(x, y) \in A^2$ with $(x \to y) \notin F$ and $(x \to y) \notin F$ by ordinals $\lambda \in I$. Let $F_0 = \{1\}$, $F_\lambda = \cup_{\mu \leq \lambda} F_\mu$ if λ is a limit ordinal. If λ is a successor ordinal, i.e., $\lambda = \mu + 1$ then F_λ is constructed from F_μ in such away that $a \notin F_\lambda$ and if possible $(x_\mu \to y_\mu) \in F_\lambda$ otherwise $(y_\mu \to x_\mu) \in F_\lambda$. Now, we claim that $P = \cup_{\lambda \in I} F_\lambda$ is a K-modal prime filter not containing a: Clearly, P is a K-modal filter, since the set $\{F_\lambda : \lambda \in I\}$ is a chain setof K-modal filters F_λ. By construction of F_λ's we see that for each pair $(x, y) \in A^2$: $(x \to y) \in F_\lambda$ or $(y \to x) \in F_\lambda$ for some $\lambda \in I$. Hence $(x \to y) \in P$ or $(y \to x) \in P$, i.e., P is prime. Clearly $a \notin F_\lambda$ for all $\lambda \in I$. Thus $a \notin P$.

Example 5.17. Consider the unit interval $I = [0,\ 1]$. We define binary operations $*, \rightarrow$ and unary operator $\Box$ on I as follows:

$$x * y = x \cap y;\ x \rightarrow y = \begin{cases} 1, & x \le y \\ y, & otherwise \end{cases};\ \Box x = \begin{cases} 0, & 0 \le x < \frac{1}{3} \\ \frac{1}{3}, & \frac{1}{3} \le x < \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{2} \le x < 1 \\ 1, & x = 1. \end{cases}$$

We can easily verify that $\mathcal{I} = (I,\ \Box)$ is a K-modal BL-algebra which satisfies the conditions $\Box 4 - \Box 7$ and $\frac{1}{3} \notin \left[\frac{1}{2},\ 1\right] \subseteq \left[\frac{1}{2},\ 1\right]$, which $\left[\frac{1}{2},\ 1\right]$ is a K-modal prime filter on $\mathcal{I}$.

Remark 5.18. If we define $G = \{x \in A : \Box x = x\}$, then $G \neq \emptyset$, since $1 \in G$. We can show that G is not a filter of BL-algebra $\mathcal{A}$, necessarily.

Example 5.19. Let $\mathcal{M} = (\mathcal{A}, \Box)$ be as in the Example 5.17. Clearly the set G is the form $G = \{x \in A : \Box x = x\} = \left\{0,\ \frac{1}{2},\ \frac{1}{3},\ 1\right\}$. $\frac{1}{3} \in G$ and $\frac{1}{3} \le \frac{2}{5}$ but $\frac{2}{5} \notin G$, since $\frac{1}{3} = \Box \frac{2}{5} \neq \frac{2}{5}$.

In the sequel we show that the conditions $\Box 4 - \Box 7$ in the previous theorem are necessary.

Example 5.20. (Iorgulescu, 2008) Consider $\mathcal{A} = (\{0,\ a,\ b,\ c,\ d,\ 1\}, \cap, \cup, *, \rightarrow,\ 0,\ 1)$ with lattice order $0 < a < c < 1, 0 < b < d < 1$ and $b < c$. This structure together with the operations of Table 11, is a BL-algebra:

Table 11. The operations $\rightarrow$ and $*$ of Example 5.20

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

We define the unary operation $\Box$ on $\mathcal{A}$ as Table 12.

Table 12. The operation $\Box$ of Example 5. 20

x	0	a	b	c	d	1
$\Box$	0	1	0	1	0	1

We can easily verify that $\mathcal{M} = (\mathcal{A}, \Box)$ is a K-modal BL-algebra which satisfies all of the conditions of Theorem 5.16 except $\Box 5$. $1 \neq a \notin \{c, 1\} \subseteq \{c, d, 1\}$, i.e., there is no K-modal prime filter F' such that contains $\{c, 1\} \subseteq F'$ and $a \notin F'$.

Example 5.21. (Iorgulescu, 2008) Consider $\mathcal{A} = (\{0, a, b, c, d, e, 1\}, \cap, \cup, *, \rightarrow, 0, 1)$ with lattice order $0 < a < c < 1, 0 < b < d < 1$ and $b < c$. This structure together with the operations of Table 13, is a BL-algebra:

Table 13. The operations $\rightarrow$ and $*$ of Example 5.21

$\rightarrow$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	e	1
c	0	d	d	1	1	1	1
d	0	c	d	e	1	e	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$*$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	b	a	b	a	b
c	0	a	a	a	a	c	c
d	0	a	b	a	b	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

We define the unary operation $\Box$ on $\mathcal{A}$ as Table 14.

Table 14. The operation $\Box$ of Example 5.21

x	0	a	b	c	d	e	1
$\Box$	0	0	0	0	d	0	1

We can easily verify that $\mathcal{M} = (\mathcal{A}, \Box)$ is a K-modal BL-algebra which satisfies all of the conditions of Theorem 5.16 except $\Box 7$, since $\Box(b \cup c) = \Box d = d \neq 0 = \Box b \cup \Box c$. $1 \neq b \notin \{d, 1\}$, i.e., there is no K-modal prime filter F' such that contains $\{d, 1\} \subseteq F'$ and $b \notin F'$.

Finally we prove:

Theorem 5.22. Let $\mathcal{A}$ be a BL-algebra with unary operator $\square$ satisfying $\square 3 - \square 7$. The construction $\mathcal{M} = (\mathcal{A}, \square)$ as a special K-modal BL-algebra is a sub-direct product of linearly ordered K-modal BL-algebras.

Proof. Recall that $\square 4$ implies the axioms $\square 1$ and $\square 2$. Let $\mathcal{U}$ be the system of all K-modal prime filters on $\mathcal{M}$; let $\mathcal{M}_F = \mathcal{M}_{/\equiv_F}$ and $\mathcal{M}^* = \prod_{F \in \mathcal{U}} \mathcal{M}_F$. Each $\mathcal{M}_F$ is linearly ordered by Theorem 5.13. Consider $i(x) = ([x]_F)_{F \in \mathcal{U}}$ or each $x \in \mathcal{M}$, i.e., i is a map of $\mathcal{M}$ to $\mathcal{M}^*$. We show that i is one to one. If $x, y \in \mathcal{M}$ and $x \neq y$, then $x \nleq y$ or $y \nleq x$. Thus $(x \to y) \neq 1$ or $(x \to y) \neq 1$, respectively. Suppose $(x \to y) \neq 1$; then there is a K-modal prime filter F such that $(x \to y) \notin F$ by Theorem 5.16. Hence $[x]_F \nleq [y]_F$ in $\mathcal{M}_F$, i.e., $[x]_F \neq [y]_F$ in $\mathcal{M}_F$. Thus $i(x) \neq i(y)$, i.e., i is one to one.

Conclusion and future research

Some modal axioms of (normal) modal logics are the following (Hughes & Cresswell, 1996; Gabbay *et al.,* 2003):

$K: \square(\phi \Rightarrow \psi) \Rightarrow (\square\phi \Rightarrow \square\psi)$;

$T: \square\phi \Rightarrow \phi$;

$4: \square\phi \Rightarrow \square\square\phi$;

$5: \lozenge\phi \Rightarrow \square\lozenge\phi$;

The minimal modal logic is a modal logic that satisfying only the axiom $K: \square(\phi \Rightarrow \psi) \Rightarrow (\square\phi \Rightarrow \square\psi)$ among the modal axioms. Every other modal logic can be obtained by extending this system with a (possibly infinite) set of extra axioms (Gabbay *et al.,* 2003).The modal logic T is $K + (\phi \Rightarrow \phi)$, the modal logic $S4$ is $KT + (\square\phi \Rightarrow \square\square\phi)$ and the modal logic $S5$ is $KT + (\lozenge\phi \Rightarrow \square\lozenge\phi)$ (Gabbay *et al.,* 2003; Hughes & Cresswell, 1996). In this paper, we introduced the K-modal BL-algebra $\mathcal{M} = (\mathcal{A}, \square)$. Since $\mathcal{A}$ is a BL-algebra as an algebraic counterpart of fuzzy logic (Hajek, 1998), we propose that $\mathcal{M}$ is an algebra appropriate to resemble the fuzzy minimal modal logic. To prove some important theorems we needed to add some conditions on the given modal operator. In future work we may obtain an algebraic counterpart of fuzzy modal logic $S5$ in the Hajek's sense used in Hajek (1998) and Hajek (2010).

References

Abbasloo, M. & Borumand Saeid, A., (2014) Coatoms and comolecules of BL-algebras, Journal of intelligent and Fuzzy Systems, **27**, No.1, 351-361.

Abramsky, S., Gabbay, D.M. & Maibaum, T.S.E. (eds). (1992) Handbook of logic in computer science, Clarendon Press-Oxford, 1-3.

Baaz, M. & Hajek, P. (1996). Infinite-valued godel logics with 0-1-projections and relativizations. In GODEL'96 - Logical foundations of mathematics, computer science and physics; Lecture Notes in Logic 6, Springer- Verlag, 23-33.

Belohlavek, R. & Vychodil, V. (2005) Fuzzy equational logic, studies in fuzziness and soft computing, 186.

Blackburn. P., De Rijke. M. & Venema, Y. (2001) Modal logic, Cambridge University Press.

Chakraborty. M.K. & Sen. J. (1998) MV-algebras embedded in a CL-algebra, Elsevier, 18, 217-229.

Fitting, M. (1991) Many-valued modal logics, Fundamental Informaticae, 15, 235-54.

Fitting, M. (1992) Many-valued modal logics, II, Fundamental Informaticae, 17, 55-73.

Fitting, M. & Richard, L.M. (1998) First-order Modal Logic, Kluwer Academic Publisher.

Gabbay, D.M., Kurucz, A., Wolter, F. & Zakharyaschev, M. (2003) Many-dimensional modal logics : Theory and Applications, Elsevier Science B.V., 148.

Gabbay, D.M., Hogger, C.J. & Robinson, J.A. (1994) Handbook of logic in artificial intelligence and logic programming, Clarendon Press-Oxford, 1-4.

Hajek, P. (1998) Metamathematics of fuzzy logic, Kluwer Academic Publishers, Dordrecht.

Hajek, P. (2010) On fuzzy modal logics $S5(C)$, Fuzzy Sets and System, 161, 2389-2396.

Hughes, G.E. & Cresswell, M.J. (1996) A new introduction to modal logic, Routledge, London and New York.

Iorgulescu, A. (2008) Algebras of logic as BCK algebras, Bucharest University of Economics, Bucharest, ROMANIA.

Ko, J.M. & Kim, Y.C. (2004) Closure operators on BL-algebras, Commun. Korean Mathematical Society, 19, 219-232.

Kowalski, T. & Ono, H. (2001) Residuated lattices: an algebraic glimpse at logic without contraction.

Ma, X., Zhan, J. & Dudek, W.A. (2009) Some kinds of $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ −fuzzy filters of BL-algebras, Computers & Mathematics with Applications, **58**(2):248-256.

Magdalena, H. & Ruchunek, J. (2006) Modal operators on MV-algebras, Mathematical Bohemica, **131**: 39-48.

Moss, L.S. & Tiede, Hans-Jorg. (2007) Applications of modal logic in Linguistics, Ch. 19 of P. Blackburn, J. Van Benthem and F. Wolter (eds.), Handbook of Modal Logic, Elsevier, 299-341.

Ono, H. (2005) Proceedings of the 39th MLG meeting at Gamagori, Japan, 36-38.

Piciu, D. (2007) Algebras of fuzzy logic, Ed. Universitaria, Craiova.

Tayebi Khorami, R. & Borumand Saeid, A. (2014) Multiplier in BL-algebras, Iranian Journal of Science and Technology, **38**(A2):95-103.

Pratt Vaughan, R. (1980) Application of modal logic to programming, Studia Logica 39(2-3), 257- 274.

Zhan, J., Jun, Y.B. & Kim, H.S. (2014) Some types of falling fuzzy filtersof BL-algebras and its applications, Journal of Intelligent and Fuzzy Systems, **26**(4):1675-1685.

Zhan, J., Dudek, W.A. & Jun, Y.B. (2009) Interval valued $(\epsilon, \epsilon \vee q)$ −fuzzy filters of pseudo BL-algebras, Soft Computing, **13**(1):13-21.

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جبريات *BL* شكليتها *K*

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خلاصة

نقوم في هذا البحث بتقديم جبرية BL ذات الشكلية K ونبحث في خصائص الجبرية الجديدة . و بناء على ذلك لابد أن نتعامل مع مرشحات شكليتها K ومرشحات مصدوقية لتكون مرشحات لجبريات BL ذات الشكلية K سنثبت أن هذا الصنف من الجبريات هو متنوعة جبرية . هدفنا النهائي في هذا البحث هو أن نثبت هو أن جبرية BL ذات الشكلية K هي جبرية جزئية لجداء مباشر لنظام من جبريات BL مرتبة خطياً و ذات شكلية K و ذلك تحت شروط خاصة.