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Cyclic ϕ-contractions on S-complete Hausdorff uniform Spaces

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ABSTRACT

In this paper, we apply the concept of cyclic ϕ-contraction for presenting a fixed point theorem on a Hausdorff uniform space.

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INTRODUCTION

Let X be a nonempty set and let ϑ be a nonempty family of subsets of $X \times X$. The pair (X, ϑ) is called a uniform space if it satisfies the following properties:

(i) if G is in ϑ, then G contains the diagonal $\{(x, x) : x \in X\}$;

(ii) if G is in ϑ and H is a subset of $X \times X$ which contains G, then H is in ϑ;

(iii) if G and H are in ϑ, then $G \cap H$ is in ϑ;

(iv) if G is in ϑ, then there exists H in ϑ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in G;

(v) if G is in ϑ, then $\{(y, x) : (x, y) \in G\}$ is also in ϑ.

ϑ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings. In Bourbaki (1998) and Zeidler (1986), (X, ϑ) is called a quasiuniform space if property (v) is omitted. Some authors studied the theory of fixed point or common fixed point for contractive selfmappings in uniform space (Altun (2011); Kubiak & Cho (1993); Turkoglu (2010); Wlodarczyk & Plebaniak (2011); Vályi (1985))

Later, Aamri & El Moutawakil (2004, 2005) proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

For any set X, the diagonal $\{(x,x): x \in X\}$ will be denoted by Δ if no confusion occurs. If $V, W \in X \times X$, then $V \circ W = \{(x, y):$ there exists $z \in X$ such that $(x,z) \in W$ and $(z, y) \in V\}$ and $V^{-1} = \{(x, y) : (y, x) \in V\}$.

If $V \in \vartheta$ and $(x, y) \in V, (y, x) \in V$, x and y are said to be V-close, and a sequence $\{x_n\}$ in X is a Cauchy sequence for ϑ, if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

A sequence $\{x_n\}$ in X is convergent to x for ϑ, if for any $V \in \vartheta$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$ and denote by $\lim_{n\to\infty} x_n = x$. A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X, i.e., if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close, then for our purpose, we assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they always refer to the topological space $(X, \tau(\vartheta)$.

PRELIMINARIES

Definition 1. (Aamri & El Moutawakil (2004)) Let (X, ϑ) be a uniform space. A function $p: X \times X \to [0, \infty)$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2. (Aamri & El Moutawakil (2004)) Let (X, ϑ) be a uniform space. A function $p: X \times X \to 0, \infty)$ is said to be an E-distance if

- (p_1) P is an A-distance,
- (p_2) $p(x, y) \leq p(x, z) + p(z, y).$ $\forall x, y, z \in X.$

There are very nice examples of *E*-distances in Aamri & El Moutawakil (2004). Some of these examples compare the concept of *E*-distance with *W*-distance, which is introduced by Montes & Charris (2001), on uniform spaces. Every *W*-distance is an *E*-distance, but the converse may not be true.

The following Lemma contain some useful properties of *A*-distances. It is stated in Aamri & El Moutawakil (2004). The proof is straightforward.

Lemma 1. Let (X,ϑ) be a Hausdorff uniform space and p be an *A*-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0. Then, for $x,y,z\in X$, the following holds:

(a) If $p(x_n,y)\le\alpha_n$ and $p(x_n,z)\le\beta_n$ for all $n\in\mathbb{N}$, then $y=z$. In particular, if $p(x,y)=0$ and $p(x,z)=0$, then $y=z$,

(b) if $p(x_n,y_n)\le\alpha_n$ and $p(x_n,z)\le\beta_n$ for all $n\in\mathbb{N}$, then $\{y_n\}$ converges to *z*,

(c) if $p(x_n,x_m)\le\alpha_n$ for all $n,m\in\mathbb{N}$ with $m>n$, then $\{x_n\}$ is a Cauchy sequence in (X,ϑ).

Let (X,ϑ) be a uniform space with an *A*-distance *P*. A sequence in *X* is *P*-Cauchy if it satisfies the usual metric condition. That is, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $p(x_n,x_m)<\varepsilon$ for all $n,m\ge n_0$. There are several concepts of completeness in this setting.

Definition 3. Let (X,ϑ) be a uniform space and *P* be an *A*-distance on *X*.

- *X* is *S*-complete uniform space if every *P*-Cauchy sequence $\{x_n\}$, there exists *x* in *X* with $\lim_{n\to\infty} p(x_n,x)=0$.
- *X* is *P*-Cauchy complete if every *P*-Cauchy sequence $\{x_n\}$, there exists *x* in *X* with $\lim_{n\to\infty} x_n=x$ with respect to $\tau(\vartheta)$.
- $T:X\to X$ is said to be *P*-continuous if $\lim_{n\to\infty} p(x_n,x)=0$ implies that $\lim_{n\to\infty} p(Tx_n,Tx)=0$.

Remark 1. Let (X,ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a *P*-Cauchy sequence. Suppose that *X* is *S*-complete, then there exists $x\in X$ such that $\lim_{n\to\infty} p(x_n,x)=0$. Lemma 1 (b) then gives $\lim_{n\to\infty} x_n=x$ with respect to the topology $\tau(\vartheta)$. Therefore *S*-completeness implies *P*-Cauchy completeness.

We recall the concept of a cyclic ϕ-contraction on a metric space and some classes of comparison functions.

Definition 4. (Kirk et al. (2003)) Let *X* be a nonempty set, *m* a positive integer and $T:X\to X$ a mapping. $X=\cup_{i=1}^{m} A_i$ is said to be a cyclic representation of *X* with respect to *T* if

- $A_i, i = 1,2,\cdots,m$ are nonempty sets,
- $T(A_1) \subset A_2, \cdots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Definition 5. (Pacurar & Rus(2010)) Let (X,d) be a metric space, m a positive integer, $A_1, A_2, \cdots, A_m$ nonempty subsets of X and $X = \cup_{i=1}^{m} A_i$. An operator $T : X \to X$ is a cyclic ϕ-contraction if

- $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T,
- $d(Tx, Ty) \leq \varphi(d(x,y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\cdots,m$, where $A_{m+1} = A_1$ and $\phi : [0,\infty) \to [0,\infty)$ a non-decreasing, continuous function satisfying $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Definition 6. (Berinde (1997)) A function $\varphi : [0,\infty) \to [0,\infty)$ is called a comparison function if it satisfies:

- ϕ is increasing,
- $\{\phi^n(t)\}$ converges to 0 as $n \to \infty$, for all $t \in [0,\infty)$.

Definition 7. (Berinde (1997)) A function $\varphi : [0,\infty) \to [0,\infty)$ is called a (c)-comparison function if:

- ϕ is increasing,
- there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that
$$\phi^{k+1}(t) \leq a\phi^k(t) + v_k,$$
for $k \geq k_0$ and any $t \in [0,\infty)$.

In Berinde (1997) the following are also proved:

Lemma 2. (Berinde (1997)) If $\varphi : [0,\infty) \to [0,\infty)$ is a (c)-comparison function, then the following hold:

- ϕ is comparison function,
- $\phi(t) < t$, for any $t > 0$,
- ϕ is continuous at 0 and $\phi(0) = 0$,
- the series $\sum_{k=0}^{\infty} \phi^k(t)$ converges for any $t \in (0,\infty)$.

MAIN RESULT

Before we state our main results, we give a formulation of cyclic ϕ-contraction in the setting of uniform spaces.

Definition 8. Let (X,ϑ) be a uniform space, *m* a positive integer, $A_1, A_2, \cdots, A_m$ nonempty subsets of *X* and $X = \cup_{i=1}^{m} A_i$. Let *P* be an *E*-distance on *X*. An operator $T: X \to X$ is a cyclic ϕ-contraction if

- $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of *X* with respect to *T*,
- $p(Tx, Ty) \le \varphi(p(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\cdots,m$, where $A_{m+1} = A_1$ and $\varphi:[0,\infty) \to [0,\infty)$ a non-decreasing, continuous function satisfying $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 1. Let (X,ϑ) be an *S*-complete Hausdorff uniform space such that *P* be a *E*-distance on *X* and *m* a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of *X* respect to the topological space $(X, \tau(\vartheta)$ and $X = \cup_{i=1}^{m} A_i$. Let $\varphi:[0,\infty) \to 0,\infty)$ is a (c)-comparison function and $T: X \to X$ be a cyclic ϕ -contraction and *P*-continuous. Then *T* has a unique fixed point $z \in \cap_{i=1}^{m} A_i$.

Proof. Take $x_0 \in X$ and consider the sequence given by

$$x_{n+1} = Tx_n, n = 0,1,2,\cdots.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then, since $x_{n_0+1} = Tx_{n_0} = x_{n_0}$, the part of existence of the fixed point is proved. Suppose that $x_{n+1} \ne x_n$ for any $n \in \mathbb{N}$. Then, since $X = \cup_{i=1}^{m} A_i$, for any $n > 0$ there exists $i_n \in \{1,2,\cdots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since *T* is a cyclic ϕ-contraction, we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\le \phi(p(x_{n-1}, x_n) \end{aligned} \qquad (0.1)$$

From (1) and taking into account that the monotonicity of ϕ, we get

$$\begin{aligned} p(x_n, x_{n+1}) &\le \phi(p(x_{n-1}, x_n) \\ &\le \phi(\phi(p(x_{n-2}, x_{n-1}) \\ &\vdots \\ &\le \phi^n(p(x_0, x_1)), \end{aligned}$$

for any $n \in \mathbb{N}$. Since *P* is an *E*-distance we obtain that

$$p(x_n, x_m) \le p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m),$$

so for $q \geq 1$ we have that

$$p(x_n, x_{n+q}) \leq \phi^n(p(x_0, x_1) + \cdots + \phi^{n+q-1}(p(x_0, x_1)).$$

In the sequel, we will prove that $\{x_n\}$ is a *P*-Cauchy sequence. Denoting

$$S_n = \sum_{k=0}^{n} \varphi^k (p(x_0, x_1)), n \geq 0,$$

then we have

$$p(x_n, x_{n+q}) \leq S_{n+q-1} - S_{n-1}. \tag{0.2}$$

As ϕ is a (c)-comparison function, supposing $p(x_0, x_1) > 0$, by Lemma 2, (iv), it follows that

$$\sum_{k=0}^{\infty} \phi^k (p(x_0, x_1) < \infty,$$

so there is $S \in 0, \infty)$ such that

$$\lim_{n\to\infty} S_n = S.$$

Then by (2) we obtain that

$$p(x_n, x_{n+q}) \to 0 \text{ as } n \to \infty,$$

which shows that $\{x_n\}$ is a *P*-Cauchy sequence in the *S*-complete space *X*. So there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. In what follows, we prove that *x* is a fixed point of *T*. In fact, since $\lim_{n\to\infty} x_n = x$ and, as $X = \cup_{i=1}^m A_i$ is a cyclic representation of *X* with respect to *T*, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \cdots, m\}$. Since A_i is closed for every *i*, it follows that $x \in \bigcap_{i=1}^m A_i$, thus we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the above mentioned comment). Since *T* is *P*-continuous we have

$$\lim_{n\to\infty} p(x_{n+1}, Tx) = \lim_{n\to\infty} p(Tx_n, Tx) = 0.$$

From Lemma 1 (a) we have *x* = *Tx* and, therefore, *x* is a fixed point of *T*.

Finally, in order to prove the uniqueness of the fixed point, suppose that $y, z \in X$ with *y* and *z* fixed points of *T*. The cyclic character of *T* and the fact that $y, z \in X$ are fixed points of *T*, imply that $y, z \in \cap_{i=1}^m A_i$. Using the contractive condition we obtain

$$p(y, z) = p(Ty, Tz) \leq \varphi(p(y, z)) < p(y, z), \text{if} p(y, z) > 0.$$

From the last inequality we get

$$p(y,z)=0.$$

Hence, also $p(y,y)=0$ and, consequently, $y=z$. This finishes the proof.

Corollary 1. Let (X,d) be a complete metric space and m a positive integer, $A_1,A_2,\cdots,A_m$ nonempty closed subsets of X and $X=\cup_{i=1}^{m}A_i$. Let $T:X\to X$ be a cyclic ϕ-contraction and P-continuous. Then T has a unique fixed point $z\in\cap_{i=1}^{m}A_i$.

Proof. By Theorem 1, it is enough set $\vartheta=\{U_\varepsilon \mid \varepsilon>0\}$.

Corollary 2. Let (X,ϑ) be a S-complete Hausdorff uniform space such that P be a E-distance on X and m a positive integer, $A_1,A_2,\cdots,A_m$ nonempty closed subsets of X respect to the topological space $(X,\tau(\vartheta))$. Let $T:X\to X$ be a and P-continuous operator such that

- $X=\cup_{i=1}^{m}A_i$ is a cyclic representation with respect to T and
- $p(Tx,Ty)\le kp(x,y)$ for any $x\in A_i$, $y\in A_{i+1}$, $i=1,2,\cdots,m$, where $A_{m+1}=A_1$ and $0<k<1$.

Then T has a unique fixed point $z\in\cap_{i=1}^{m}A_i$.

Proof. By Theorem 1, it is enough set $\phi(t)=kt$.

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التقلصات الدوريةعلى فضاءات هاوسدورف المنتظمة

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خلاصة

نقوم في هذا البحث بتطبيق مبدأ التقلص الدوري للحصول على مبرهنة النقطة الصامدة في فضاء هاوسدورف المنتظم.