Cyclic $\phi$-contractions on S-complete Hausdorff uniform Spaces

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ABSTRACT

In this paper, we apply the concept of cyclic $\phi$-contraction for presenting a fixed point theorem on a Hausdorff uniform space.

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INTRODUCTION

Let $X$ be a nonempty set and let $\mathcal{G}$ be a nonempty family of subsets of $X \times X$. The pair $(X, \mathcal{G})$ is called a uniform space if it satisfies the following properties:

(i) if $G$ is in $\mathcal{G}$, then $G$ contains the diagonal $\{(x, x) : x \in X\}$;

(ii) if $G$ is in $\mathcal{G}$ and $H$ is a subset of $X \times X$ which contains $G$, then $H$ is in $\mathcal{G}$;

(iii) if $G$ and $H$ are in $\mathcal{G}$, then $G \cap H$ is in $\mathcal{G}$;

(iv) if $G$ is in $\mathcal{G}$, then there exists $H$ in $\mathcal{G}$, such that, whenever $(x, y)$ and $(y, z)$ are in $H$, then $(x, z)$ is in $G$;

(v) if $G$ is in $\mathcal{G}$, then $\{(y, x) : (x, y) \in G\}$ is also in $\mathcal{G}$.

$\mathcal{G}$ is called the uniform structure of $X$ and its elements are called entourages or neighbourhoods or surroundings. In Bourbaki (1998) and Zeidler (1986), $(X, \mathcal{G})$ is called a quasiuniform space if property (v) is omitted. Some authors studied the theory...
of fixed point or common fixed point for contractive selfmappings in uniform space (Altun (2011); Kubiak & Cho (1993); Turkoglu (2010); Wlodarczyk & Plebaniak (2011); Vályi (1985))

Later, Aamri & El Moutawakil (2004, 2005) proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an $A$-distance and an $E$-distance.

For any set $X$, the diagonal $\{(x, x) : x \in X\}$ will be denoted by $\Delta$ if no confusion occurs. If $V, W \in X \times X$, then $V \circ W = \{(x, y) : \text{there exists } z \in X \text{ such that } (x, z) \in W \text{ and } (z, y) \in V\}$ and $V^{-1} = \{(x, y) : (y, x) \in V\}$.

If $V \in \mathcal{G}$ and $(x, y) \in V, (y, x) \in V$, $x$ and $y$ are said to be $V$-close, and a sequence $\{x_n\}$ in $X$ is a Cauchy sequence for $\mathcal{G}$, if for any $V \in \mathcal{G}$, there exists $N \geq 1$ such that $x_n$ and $x_m$ are $V$-close for $n, m \geq N$. An uniformity $\mathcal{G}$ defines a unique topology $\tau(\mathcal{G})$ on $X$ for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when $V$ runs over $\mathcal{G}$.

A sequence $\{x_n\}$ in $X$ is convergent to $x$ for $\mathcal{G}$, if for any $V \in \mathcal{G}$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V(x)$ for every $n \geq n_0$ and denote by $\lim_{n \to \infty} x_n = x$. A uniform space $(X, \mathcal{G})$ is said to be Hausdorff if and only if the intersection of all the $V \in \mathcal{G}$ reduces to the diagonal $\Delta$ of $X$, i.e., if $(x, y) \in V$ for all $V \in \mathcal{G}$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \mathcal{G}$ is said to be symmetrical if $V = V^{-1}$. Since each $V \in \mathcal{G}$ contains a symmetrical $W \in \mathcal{G}$ and if $(x, y) \in W$ then $x$ and $y$ are both $W$ and $V$-close, then for our purpose, we assume that each $V \in \mathcal{G}$ is symmetrical. When topological concepts are mentioned in the context of a uniform space $(X, \mathcal{G})$, they always refer to the topological space $(X, \tau(\mathcal{G}))$.

**PRELIMINARIES**

**Definition 1.** (Aamri & El Moutawakil (2004)) Let $(X, \mathcal{G})$ be a uniform space. A function $p : X \times X \to [0, \infty)$ is said to be an $A$-distance if for any $V \in \mathcal{G}$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

**Definition 2.** (Aamri & El Moutawakil (2004)) Let $(X, \mathcal{G})$ be a uniform space. A function $p : X \times X \to [0, \infty)$ is said to be an $E$-distance if

- $(p_1)$ $p$ is an $A$-distance,

- $(p_2)$ $p(x, y) \leq p(x, z) + p(z, y)$. $\forall x, y, z \in X$. 

There are very nice examples of $E$-distances in Aamri & El Moutawakil (2004). Some of these examples compare the concept of $E$-distance with $W$-distance, which is introduced by Montes & Charris (2001), on uniform spaces. Every $W$-distance is an $E$-distance, but the converse may not be true.

The following Lemma contain some useful properties of $A$-distances. It is stated in Aamri & El Moutawakil (2004). The proof is straightforward.

**Lemma 1.** Let $(X, ℋ)$ be a Hausdorff uniform space and $p$ be an $A$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. Then, for $x, y, z \in X$, the following holds:

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$.

(c) if $p(x_n, x_m) \leq \alpha_n$ for all $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in $(X, ℋ)$.

Let $(X, ℋ)$ be a uniform space with an $A$-distance $P$. A sequence in $X$ is $P$-Cauchy if it satisfies the usual metric condition. That is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$. There are several concepts of completeness in this setting.

**Definition 3.** Let $(X, ℋ)$ be a uniform space and $P$ be an $A$-distance on $X$.

- $X$ is $S$-complete uniform space if every $P$-Cauchy sequence $\{x_n\}$, there exists $x$ in $X$ with $\lim_{n \to \infty} p(x_n, x) = 0$.
- $X$ is $P$-Cauchy complete if every $P$-Cauchy sequence $\{x_n\}$, there exists $x$ in $X$ with $\lim_{n \to \infty} x_n = x$ with respect to $\tau(ℋ)$.
- $T : X \to X$ is said to be $P$-continuous if $\lim_{n \to \infty} p(x_n, x) = 0$ implies that $\lim_{n \to \infty} p(Tx_n, Tx) = 0$.

**Remark 1.** Let $(X, ℋ)$ be a Hausdorff uniform space and let $\{x_n\}$ be a $P$-Cauchy sequence. Suppose that $X$ is $S$-complete, then there exists $x \in X$ such that $\lim_{n \to \infty} p(x_n, x) = 0$. Lemma 1 (b) then gives $\lim_{n \to \infty} x_n = x$ with respect to the topology $\tau(ℋ)$. Therefore $S$-completeness implies $P$-Cauchy completeness.

We recall the concept of a cyclic $\phi$-contraction on a metric space and some classes of comparison functions.

**Definition 4.** (Kirk et al. (2003)) Let $X$ be a nonempty set, $m$ a positive integer and $T : X \to X$ a mapping. $X = \bigcup_{i=1}^{m} A_i$ is said to be a cyclic representation of $X$ with respect to $T$ if
• \( A_i, i = 1, 2, \ldots, m \) are nonempty sets,
• \( T(A_i) \subseteq A_2, \ldots, T(A_{m-1}) \subseteq A_m, T(A_m) \subseteq A_1 \).

**Definition 5.** (Pacurar & Rus(2010)) Let \( (X, d) \) be a metric space, \( m \) a positive integer, \( A_1, A_2, \ldots, A_m \) nonempty subsets of \( X \) and \( X = \bigcup_{i=1}^{m} A_i \). An operator \( T : X \to X \) is a cyclic \( \phi \)-contraction if
- \( X = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T \),
- \( d(Tx, Ty) \leq \phi(d(x, y)) \), for any \( x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m \), where \( A_{m+1} = A_1 \) and \( \phi : [0, \infty) \to [0, \infty) \) a non-decreasing, continuous function satisfying \( \phi(t) > 0 \) for all \( t > 0 \) and \( \phi(0) = 0 \).

**Definition 6.** (Berinde (1997)) A function \( \phi : [0, \infty) \to [0, \infty) \) is called a comparison function if it satisfies:
- \( \phi \) is increasing,
- \( \{\phi^n(t)\} \) converges to 0 as \( n \to \infty \), for all \( t \in [0, \infty) \).

**Definition 7.** (Berinde (1997)) A function \( \phi : [0, \infty) \to [0, \infty) \) is called a \((c)\)-comparison function if:
- \( \phi \) is increasing,
- there exist \( k_0 \in \mathbb{N}, \ a \in (0,1) \) and a convergent series of nonnegative terms \( \sum_{k=1}^{\infty} v_k \) such that \( \phi^{k+1}(t) \leq a \phi^k(t) + v_k \), for \( k \geq k_0 \) and any \( t \in [0, \infty) \).

In Berinde (1997) the following are also proved:

**Lemma 2.** (Berinde (1997)) If \( \phi : [0, \infty) \to [0, \infty) \) is a \((c)\)-comparison function, then the following hold:
- \( \phi \) is comparison function,
- \( \phi(t) < t \), for any \( t > 0 \),
- \( \phi \) is continuous at 0 and \( \phi(0) = 0 \),
- the series \( \sum_{k=0}^{\infty} \phi^k(t) \) converges for any \( t \in (0, \infty) \).
MAIN RESULT

Before we state our main results, we give a formulation of cyclic $\phi$-contraction in the setting of uniform spaces.

**Definition 8.** Let $(X, \mathcal{U})$ be a uniform space, $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty subsets of $X$ and $X = \bigcup_{i=1}^{m} A_i$. Let $P$ be an $E$-distance on $X$. An operator $T : X \to X$ is a cyclic $\phi$-contraction if

- $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $X$ with respect to $T$,
- $p(Tx, Ty) \leq \phi(p(x, y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\phi : [0, \infty) \to [0, \infty)$ a non-decreasing, continuous function satisfying $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

**Theorem 1.** Let $(X, \mathcal{U})$ be an $S$-complete Hausdorff uniform space such that $P$ be a $E$-distance on $X$ and $m$ a positive integer, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of $X$ respect to the topological space $(X, \tau(\mathcal{U}))$ and $X = \bigcup_{i=1}^{m} A_i$. Let $\phi : [0, \infty) \to [0, \infty)$ is a (c)-comparison function and $T : X \to X$ be a cyclic $\phi$-contraction and $P$-continuous. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

**Proof.** Take $x_0 \in X$ and consider the sequence given by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots.$$ 

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then, since $x_{n_0+1} = Tx_{n_0} = x_{n_0}$, the part of existence of the fixed point is proved. Suppose that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N}$. Then, since $X = \bigcup_{i=1}^{m} A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n+1} \in A_{i_n}$ and $x_n \in A_{i_n+1}$. Since $T$ is a cyclic $\phi$-contraction, we have

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \leq \phi(p(x_{n-1}, x_n)) \quad (0.1)$$

From (1) and taking into account that the monotonicity of $\phi$, we get

$$p(x_n, x_{n+1}) \leq \phi(p(x_{n-1}, x_n)) \leq \phi(\phi(p(x_{n-2}, x_{n-1}))) \leq \cdots \leq \phi^n(p(x_0, x_1)),$$

for any $n \in \mathbb{N}$. Since $P$ is an $E$-distance we obtain that

$$p(x_n, x_m) \leq p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m),$$
so for \( q \geq 1 \) we have that

\[
p(x_n, x_{n+q}) \leq \phi^n(p(x_0, x_1) + \cdots + \phi^{n+q-1}(p(x_0, x_1)).
\]

In the sequel, we will prove that \( \{x_n\} \) is a \( P \)-Cauchy sequence. Denoting

\[
S_n = \sum_{k=0}^{n} \phi^k(p(x_0, x_1)), n \geq 0,
\]

then we have

\[
p(x_n, x_{n+q}) \leq S_{n+q-1} - S_{n-1}.
\]

As \( \phi \) is a \((c)\)-comparison function, supposing \( p(x_0, x_1) > 0 \), by Lemma 2, (iv), it follows that

\[
\sum_{k=0}^{\infty} \phi^k(p(x_0, x_1)) < \infty,
\]

so there is \( S \in \mathbb{0}, \infty \) such that

\[
\lim_{n \to \infty} S_n = S.
\]

Then by (2) we obtain that

\[
p(x_n, x_{n+q}) \to 0 \text{ as } n \to \infty,
\]

which shows that \( \{x_n\} \) is a \( P \)-Cauchy sequence in the \( S \)-complete space \( X \). So there exists \( x \in X \) such that \( \lim_{n \to \infty} p(x_n, x) = 0 \). In what follows, we prove that \( x \) is a fixed point of \( T \). In fact, since \( \lim_{n \to \infty} x_n = x \) and, as \( X = \bigcup_{i=1}^{m} A_i \) is a cyclic representation of \( X \) with respect to \( T \), the sequence \( \{x_n\} \) has infinite terms in each \( A_i \) for \( i \in \{1, 2, \cdots, m\} \). Since \( A_i \) is closed for every \( i \), it follows that \( x \in \bigcap_{i=1}^{m} A_i \), thus we take a subsequence \( x_{n_k} \) of \( \{x_n\} \) with \( x_{n_k} \in A_{i-1} \) (the existence of this subsequence is guaranteed by the above mentioned comment). Since \( T \) is \( P \)-continuous we have

\[
\lim_{n \to \infty} p(x_{n+1}, Tx_n) = \lim_{n \to \infty} p(Tx_n, Tx) = 0.
\]

From Lemma 1 (a) we have \( x = Tx \) and, therefore, \( x \) is a fixed point of \( T \).

Finally, in order to prove the uniqueness of the fixed point, suppose that \( y, z \in X \) with \( y \) and \( z \) fixed points of \( T \). The cyclic character of \( T \) and the fact that \( y, z \in X \) are fixed points of \( T \), imply that \( y, z \in \bigcap_{i=1}^{m} A_i \). Using the contractive condition we obtain

\[
p(y, z) = p(Ty, Tz) \leq \phi(p(y, z)) < p(y, z), \text{if} p(y, z) > 0.
\]
Cyclic $\phi$-contractions on S-complete Hausdorff uniform Spaces

From the last inequality we get

$$p(y, z) = 0.$$  

Hence, also $p(y, y) = 0$ and, consequently, $y = z$. This finishes the proof.

**Corollary 1.** Let $(X, d)$ be a complete metric space and $m$ a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of $X$ and $X = \bigcup_{i=1}^m A_i$. Let $T : X \to X$ be a cyclic $\phi$-contraction and $P$-continuous. Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

**Proof.** By Theorem 1, it is enough set $\mathcal{G} = \{U_\varepsilon | \varepsilon > 0\}$.

**Corollary 2.** Let $(X, \mathcal{G})$ be a S-complete Hausdorff uniform space such that $P$ be a $E$-distance on $X$ and $m$ a positive integer, $A_1, A_2, \cdots, A_m$ nonempty closed subsets of $X$ respect to the topological space $(X, \tau(\mathcal{G}))$. Let $T : X \to X$ be a and $P$-continuous operator such that

- $X = \bigcup_{i=1}^m A_i$ is a cyclic representation with respect to $T$ and
- $p(Tx, Ty) \leq kp(x, y)$ for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \cdots, m$, where $A_{m+1} = A_1$ and $0 < k < 1$.

Then $T$ has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

**Proof.** By Theorem 1, it is enough set $\phi(t) = kt$.

**REFERENCES**


Altun, I. 2011. Common fixed point theorems for weakly increasing mappings on ordered uniform spaces, Miskolc Mathematical Notes 12: 3-10.


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التبقيات الدورية على فضاءات هاوسدورف المنتظمة

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خلاصة

نقوم في هذا البحث بتطبيق مبدأ التبقيات الدوري للحصول على مبرهنة النقطة الصامدة في فضاء هاوسدورف المنتظم.
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ومن أبوابها
- البحوث العربية.
- البحوث الإنجليزية.
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