Asymptotic Behavior and Existence of Similarity Solutions for a Boundary Layer Flow Problem

Samer Al-Ashhab*
Dept. of Mathematics and Statistics, Al Imam Mohammad Ibn Saud Islamic University (IMSIU),
P.O. Box 90950, Riyadh 11623, Saudi Arabia.
**Corresponding author: ssashhab@imamu.edu.sa*

Abstract

The problem of boundary layer flow of a non-Newtonian power-law fluid (which is assumed to be incompressible) is considered. Existence and uniqueness of similarity solutions are considered for all values of the power-law index $n>0$. Conditions are determined (values of n and various parameters within the problem) where existence and uniqueness of solutions hold and where they do not hold. Exact solutions in some cases are exhibited. The asymptotic behavior of solutions is also determined for all values of $n>0$ of the non-Newtonian fluid.

Keywords: Asymptotic behavior; boundary layer flow; existence of solutions; power-law fluid; singular non-linear boundary value problem.

2010 Mathematics Subject Classification: 34B16,76A05

1. Introduction

The problem of boundary layer flow of non-Newtonian fluids has been the focus of many studies in the last couple of decades as it has appeared in many applications relating to industry, mechanics, and applied mathematics. This scientific problem that arises from applications to fluid mechanics is modeled by a boundary value problem on a semi-infinite domain which involves a third order non-linear ordinary differential equation. A transformation to a finite domain naturally results in a non-linear singular boundary value problem of the third order.

Earlier modeling of boundary-layer power-law fluid problems can be found in Acrivos *et al.* (1960). However, more on the modeling and derivation of the problem can be found in Pakdemirli (1994), Astarita & Marrucci (1974), Schlichting (1979), and Bohme (1987). We remark that using a Crocco variable formulation Nachman & Talliafero (1979) studied existence and uniqueness for the mathematical problem involving a Newtonian fluid. Existence and uniqueness for both Newtonian and non-Newtonian fluids were examined by studying the problem within its semi-infinite domain by Guedda & Hammouch (2008) and Guedda (2009). Wei & Al-Ashhab (2014) applied a transformation to a finite domain to establish existence and uniqueness and study properties of solutions for non-Newtonian fluids.

Many other authors contributed to this rich problem that is difficult to study in full generality, see for example Ece & Büyük (2002), Denier & Dabrowski (2004), and Blasius (1908) to mention a few. It is noted that Rahman *et al.* (2017) studied a time varying fluid flow problem in the presence of a magnetic field, while a similar mathematical model was considered in Marin & Lupu (1998) in the context of thermoelasticity and harmonic vibrations. The work by Howell *et al.* (1997) on power-law fluids discussed this non-Newtonian problem in the context of momentum and heat transfer, and it was an important physical contribution. Indeed however, many questions remain unanswered.

The most commonly used model in non-Newtonian fluid mechanics is the Ostwald-de Waele model with a power-law rheology which is characterized by a power-law index n. The value $n=1$ corresponds to a Newtonian fluid, while $n>1$ describes a dilatant or shear-thickening fluid and $0<n<1$ describes pseudo-plastic or shear-thinning fluid.

In this paper we look to apply a Crocco variable transformation to study properties of solutions, existence and uniqueness, as well as the asymptotic behavior for a general version (relatively speaking) of the problem involving a non-Newtonian fluid with initial conditions not set a priori, but rather left arbitrary. It is important to note that in Al-Ashhab (2015), a somewhat similar technique was utilized to establish a single equation for both positive and negative curvatures, and to investigate the asymptotic behavior but for a simpler version of the problem with a somewhat simpler governing ODE. This enables the discovery of new results and aspects of the problem, and also enables the analysis of the asymptotic behavior of solutions. This also enables the exhibition of exact solutions in some cases. Crocco variables within this context also enable the exploration of a relatively new and peculiar result that uniqueness of solutions do not always hold. Those results are further discussed using numerical evidence (in the last section of the paper).

2. Governing equations

A brief derivation of the problem is given here: The model used here is the Ostwald-de Waele model with a power-law rheology, where the relationship between the shear stress τ_{xy} and the strain rate $u_y = \frac{\partial u}{\partial y}$ is governed by (see for example Pakdemirli (1994), Astarita & Marrucci (1974), or Schlichting (1979) for full physical derivation and more details):

$$\tau_{xy} = k(|u_y|^{n-1} u_y) . \tag{1}$$

The shear stress τ_{xy} here is a component of the stress tensor, however a discussion of the stress tensor is beyond the scope of this paper. The physical problem is defined by a two dimensional incompressible non-Newtonian steady-state laminar fluid flow on a semi-infinite plate, where the flow is governed by equation (1). The x-direction is parallel to a bounding plate situated at $y=0$, while the y-direction is perpendicular to this bounding plate. The (so-called boundary layer) governing equations are the continuity and momentum equations as follows:

$$u_x + v_y = 0, \tag{2}$$

$$uu_x + vu_y = \nu\, (|u_y|^{n-1} u_y)_y \tag{3}$$

where u and v are the velocity components in the x and y directions respectively. Observe that, physically, the bounding plate is at $y=0$ where a zero velocity parallel to the plate $u(x,0)$ defines a no-slip condition at this bounding plate. On the other hand a zero velocity perpendicular to the plate $v(x,0)$ implies an impermeable bounding plate. In what follows it shall be assumed that $v(x,0) \neq 0$, which implies that the bounding plate is porous. The boundary conditions are:

$$u(x,0) = U_w(x), \quad v(x,0) = V_w(x), \qquad u(x,y) \to 0 \text{ as } y \to \infty \tag{4}$$

where $U_w(x)=u_w x^m$ (which represents the stretching velocity) and $V_w(x) = v_w x^{\frac{m(2n-1)-n}{n+1}}$ (which represents the suction/injection velocity).

Now let $\psi=\psi(x,y)$ be a function that satisfies $u=\psi_y$, $v=-\psi_x$ where ψ is referred to as the stream function). This transforms problem (2-4) into:

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu\left(|\psi_{yy}|^{n-1}\psi_{yy}\right)_y \tag{5}$$

with conditions

$$\psi_y(x,0) = u_w x^m, \;\; \psi_x(x,0) = -v_w x^{\frac{m(2n-1)-n}{n+1}} \tag{6}$$

$$\psi_y(x,y) \to 0 \quad \text{as} \quad y \to \infty$$

Introduce a function f and parameter η via:

$$\psi(x,y) = Dx^\alpha f(\eta), \quad \eta = E\frac{y}{x^\beta} \tag{7}$$

This is called a similarity transformation, where f is referred to as the dimensionless stream function and η is the similarity variable. Substituting (7) into (5) yields the following equation:

$$\nu D^n E^{2n+1} x^{(\alpha-2\beta)n-\beta}(|f''|^{n-1}f'')' + \alpha D^2E^2x^{2(\alpha-\beta)-1}ff'' = (\alpha-\beta)D^2E^2x^{2(\alpha-\beta)-1}\,(f')^2$$

which simplifies to an ordinary differential equation, namely:

$$(|f''|^{n-1}f'')' + \alpha ff'' = (\alpha-\beta)\,(f')^2 \tag{8}$$

if and only if $\alpha(2-n)+\beta(2n-1)=1, \alpha-\beta=m$, and where in this context one assumes $\nu D^{(n-2)} E^{(2n+1)}=1$. The boundary conditions are transformed to:

$$f(0) = -\frac{v_w}{\alpha D}, \quad f'(0) = \frac{u_w}{DE}, \qquad f'(\infty) = \lim_{\eta\to\infty} f'(\eta) = 0 \tag{9}$$

This power-law problem (8-9) has in fact been studied by many authors, but not in its full generality. Some authors set some boundary conditions (and/or some of the other parameters) to zero. Others fixed the values of some of the parameters a priori, or considered certain values/ranges of the power-law index n.

It is worthwhile noting here that for the special case of Newtonian fluids $n=1$ with $m=u_w=v_w=0$, and an adjustment of the condition at infinity to $\lim_{\eta\to\infty} f'(\eta) = C \neq 0$ one obtains the Blasius equation (see Blasius (1908)). In Guedda (2009) and references therein the authors considered a power-law velocity profile where they adjusted the condition at infinity to the following: $\lim_{\eta\to\infty} f'(\eta) = C\eta^\sigma$ but with $u_w=v_w=0$. A full derivation can be found in those references mentioned above, and where we note that the researchers added the condition $\alpha=\beta(1+\sigma)$ which resulted from the derivation process, and determined existence/non-existence of similarity solutions for certain ranges of σ and n. It should be emphasized here that the case $C=0$ was not included as a special case of their solution (due to the approach that they used).

We shall study the general case $m\neq0, u_w\neq0$, and $v_w\neq0$ (we may allude to the case $m=0$ briefly to introduce our approach but then move on to the general case with $m\neq0$). All values of $n>0$ are considered. In Guedda & Hammouch (2009) the authors studied a somewhat similar problem but considered values of $n>1$ only, and a certain range of the parameters α and m. We look to expand the range of those parameters, where we use a different approach and study solutions as well as their asymptotic behavior. Illustrations to highlight the results are also included. Now by setting $a=-\frac{v_w}{\alpha D}$, $\epsilon=\frac{u_w}{DE}$ the power-law problem (8-9) takes the form:

$$(|f''|^{n-1}f'')'+\alpha ff''-m(f')^2=0 \tag{10}$$

subject to

$$f(0)=a,\ f'(0)=\epsilon,\ f'(\infty)=0 \tag{11}$$

We assume negative curvature solutions: $f''<0$. This is the assumption in much of the literature, however positive curvatures are also considered frequently. In the case where $f''<0$ we must have $\epsilon>0$ (this is the case when for example $u_w>0$, $D>0$, and $E>0$ are all positive) and $f'\geq 0$ on the entire solution domain.

Remark We note that replacing a by $-a$ and ϵ by $-\epsilon$ yields a transformation of negative curvature solutions to positive curvature solutions since if f, f', and f'' all change sign, then the last two terms of the ODE in (10) will not change sign. Yet, the first term will change sign. Transferring the negative sign introduced in the first term to the other two will then show that a negative f'' increases whereas a positive f'' decreases with both eventually reaching 0.

3. Asymptotic behavior and existence of solutions

In this section, we study the asymptotic behavior of solutions for different values of the power-law index n. Apply a Crocco variable transformation to problem (10-11) with the following variables:

$$z=f'(\eta),\ \ h(z)=\left(-f''(\eta)\right)^n \tag{12}$$

This leads to the problem consisting of the equation:

$$h''(z)=((-\alpha+2m)z-\frac{mz^2h'(z)}{nh(z)})\,h^{-\frac{1}{n}}(z), \tag{13}$$

for $0<z<\epsilon$, subject to: $h(0)=0$, $h'(\epsilon)=\alpha a-\frac{m\epsilon^2}{f''(0)}$. Observe that $h(z)$ must be a positive function and $z>0$, therefore the only solutions of relevance here are the ones in the first quadrant. By letting $A=-\alpha+2m$, $B=-\frac{m}{n}$ we obtain and study the equivalent problem:

$$h''(z)=\left(Az+Bz^2\frac{h'(z)}{h(z)}\right)h^{-\frac{1}{n}}(z) \tag{14}$$

for $0<z<\epsilon$, subject to

$$h(0)=0,\ \ h'(\epsilon)=\alpha a-\frac{m\epsilon^2}{f''(0)} \tag{15}$$

3.1 The case B=0

We first consider the case where B=0. This happens when m=0 in our derivation process since $B=-\frac{m}{n}$. To discuss the asymptotic behavior of f' (and consequently f) as $\eta\to\infty$ let $h(z)$ be represented by $h(z)\approx kz^p$ for z close to 0 ($z>0$), and for some parameters k and p. Observe that k must be positive since $h(z)$ is a positive function so that for $A<0$ and $0<n<\frac{1}{2}$ we have:

$$p=\frac{3n}{n+1},\qquad k^{1+\frac{1}{n}}=\frac{A}{p(p-1)} \tag{16}$$

This implies that $p<1$ and consequently $k>0$ which is consistent with the fact that $h(z)$ must be a positive function. In fact $h(z)\approx kz^p$ in its own right is an exact solution to (14) which satisfies (15) if $h'(\epsilon)=kp\epsilon^{p-1}=\alpha a=\alpha f(0)>0$.

Substituting back the values of z and $h(z)$ in terms of the Crocco variables (derivatives of f as given above in (12)) and integrating the resulting equation, yields:

$$f'\approx\left(\left(\frac{2-n}{n+1}k^{\frac{1}{n}}\right)\eta+K\right)^{\frac{n+1}{n-2}} \tag{17}$$

for large η and where K is a constant (of integration). In other words:

$$f'\to c\cdot\eta^{\frac{n+1}{n-2}}\ \text{ as } \eta\to\infty \tag{18}$$

for $0<n<\frac{1}{2}$ and for some constant $c>0$, where in fact $c=\left(\frac{2-n}{n+1}k^{\frac{1}{n}}\right)^{\frac{n+1}{n-2}}$ Observe that f' tends to zero as $\eta\to\infty$, while

$$f\to\frac{n-2}{2n-1}c\cdot\eta^{\frac{2n-1}{n-2}}+L \tag{19}$$

where L is a constant. Note that f does not tend to constant as $\eta\to\infty$ since the exponent $\frac{2n-1}{n-2}>0$.

Now for $n>\frac{1}{2}$, let $h(z)\approx kz+\kappa z^p$ for z close to 0 ($z>0$). This yields a value of $p=3-\frac{1}{n}$ observe that $p>1$), and it can be shown that the equation $\kappa p(p-1)\kappa^{\frac{1}{n}}=A$ relates $k>0$ to κ. Observe that this works for positive A as well as negative A. This is the case where κ is positive when A is positive, and it is negative when A is negative. However, k is positive in both cases. Substituting back the values of z and $h(z)$ in terms of the Crocco variables (12) and integrating the resulting equation yields:

$$f'\approx\left(\left(\frac{1-n}{n}(k)^{\frac{1}{n}}\right)\eta+K\right)^{\frac{n}{n-1}} \tag{20}$$

for large η and where K is a constant. Therefore we have:

$$f'\to c\cdot\eta^{\frac{n}{n-1}}\ \text{ as }\ \eta\to\infty, \tag{21}$$

for $\frac{1}{2}<n<1$ and for some constant $c>0$, which in turn implies that:

$$f\to\frac{n-1}{2n-1}\cdot c\cdot\eta^{\frac{2n-1}{n-1}}+f_\infty \tag{22}$$

so that f tends to a constant f_∞ as $\eta\to\infty$ since the exponent on η is negative. On the other hand, observe that if $n>1$ the first term in (20) is negative, and then in the case of even radicals on exponents the equation will terminate and cannot be extended with infinite η, otherwise f' will be negative or become unbounded which is a contradiction: In fact equation (20) suggests that f' and f'' reach zero at a finite value of η when the expression in parentheses reaches zero. This shows the natural and crucial result that for $n>1$, f' goes to zero very rapidly and may reach zero at a finite η which is consistent with the results obtained in Wei & Al-Ashhab (2014) for a similar equation. Finally observe that, in this case of $n>1$, f tends to a constant as $\eta\to\infty$ since f' reaches zero at finite η as discussed above.

For $n=1/2$, observe that we may assume an approximation of the form $h(z)\approx kz(\ln z)^p$ near $z=0$, where substituting back into (14) yields $p=\frac{1}{3}$, and $k=\left(\frac{A}{p}\right)^{\frac{1}{3}}=(3A)^{\frac{1}{3}}$ which for negative A does yield the positive (since $z\approx 0$ with $z>0$) approximate solution $h(z)\approx(3A)^{\frac{1}{3}}z(\ln z)^{\frac{1}{3}}$, and where it can be concluded that a solution satisfying (14-15) exists, but with possibly additional conditions on the parameters of the problem. This in turn yields an asymptotic behavior of the form $f'\to(k^2\eta+K)^{-1}$.

Remark As for the case $n>0$, notice that for $n>\frac{1}{2}$ we have $p>1$ since $p=\frac{3n}{n+1}$ from (16) above, and that works with positive k, so it does lead to a solution. The asymptotic behavior then follows equations (17) and (18) for $\frac{1}{2}<n<2$. As for $n>2$, equation (17) suggests very rapid decline to zero for f', since the first term in (17) is negative. Therefore, K must be positive since by assumption $f'(0)=\epsilon>0$, $f'>0$ on the entire solution domain. Thus, this fact suggests that f' reaches zero at some finite η which makes the expression in parentheses equal zero. Lastly, observe that for $n=2$ we have $h(z)=kz^2$, $f''(\eta)=\sqrt{k}f'(\eta)$ and the asymptotic behavior takes the form $f'\to ce^{-\sqrt{k}\eta}$, again showing a rapid decline of $f'(\eta)$ to zero. These solutions require $h'(\epsilon)=\alpha a>0$ (here $m=0$ since $B=0$ and $h(z)=(-f''(\eta))^n$ from (12) above so that as f' and f'' strictly decrease to 0 then $h(z)$ strictly decreases to 0. Observe in our arguments z moves left from ϵ to 0 and therefore $h'(z)>0$ on $(0,\epsilon]$. If $A>0, n<\frac{1}{2}$ solutions do not exist as will be shown shortly.

3.1.1. Proof of existence and uniqueness for B= 0

We choose to include a proof on existence and uniqueness, but avoid rigorous mathematical styles. To this end, observe that while it may not be difficult to establish existence and uniqueness of solutions to (14-15) that extend back to a point $(0,b>0)$, we still need to discuss existence and uniqueness of solutions that approach the origin (0,0). Existence of a solution that approaches (0,0) can be established by simple arguments (as a limiting case of solutions through $(0,b)$ where $b>0$). As for uniqueness of such a solution observe that multiplying both sides of (14) by $h'(z)$ and integrating yields:

$$(h'(z))^2-(h'(a))^2=\frac{2A}{1-\frac{1}{n}}\left(zh^{1-\frac{1}{n}}(z)-ah^{1-\frac{1}{n}}(a)-\int_a^z h^{1-\frac{1}{n}}(\mu)d\mu\right) \tag{23}$$

Therefore, if there are two solutions approaching the origin but with different initial conditions (say same $h(z)$ but different $h'(z)$ then that will lead to a contradiction: Suppose that h_1 is a solution through the origin. Now, take another solution h_2 to (14) with initial conditions $h_2(z)=h_1(z)$ and $h_2'(z)>h_1'(z)$, i.e. a larger $h_2'(z)$. The two solutions may not intersect at any point a<z, since if they did we should have $h_1'(a)>h_2'(a)$ (the condition $h_2'(z)>h_1'(z)$, implies that $h_2(\mu)<h_1(\mu)$ within the interval (a,z) so that geometrically h_2 is under h_1 within (a,z)). Now the first two terms on the right hand side of (23), namely $zh^{1-\frac{1}{n}}(z)-ah^{1-\frac{1}{n}}(a)$, would be the same for h_1 and h_2 and therefore we have:

$$(h_2'(a))^2-(h_1'(a))^2=(h_2'(z))^2-(h_1'(z))^2+\frac{2A}{1-\frac{1}{n}}\int_a^z\left(h_2^{1-\frac{1}{n}}(\mu)-h_1^{1-\frac{1}{n}}(\mu)\right)d\mu \tag{24}$$

which implies that $h_1'(a)>h_2'(a)$ since the right hand side is positive for $n>1$ and for $\frac{1}{2}<n<1$ ($h_2(\mu)<h_1(\mu)$ within the interval (a,z) and $A<0$). In fact this argument can be made with $a=0$ where the two solutions cannot cross each other or meet at the origin. Observe that in our arguments if $n>1$ then the second term and the integral in (23) are finite, the same can be said for $\frac{1}{2}<n<1$ (but not for $0<n<\frac{1}{2}$) due to the asymptotic behavior discussed earlier. Thus we have established:

Theorem 1 There is a unique solution to (14) subject to (15) for $n>\frac{1}{2}$, $A<0$ and $B=0$.

Remark For $n=1$, notice that equation (23) is replaced by

$$(h'(z))^2-(h'(a))^2=2A\left(z\ln(h(z))-a\ln(h(a))-\int_a^z\ln(h(\mu))\,d\mu\right)$$

Using similar arguments to the ones above, the same uniqueness result can be established for the Newtonian case where $n=1$.

Now, for $A>0$ and $n<\frac{1}{2}$, a solution does not exist due to the following argument. That is, recall from our analysis that for $A<0, n<\frac{1}{2}$, solutions exhibit infinite first and second order derivatives of $h(z)$. If a solution approaching the origin exists for $A>0$ its second derivative should be larger, or "more infinite", than was the case for $A<0$ (in absolute values) since now the fact that $h''(z)>0$ implies that (from (14) above, $B=0$) at the same z (z close to 0) $h(z)$ is smaller for any solution extending to the origin with $A>0$. To make this argument more precise, consider a solution to equation (14) with $B=0$, that satisfies the (initial) condition $h(z)=h_0, h'(z)=h_0'$ with $z>0, h_0>0, h_0'>0$, which obviously is an arbitrary solution. This solution cannot be extended back to reach the origin for the following reason: Since $h''(\mu)>0$ we must have that $h(\mu)<k\mu$ for all $\mu\in(0,z)$ and for some $k>0$. Now the first term in (23) is finite, the second term should approach infinity as $a\to0^+$ (geometrically these two terms in fact yield the area of the rectangle $(z-a)h^{1-\frac{1}{n}}(z)$ minus the area of the rectangle $a\left(h^{1-\frac{1}{n}}(a)-h^{1-\frac{1}{n}}(z)\right)$ which results in a finite answer or $-\infty$ (it is finite for n= $\frac{1}{2}$, but it is $-\infty$ for n $>\frac{1}{2}$ since $h(z)<kz$) and the last term (the integral) should approach infinity as $a\to0^+$. With $A>0$ the right hand side then approaches positive infinity when $a\to0^+$ but $h'(z)=h_0{}'$ is finite which is a contradiction and it implies that $h'(a)$ $(a)=0$ for some $a>0$ and hence solutions $z\mapsto h(z)$ will actually turn away from the origin and be directed into larger values of $h(z)>0$ as $z\to0^+$. This establishes that

Theorem 2 Equation (14) has no solution in the first quadrant ($z>0, h(z)>0$) that converges to (0,0) for $n<\frac{1}{2}$,$A>0$ and $B=0$.

Corollary 3 Equation (14) has no solution in the first quadrant ($z>0, h(z)>0$) that converges to (0,0) for n< $\frac{1}{2}$,$A>0$ and $B>0$.

3.2. The case B≠0

For $B\neq0$, let $h(z)$ be represented by $h(z)\approx kz^p$ for z close to 0 ($z>0$), and for some parameters k and p. We have:

$$p=\frac{3n}{n+1},\quad k^{1+\frac{1}{n}}=\frac{A+pB}{p(p-1)} \tag{25}$$

This does yield a positive value of k (and positive $h(z)$) for $0<n<\frac{1}{2}$ if $A+pB<0$. Notice that $h(z)=kz^p$ in this case is an exact (positive) solution that satisfies (14), and in fact it also satisfies (15) if

$$\begin{aligned} h'(\epsilon)=kp\epsilon^{p-1}&=\left(\frac{A+pB}{p(p-1)}\right)^{\frac{n}{n+1}}p\epsilon^{p-1}\\ &=\alpha f(0)-m\,\epsilon^2/f''(0). \end{aligned} \tag{26}$$

This exhibits the existence of solutions for n $<\frac{1}{2}$, therefore (by continuity with respect to initial conditions):

Theorem 4 A solution to (14) subject to (15) exists for 0<n< $\frac{1}{2}$ and $A+pB<0$ where $p=\frac{3n}{n+1}$.

Observe that for the given range of n we have that $0<p<1$, so that the corresponding solutions have infinite derivatives at the origin. On the other hand, the asymptotic behavior of solutions follows equations (17), (18) and (19) but with the new value of k. On the other hand, for n>1/2, let $h(z)\approx kz+\kappa z^p$ for z close to 0 ($z>0$). This yields a value of $p=3-\frac{1}{n}$ $(p>1)$ and the following equation relates $k>0$ to κ:

$$\kappa p(p-1)k^{\frac{1}{n}}=A+B. \tag{27}$$

This holds whether $A+B$ is positive or negative, and the asymptotic behavior in this case follows equations (20), (21) and (22) with the new value of k.

3.2.1. A non-uniqueness result

Observe that if $A+B=0$ then (27) implies that $\kappa=0$, and then in fact $h(z)=kz$ is a solution to (14) for all values of n (not only for n $>\frac{1}{2}$), where a value of $k=\alpha f(0)-\frac{m\epsilon^2}{f''(0)}>0$ yields a solution to (14) that satisfies (15). In fact, since $p=\frac{3n}{n+1}<1$ in (25) above for 0<n< $\frac{1}{2}$, we have another exact solution, namely $h(z)=kz^p$ with k,p given in (25). This leads to the following proposition.

Proposition 5 Solutions to (14) subject to (15) are not unique for $0<n<\frac{1}{2}$, $A<0$ and $A+B=0$.

Observe that the conditions $A<0$ and $A+B=0$ imply the $A+pB<0$ since $p<1$. Even though the result given above requires a few conditions, it does exhibit a peculiar non-uniqueness result for this kind of problem. In fact this result maybe more general. The (Crocco variable) solution $h(z)=\left(\alpha f(0)-\frac{m\epsilon^2}{f''(0)}\right)z$ results in the following solution to the original problem for $f'(\eta)$ and $f(\eta)$:

$$f'(\eta)=\left(\left(\frac{1}{n}-1\right)\left(\alpha f(0)-\frac{m\epsilon^2}{f''(0)}\right)^{\frac{1}{n}}\eta+\epsilon^{1-\frac{1}{n}}\right)^{\frac{n}{n-1}}$$

$$f(\eta)=f(0)+n\frac{\left(\left(\frac{1}{n}-1\right)\left(\alpha f(0)-\frac{m\epsilon^2}{f''(0)}\right)^{\frac{1}{n}}\eta+\epsilon^{1-\frac{1}{n}}\right)^{\frac{2n-1}{n-1}}-\epsilon^{\frac{2n-1}{n}}}{(2n-1)\left(\alpha f(0)-\frac{m\epsilon^2}{f''(0)}\right)^{\frac{1}{n}}}$$

It can be directly checked that this is a solution to (10-11) provided $\alpha f(0)-\frac{m\epsilon^2}{f''(0)}=\frac{\left(-f''(0)\right)^n}{\epsilon}$, a condition that can also be obtained from the Crocco variables with $h(z)$ as given above where:

$$h(\epsilon)=\left(\alpha f(0)-\frac{m\epsilon^2}{f''(0)}\right)\epsilon=\left(-f''(0)\right)^n.$$

On the other hand, the other (Crocco variable) solution $h(z)=kz^p$ results in the solution:

$$f'(\eta)=\left(\frac{2-n}{n+1}\sqrt[n]{k}\,\eta+\epsilon^{\frac{n-2}{n+1}}\right)^{\frac{n+1}{n-2}}$$

$$f(\eta)=\frac{n+1}{(1-2n)\sqrt[n]{k}}\left[\left(\frac{2-n}{n+1}\sqrt[n]{k}\,\eta+\epsilon^{\frac{n-2}{n+1}}\right)^{\frac{2n-1}{n-2}}-\epsilon^{\frac{2n-1}{n+1}}\right]+f(0),$$

and where $-\left(h(\epsilon)\right)^{\frac{1}{n}}=f''(0)=-\sqrt[n]{k}\,\epsilon^{\left(\frac{3}{n+1}\right)}$.

Observe that both solutions satisfy the boundary conditions: $f'(0)=\epsilon$ and $f'(\infty)=0$, and then choosing the same $f(0)=a$, results in two different values for $f''(0)$ namely: $f''(0)=\frac{p^{\frac{1}{n}}(A+pB)}{m(p-1)}$ for the first and $f''(0)=\frac{(A+pB)}{m(p-1)}$ for the second. This establishes non-uniqueness of solutions for problem (10-11).

3.2.2. Exact solutions and asymptotic behavior for $B\neq0, n=\frac{1}{2}$

For $B\neq0, n=\frac{1}{2}$, assume an asymptotic solution of the form $h(z) \approx kz(\ln z)^p$. Substituting back into (14) yields $p = \frac{1}{3}, k = \left(\frac{A+B}{p}\right)^{1/3} = (3(A+B))^{1/3}$ which for negative A+B does yield a positive solution ($k< 0, z >0$, $\ln z<0$) for $h(z)$. This in turn yields the asymptotic behavior, $f' \to (k^2\eta + K)^{-1}$ In fact, $h(z) = kz\sqrt[3]{\ln z}$ an exact solution to (14) if $A = -\frac{5B}{2}, \alpha = -3m, k = \sqrt[3]{18m}$. It satisfies (15) if $\frac{k}{3}(3(\ln\epsilon)^{1/3} + (\ln\epsilon)^{-\frac{2}{3}}) = \alpha f(0) - \frac{m\epsilon^2}{f''(0)}$ and where the additional condition $k\epsilon(\ln\epsilon)^{1/3} = \sqrt{-f''(0)}$ ensures that the resulting (implicit) solution: $\int_\epsilon^{f'(\eta)} \frac{dz}{z^2(\ln z)^{\frac{2}{3}}} = -81m^2\eta$ satisfies (10-11) for $\epsilon <1$.

3.2.3. Proof of existence and uniqueness for $B \neq 0$

For $B \neq 0$ integrating (14) yields:

$$h'(z) - h'(a) = -nB\left(z^2 h^{-\frac{1}{n}}(z) - a^2 h^{-\frac{1}{n}}(a)\right) + (A+2nB)\int_a^z \mu h^{-\frac{1}{n}}(\mu)d\mu \qquad (28)$$

Observe that -nB=-m and A+2nB=-α. Equation (28) can be used as was done with equation (23), and with similar arguments to establish uniqueness of solutions that converge to (0,0) for n>1/2 and A+2nB<0:

Suppose that h_1 is a solution through the origin. Take another solution h_2 to (14) with initial conditions $h_2(z)=h_1(z)$ but a larger $h'_2(z)$, i.e., $h'_2(z) > h'_1(z)$. The two solutions may not intersect at any point $a < z$, since if they did we should have $h'_1(a) > h'_2(a)$ (the condition $h'_2(z) > h'_1(z)$ implies that $h_2<h_1$ within the interval (a,z) so that geometrically h_2 is under h_1). Now, the first term on the right hand side of (28) would be the same for h_1 and h_2, and therefore we have:

$$h_2'(a) - h_1'(a) = h_2'(z) - h_1'(z) - (A+2nB)\int_a^z \mu\left(h_2^{-\frac{1}{n}}(\mu) - h_1^{-\frac{1}{n}}(\mu)\right)d\mu \qquad (29)$$

which implies that $h'_2(a) > h'_1(a)$ since the right-hand side is positive for $n >1/2$ ($h_2(\mu) < h_1(\mu)$ within the interval (a,z) and $A+2nB<0$). This is a contradiction, so therefore we have the following crucial uniqueness result stated in Theorem 6 (note that $A+2nB=-\alpha<0$ in many applied problems).

Theorem 6 There is a unique solution to (14) subject to (15) for $n >1/2$ and $A+2nB<0$.

4. Numerical solutions

The established results were confirmed by numerical solutions where some figures are included as illustrations. Figure 1 is an illustration of Theorem 2 where a solution in the first quadrant does not exist for n $< \frac{1}{2}$, B=0. The figure shows different values of $h'(1)$, however note that higher values of $h'(1)$ take the values of $h(0)$ down (curves marked with 'o', '-', and '·') but never reaching the boundary condition $h(0)=0$, as they eventually take a sharp turn for higher values of h(0) (curves marked with '-· ', and '*'). Figure 2 is an illustration of Theorem 6 in which a solution does exist satisfying the required boundary conditions for this singular problem at z=0.

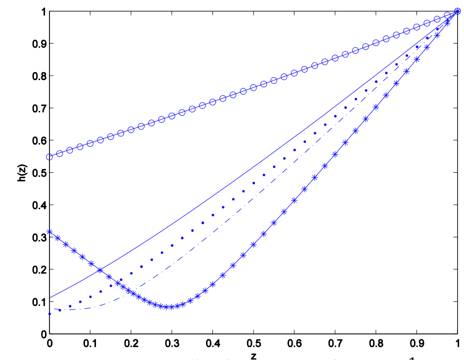

Fig. 1. No solution case for $n< \frac{1}{2}$;B=0 (Theorem 2). [n=0.4,A=0.1,B=0. I.C.'s h(1)=1, $h'(1)$=0.5,1,1.1,1.2,1.5].

Figure 3 is an illustration of Theorem 4. It can be observed that a solution exists here, yet numerical results are difficult to illustrate exactly since the derivatives are infinite around x=0 ($y\sim x^{0.5}$). In fact, the “bifurcation” here between the two curves results from a slight change of the initial condition $h(1)$ at z=1, this sharpness can be attributed to the small value of n=0.2. Advanced integrators will be needed to extend the numerical solution to the singularity at (0,0).

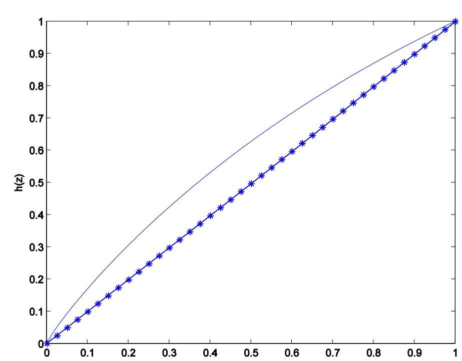

Fig. 2. Existence of solutions for $n >1/2$ and $A+2nB<0$ (Theorem 6). [$h(1)=1, m=0.6, \alpha=1$: pseudo-plastic fluid (solid line) $n=0.5086, h'(1)=0.6$ and dilatant fluid (solid line-*) $n=4$, $h'(1)=1.018$].

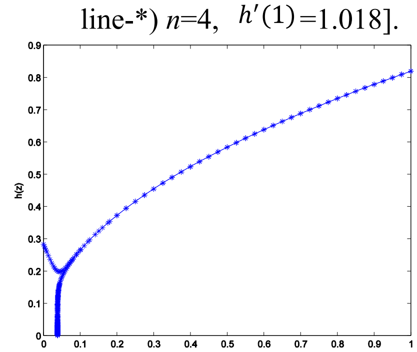

Fig. 3. Solutions having infinite derivatives (Theorem 4). [$n=0.2, A=0.9, B=-2$, $h'(1)=0.4$].

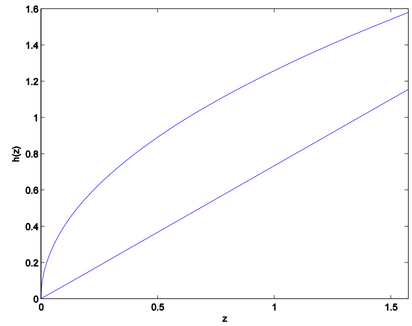

Fig. 4. Non-uniqueness of solutions case (Proposition 5). [n=0.2 (n<0.5),A=-2,B= 2, $h'\left(5\cdot(2)^{-\frac{5}{3}}\right)=5\cdot(2)^{-\frac{7}{3}}$].

Finally, Figure 4 is an illustration of the non-uniqueness result exhibited in Proposition 5. Observe that the required boundary conditions here are satisfied as $h(0)=0$ and we have the same $h'(\epsilon)$ for both solutions. The difference in $h(\epsilon)$ in this case corresponds to a difference in $f''(0)$ while the boundary conditions for $f(\eta)$ and $f'(\eta)$ are also satisfied.

Remark The values of n had to be chosen in the range $0< n < \frac{1}{2}$ for Figures 1, 3, and 4 to be consistent with the obtained results, which corresponds to pseudo-plastic fluid. For Figure 2, since $n > \frac{1}{2}$, illustrations were made for pseudo-plastic and dilatant fluids.

5. Conclusions

The asymptotic behavior of similarity solutions to a boundary layer flow problem characterized by a power-law rheology has been examined for different values of the power-law index n >0. Exact solutions to the problem were exhibited in some cases. Conditions were determined where existence and uniqueness of the problem can be established. Instances were found where uniqueness fails. Non-uniqueness of solutions has been exhibited in those instances by explicit multiple solutions that satisfy the governing differential equation and its boundary conditions. The study utilized a generalized Crocco variable transformation which enabled us to add a new understanding to the problem and establish new results.

References

Acrivos, A., Shah, M. J., & Petersen, E. E. (1960). Momentum and heat transfer in laminar boundary-layer flow of non-Newtonian fluids past external surfaces. American Institute of Chemical Engineers (AIChE) Journal, **6** (2): 312-317.

Al-Ashhab, S. (2015). A curvature-unified equation for a non-Newtonian power-law fluid flow. International Journal of Advances in Applied Mathematics and Mechanics **2**(3): 72-77.

Astarita, G., & Marrucci, G. (1974). Principles of Non–Newtonian Fluid Mechanics. McGraw-Hill Press, New York.

Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zeitschrift für angewandte Mathematik und Physik, **56**: 1-37.

Bohme, G. (1987). Non–Newtonian Fluid Mechanics. North-Holland Series in Applied Mathematics and Mechanics, Amsterdam.

Denier, J. P., & Dabrowski P. (2004). On the boundary-layer equations for power–law fluids. Proceedings of the Royal Society of London A, **460**: 3143-3158.

Ece, M. C., & Büyük, E. (2002). Similarity solutions for free convection to power-law fluids from a heated vertical plate. Applied Mathematics Letters, **15**: 1-5.

Guedda, M. (2009). Boundary-layer equations for a power-law shear driven flow over a plane surface of non-Newtonian fluids. Acta Mechanica, **202** (1): 205-211.

Guedda, M., & Hammouch, Z. (2008). Similarity flow solutions of a non-Newtonian power-law fluid flow. International Journal of Nonlinear Science **6** (3): 255-264

Howell, T. G., Jeng, D. R., & De Witt, K. J. (1997). Momentum and heat transfer on a continuous moving surface in a power-law fluid. International Journal of Heat Mass Transfer, **40** (8): 1853-1861.

Marin, M., & Lupu,M. (1998). On harmonic vibrations in thermoelasticity of micropolar bodies. Journal of Vibration and Control, **4**: 507-518.

Nachman, A., & Taliaferro, S. (1979). Mass transfer into boundary-layers for power-law fluids. Proceedings of the Royal Society of London A, **365**: 313-326.

Pakdemirli, M. (1994). Similarity analysis of boundary layer equations of a class of non-Newtonian fluids. International Journal of Non-Linear Mechanics, **29** (2): 187-196.

Rahman, S., Hayat, T., & Alsulami, H. (2017). Some new regularity criterion for MHD three-dimensional flow. Kuwait Journal of Science 44(4): 28-48.

Schlichting, H. (1979). Boundary layer theory. McGraw-Hill Press, New York.

Wei, D., & Al-Ashhab, S. (2014). Similarity Solutions for a non-Newtonian power-law fluid flow. Applied Mathematics and Mechanics (English Edition), **35** (9): 1155-1166.

Submitted :13/11/2017
Revised :28/01/2018
Acceptance :31/01/2018

السلوك المتقارب ووجود حلول تشابه لمسألة
تدفق ذات طبقة حدودية

سامر الأشهب
قسم الرياضيات، جامعة الإمام محمد بن سعود الإسلامية،
ص.ب. 90950، الرياض 11623، المملكة العربية السعودية

الملخص

يتم إعتبار مسألة تدفق بوجود طبقة حدودية لمائع غير نيوتوني خاضع لقانون القوة (والذي نفترض أنه غير قابل للضغط). يتم النظر في وجود ووحدانية حلول التشابه لجميع القيم الموجبة لمؤشر قانون القوة. هذا ويتم تحديد الشروط (قيم المؤشر وقيم ثوابت مختلفة في المسألة) حيث يتحقق كل من وجود ووحدانية الحلول وحيث لا يتحققان. ويتم إيجاد وعرض الحلول الدقيقة في بعض الحالات. يتم كذلك تحديد السلوك المتقارب للحلول وذلك لجميع القيم الموجبة لمؤشر قانون القوة للمائع غير النيوتوني.