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Hosoya polynomial of the subdivided join

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Abstract

The Hosoya polynomials of diameter 1 and diameter 2 graphs are known. We extend the concept of a vertex join of a graph to a subdivided join. Then we give the formula of the Hosoya polynomial of a subdivided join of a complete graph and the formula of the Hosoya polynomial of a subdivided join of diameter 2 graphs.

Keywords: Diameter; distance; Hosoya polynomial; subdivided join; Wiener index

1. Introduction

There is a well-developed relationship between chemistry and graph theory, such that in chemical graphs, the vertices of a graph correspond to the atoms of the molecule, and the edges represent the chemical bonds. We need the concept of distance in graph theory to be able to define the Wiener index, which is a tool for obtaining the boiling points of alkanes (Wiener, 1947). Sagan *et al*. (1996) studied the Wiener polynomial of a graph as a generating function in q and revealed that the derivative of the Wiener polynomial was the q-analog of the Wiener index of a graph. The Wiener polynomial is a counting polynomial with applications to mathematical and physical chemistry. The structure of molecules and their branching patterns are studied through their molecular graphs, which are simple connected graphs. The Wiener index correlates with chemical properties of organic compounds by quantifying the branching pattern of a molecule through its molecular graph.

Let G be a connected graph and let $V(G)=\{u_1, u_2, \dots, u_n\}$ be the vertex set of G, the Wiener index of a graph G, is given by:

$$W(G) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} d(u_i, u_j). \tag{1}$$

In addition, the hyper-Wiener index of a graph G is

$$WW(G) = W(G) + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} d^2(u_i, u_j). \tag{2}$$

Hosoya (1988) independently studied a generating function regarding distance distributing, or the Hosoya polynomial of a graph. This turned out to be equivalent to the Wiener polynomial. Hence, in the literature, we have two names for the same polynomial: Wiener and Hosoya. In this paper, we use the Hosoya polynomial. Since its inception, the study of Hosoya polynomials has been adapted in a variety of ways by chemists and mathematicians. One direction of study is finding explicit expressions for the Hosoya polynomials of certain classes of graphs (see Ali *et al*., 2011; Caporossi *et al*., 1999; Deng, 2012; Deutsch, 2014; Gutman *et al*., 2001). Another direction is to find the Hosoya polynomial of certain graph operation (see Deutsch, 2013). A new direction is solving for the roots of Hosoya polynomials, as done by Kumar *et al*. (2016) and Reyhani *et al*. (2013). Other investigators studied new topological indices, such as the Randić index (Ali *et al*., 2017), an augmented Zagreb index (Ali *et al*., 2016), and the F-index (Abdo *et al*., 2017). The diameter D of a graph G is given by $D := max_{u,v\in V(G)}\{d(u, v)\}$. The Hosoya polynomial is defined for a connected graph G as

$$H(G,z) = \sum_{w=1}^{D} d(G,w)z^w, \tag{3}$$

where $d(G,w)\geq 1$ is the number of vertex pairs at distance w. The Hosoya polynomial has applications to two important topological indices: the Wiener and hyper-Wiener.

Thus, the Wiener index is given by the first derivative of the Hosoya polynomial $H(G, z)$ shown in Equation 4 at $z=1$, That is

$$\frac{d}{dz}H(G,z)|_{z=1} = W(G) = \sum_{w=1}^{D} wd(G,w). \tag{4}$$

Estrada *et al*. 1998 showed that the hyper-Wiener index is given by half of the second derivative of the Hosoya polynomial as $zH(G, z)$ at $z=1$, that is

$$\frac{d^2}{dz^2}zH(G,z)|_{z=1} = WW(G) = W(G) + \sum_{w=1}^{D} w^2 d(G,w). \tag{5}$$

It is of special interest to note that unlike many graph polynomials, the Hosoya polynomials of even and odd cycles have different formulae. Similarly, non-isomorphic trees of the same size have different formulae of the Hosoya polynomial.

Theorem 1.1 (Sagan, 1996) The Hosoya polynomial of
(i) an even cycle C_{2n} is

$$H(C_{2n}) = 2n\sum_{i=1}^{n-1} z^i + nz^n. \qquad (6)$$

(ii) an odd cycle C_{2n} is

$$H(C_{2n+1}) = (2n+1)\sum_{i=1}^{n} z^i. \qquad (7)$$

(iii) a path P_n is

$$H(P_n) = \sum_{i=1}^{n-1}(n-i)z^i. \qquad (8)$$

In this paper, we extend the concept of a vertex join of a graph to a subdivided join. We give some properties of subdivided join of a graph. Then we give a formula for the Hosoya polynomial of a subdivided join of a complete graph. Finally, the Hosoya polynomial of a subdivided join of a graph with diameter 2 is obtained.

2. Subdivided join and diameter 2 graphs

In this section, we give a brief discussion on diameter 2 graphs and subdivided joins of these graphs. Let $d(u,v)$ denote the minimum distance between any two vertices u and v in a graph G. A graph with maximum distance equal to k between pairs of vertices is said to be a diameter k graph. In this paper we shall consider graphs with diameter 1 and diameter 2.

Let G be a graph with vertex set $V(G)=\{u_1, u_2, \dots, u_n\}$, edge set $E(G)$ and let a vertex w be a vertex not in $V(G)$. A vertex join of a graph G, is the graph denoted by $\widehat{G}$ with vertex set $V(\widehat{G})=\{u_1, u_2, \dots, u_n\}\cup\{w\}$ and edge set $E(\widehat{G})= E(G)\cup\{\{u_1,w\},\{u_2,w\}, \dots, \{u_n,w\}\}$. To ease notation, an edge $e\in\{\{u_1,w\},\{u_2,w\}, \dots, \{u_n,w\}\}$ is called a join edge and vertex w is called a join vertex. If each join edge, $\{u_i,w\}$, of a vertex join, $\widehat{G}$,is replaced by a path $P_{(q+1)}$ the resulting graph is called a subdivided join of a graph G denoted by $\hat{G}_q$. A path $P_{(q+1)}$ in a subdivided join which replaced a join edge of G is called a join path. We denote a join path by $\hat{P}$ and to ease notation, we label $\hat{P}_i$ as a join path from vertex u_i to vertex w. It is clear from definitions that, for every pair of vertices u_i and u_j in G we have classes of cycles consisting of the shortest path between u_i and u_j and the two join paths $\hat{P}_i$ and $\hat{P}_j$. We define a transversal to be a set of cycles such that no cycles belong to the same class. The element of the transversal are called the join cycles of $\hat{G}_q$. We denote a join cycle with one edge of G by $\hat{C}_1$ and a join cycle with two edges of G by $\hat{C}_2$. We now state some properties of a subdivided join which are useful in some proofs in this paper.

Lemma 2.1 Let G be a diameter 2 graph of order n and size m. Let $\hat{G}_q$ be the subdivided join of G. Then
(i) the order of $\hat{G}_q$ is equal to $qn+1$.
(ii) the number of join paths in $\hat{G}_q$ is equal to n.
(iii) the total number of join cycles, $\hat{C}_1$ in $\hat{G}_q$ is m.
(iv) the total number of join cycles, $\hat{C}_2$ in $\hat{G}_q$ is $\binom{n}{2}-m$.
(v) the total number of join cycles in $\hat{G}_q$ is equal to $\binom{n}{2}$.

A vertex cycle cover of a graph G is a set of cycles which are subgraphs of G and contain all vertices of G. Since in this paper we are discussing vertex pairs, we extend the vertex cycle cover terminology to vertex pair cycle cover which is a set of cycles that are subgraphs of G and contain all vertex pairs of G.

Lemma 2.2 Let G be a graph of order n and size m with diameter at most 2. Let $\hat{G}_q$ be the subdivided join of G. Then every vertex pair in $\hat{G}_q$ belong to some join cycle in $\hat{G}_q$, that is $\hat{G}_q$ has a vertex pair join cycle cover.

Proof. There are four cases to be considered:

Case 1. A pair of vertices $v_i,v_j \in V(G)$ are either at distance 1 or 2 from each other since G is a diameter at most 2 graph. By definition, each of these pairs of vertices belong to a join cycle.

Case 2. A pair of vertices v_j, $v_t \in V(\hat{P}_j)$ such that both vertices are in the same join path $\hat{P}_j$ *and* $v_t \in V(G)$. By definition of a join cycle each of these pairs of vertices belong to a join cycle.

Case 3. A pair of vertices v_i ,v_t such that $v_i \in V(G)$, $v_t \in V(\hat{P}_j)$ and $w \neq v_t \neq v_j$. By definition the join path $\hat{P}_j$ joins vertex w and vertex v_j $V(G)$ in the subdivided join. Thus $v_j \in V(\hat{P}_j)$. But by part (i) the pair v_i , v_j is on the join cycle. Since $v_t \in V(\hat{P}_j)$, then the pair v_i ,v_t is on this join cycle.

Case 4. A pair of vertices v_r,v_t such that v_r,v_t is not in $V(G)$, $v_t \neq w \neq v_r$, $v_r \in V(\hat{P}_i)$ and $v_t \in V(\hat{P}_j)$. Let v_i,v_j $V(G)$, such that v_i $V(\hat{P}_i)$ and v_j $V(\hat{P}_j)$. If $d(v_i,v_j)=1$, then v_r,v_t $\hat{C}_1$, while if $d(v_i,v_j)=2$, then v_r,v_t $\hat{C}_2$. Thus every vertex pair in Gq belong to some join cycle in $\hat{G}_q$, hence $\hat{G}_q$ has a vertex pair join cycle cover.

Lemma 2.3 Let G be a graph of order n and size m with diameter at most 2. Let $\hat{G}_q$ be the subdivided join of G. Then the shortest path between any pair of vertices in $\hat{G}_q$ lie on some join cycle of $\hat{G}_q$.

Proof. There are four cases of pairs of vertices on a join cycle to be considered:

Case 1. A pair of vertices v_i,$v_j\in V(G)$. It is clear that the construction of $\hat{G}q$ does not affect the shortest path in G. But v_i ,v_j are on the join cycle consisting of the shortest path between v_i and v_j and join paths $\hat{P}_i$ and $\hat{P}_j$.

Case 2. A pair of vertices v_s, $v_t \in V(\hat{P}_j)$, that is both v_s,

v_t are lying on the same join path $\hat{P}_j$. Let the path from v_s to v_t on $\hat{P}_j$ be called P_{st}. Assume there is another path from v_s to v_t shorter than P_{st}, say path $\acute{P}_{st}$. Then it is clear that $P_{st} \cup \acute{P}_{st}$ is a cycle. By construction of $\hat{G}_q$, the only possibility is that $P_{st} \cup P'_{st}$ is some cycle consisting of some join path, say $\hat{P}_k$ and some edges in G. Thus $\acute{P}_{st}$ is a path consisting of join path $\hat{P}_k$ and at least one more edge. Thus $|E(\acute{P}_{st})| > q+1$. But $E(P_{st}) \subseteq E(\acute{P}_j)$ where $|E(\hat{P}_k)| = q = |E(\acute{P}_j)|$, therefore $|E(P_{st'})| \leq q$. Hence P_{st} is the shortest path between v_s to v_t and is on some join cycle.

Case 3. A pair of vertices v_i, v_t such that $v_i \in V(G)$, $v_t \in V(\hat{P}_j)$ and $w \neq v_t \neq v_j$. It is clear by construction that the only paths from v_i to v_t are the join path cycle consisting of a path from v_i to v_t, $\hat{P}_i$ and $\hat{P}_j$. Hence the shortest path is on the same join cycle.

Case 4. A pair of vertices v_r, v_t such that v_r, v_t are not in $V(G)$, $v_t \neq w \neq v_r$, $v_r \in V(\hat{P}_i)$ and $v_t \in V(\hat{P}_j)$. Let $v \in V(G)$, such that $v_i \in V(\hat{P}_i)$ and $v_j \in V(\hat{P}_j)$. Let the join cycle consisting of the shortest path from v_i to v_j $\cup \hat{P}_i \cup \hat{P}_j$ be $\hat{C}_x$ and let the shortest path from v_r to v_t on $\hat{C}_x$ be $\acute{P}_{rt}$. Assume there is another path in $\hat{G}_q$ not on the join cycle $\hat{C}_x$ which is shorter than $\acute{P}_{rt}$, say path P_{rt}. Then $\acute{P}_{rt} \cup P_{rt}$ is a cycle consisting of the paths $\hat{P}_i$ and $\hat{P}_j$ and some path from v_i to v_j say path P'_{ij} in G. But the path in G from v_i to v_j in the join cycle $\hat{C}_x$, is the shortest path by definition of $\hat{C}_x$. Thus the cycle $P'_{rt} \cup P_{rt}$ is longer than the join cycle $\hat{C}_x$. Hence P'_{rt} is the shortest path between v_r and v_t and is on a join cycle.

Corollary 2.4 Let G be a graph of order n and size m with diameter at most 2. Let $\hat{G}_q$ be the subdivided join of G. Then the sum of the Hosoya polynomials of all the join cycles in $\hat{G}_q$ covers the distance of any pair of vertices in $\hat{G}_q$ at least once.

3. The Hosoya polynomial of subdivided join

In this section, we give the Hosoya polynomial of the subdivided join of a diameter 1 graph and a diameter 2 graph. Finally we state the Wiener indices of the subdivided join of a diameter 1 graph and a diameter 2 graph.

Note that all diameter 1 graphs are complete graphs, thus we find the Hosoya polynomial of the subdivided join of a complete graph.

Lemma 3.1 Let K_n be a complete graph of order n, and let $\hat{G}_q$ be the subdivided join of K_n. Then there are $\binom{n}{2}$ join cycles of the form $\hat{C}_1$ in $\hat{G}_q$.

Theorem 3.2 Let K_n be a complete graph of order n, and let $\hat{G}_q$ be the subdivided join of K_n. Then the Hosoya polynomial of $\hat{G}_q$ is

$$H(\hat{G}_q, z) = \binom{n}{2} H(\hat{C}_1, z) - n(n-2) H(\hat{P}, z). \quad (9)$$

Proof. By Lemma 2.2, we know that every vertex pair in $\hat{G}_q$ belong to some join cycle $\hat{C}_1$. By Corollary 2.4, the sum of the Hosoya polynomials of all the join cycles in $\hat{G}_q$ covers the distance of any pair of vertices in $\hat{G}_q$ at least once. We use the principle of inclusion-exclusion, Theorem 1.2. Thus the Hosoya polynomial of $\hat{G}_q$ can be found in terms of join cycles and join paths.

By Lemma 3.1 the number of join cycles of size $2q+1$ in By Lemma 3.1 the number of join cycles of size $2q+1$ in $\hat{G}_q$ is $\binom{n}{2}$. Therefore the sum of the Hosoya polynomials of all the join cycles of $\hat{G}_q$ is $\binom{n}{2} H(\hat{C}_1, z)$. Since u_i is paired with all the other *(n-1)* vertices, this implies that the join path $\hat{P}_i$ will appear in *(n-1)* join cycles. Thus the Hosoya polynomial of each join path $\hat{P}_i$ is in the Hosoya polynomials of *(n-1)* join cycles. But we need the Hosoya polynomial of $\hat{P}_i$ to contribute once in the total sum, thus we remove the *(n-2)* repetitions. We do this to all the n join paths, to get the Hosoya polynomial of $\hat{G}_q$, is $\binom{n}{2}$. Therefore the sum of the Hosoya polynomials of all the join cycles of $\hat{G}_q$ is $\binom{n}{2} H(\hat{C}_1, z)$. Since u_i is paired with all the other *(n-1)* vertices, this implies that the join path $\hat{P}_i$ will appear in *(n-1)* join cycles. Thus the Hosoya polynomial of each join path $\hat{P}_i$ is in the Hosoya polynomials of *(n-1)* join cycles. But we need the Hosoya polynomial of $\hat{P}_i$ to contribute once in the total sum, thus we remove the *(n-2)* repetitions. We do this to all the n join paths, to get the Hosoya polynomial of $\hat{G}_q$,

$$H(\hat{G}_q, z) = \binom{n}{2} H(\hat{C}_1, z) - n(n-2) H(\hat{P}, z). \quad (10)$$

We now compute the Hosoya polynomials of the subdivided join for a diameter 2 graph. The Hosoya polynomial of any graph with diameter 2 is known in the literature, see Kumar, 2016.

Proposition 3.3 (Kumar, 2016). Let G be a diameter 2 graph of order n and size m. Then the Hosoya polynomial of G is

$$H(G, z) = mz - \left[\binom{n}{2} - m\right] z^2. \quad (11)$$

Lemma 3.4 Let $G=P_3$ be a path on three vertices and let $\hat{G}_q$ be the subdivided join of P_3. Then the Hosoya polynomial of $\hat{G}_q$ is

$$H(\hat{G}_q, z) = 2H(\hat{C}_1, z) + \frac{1}{q+1} H(\hat{C}_2, z) - 3H(\hat{P}, z). \quad (12)$$

Proof. Let $V(G)=\{u_1, u_2, u_3\}$ such that $d(u_1, u_2) = d(u_2, u_3) = 1$ and $d(u_1, u_3) = 2$. It is clear that in $\hat{G}_q$, there are three join cycles. Thus the contribution of the three join cycles to the Hosoya polynomial of $\hat{G}_q$ is less than or equal to $2H(\hat{C}_1, z) + H(\hat{C}_2, z)$. We now remove all the repeated pairs of vertices in the three join cycles. Each join path appears twice in the three join cycles, so we remove remove $H(\hat{P}, z)$ from $2H(\hat{C}_1,z)$ and 2 $H(\hat{P}, z)$ from $H(\hat{C}_2,z)$. The edge of G $\{u_1, u_2\}$ appears in 2 join cycles and so is the edge $\{u_2, u_3\}$. Hence we remove the term $2z$ from $H(\hat{C}_2,z)$.

Consider all the distances from u_2 to any vertex but w in the two join paths $\hat{P}_1$, $\hat{P}_3$ and in the join cycle $\hat{C}_2$. These distances have been included already in the 2 join cycles, hence we remove the term $2z^2+2z^3+...+2z^q$ from $H(\hat{C}_2,z)$.

We note that the shortest distance between u_2 and w is q, that is via the join path $\hat{P}_2$. However in the join cycle $\hat{C}_2$ the shortest distance between u_2 and w is q+1, which is more than q , thus we must also remove $z^{(q+1)}$ from $H(\hat{C}_2,z)$.

Recall that the Hosoya polynomial of an even cycle $H(C_{2n}) = 2n\sum_{i=1}^{n-1} z^i + nz^n$.

Now we note that

$$\sum_{i=1}^{q} 2z^i + z^{q+1} = \frac{q+1}{q+1}\left[\sum_{i=1}^{q} 2z^i + z^{q+1}\right] = \frac{1}{q+1}\left[\sum_{i=1}^{q} 2(q+1)z^i + (q+1)z^{q+1}\right] = \frac{H(\hat{C}_2,z)}{q+1}. \quad (13)$$

Then substituting Equation 10 we obtain

$$H(\hat{C}_2,z) - 2H(\hat{P},z) - \frac{1}{q+1}H(\hat{C}_2,z) \quad (14)$$

which simplifies to $\frac{1}{q+1}H(\hat{C}_2,z) - 2H(\hat{P},z)$.

We combine all the results as follows

$$H(\hat{G}_q,z) = 2H(\hat{C}_1,z) - H(\hat{P},z) + \frac{1}{q+1}H(\hat{C}_2,z) - 2H(\hat{P},z)$$

$$= 2H(\hat{C}_1,z) + \frac{1}{q+1}H(\hat{C}_2,z) - 3H(\hat{P},z). \quad (15)$$

Thus we get the required results.

Theorem 3.5 Let G be a diameter 2 graph of size m and order n. Then the Hosoya polynomial of the subdivided join, $\hat{G}_q$ is

$$H(\hat{G}_q,z) = mH(\hat{C}_1,z) - \frac{q}{q+1}\left(\binom{n}{2} - m\right)H(\hat{C}_2,z) - n(n-2)H(\hat{P},z). \quad (16)$$

Proof. By Lemma 2.2 , we know that every vertex pair in $\hat{G}_q$ belong to some join cycle $\hat{C}_1$. By Corollary 2.4, the sum of the Hosoya polynomials of all the join cycles in $\hat{G}_q$ covers the distance of any pair of vertices in $\hat{G}_q$ at least once. We use the principle of inclusion-exclusion, Theorem 1.2. Thus the Hosoya polynomial of $\hat{G}_q$ can be found in terms of join cycles and join paths. By Lemma 2.1 there are m join cycles $\hat{C}_1$ 1 and $\binom{n}{2}$-m join cycles $\hat{C}_2$ in $\hat{G}_q$. We have m join cycles of the form $\hat{C}_1$ in $\hat{G}_q$. Hence these join cycles contribute the term $mH(\hat{C}_1,z)$ to the Hosoya polynomial of $\hat{G}_q$. As in the proof of Lemma 3.4 we need to remove the Hosoya polynomials of some of the join paths $\hat{P}_i$ that are found in more than 1 join cycle. Let $V(G)=\{u_1, u_2, \ldots, u_n\}$ be the vertex set of G and let the degree of vertex u_i be $a_i \geq 1$ for $1 \leq i \leq n$. Recall that the hand-shake lemma states that the sum of all the degrees of vertices of a graph are equal to twice the number of edges of a graph. Therefore for graph G of size m we have $\sum_{i=1}^{n} a_i = 2m$.

Now we consider the degree of vertices of G in order to count the number of repetitions of join paths. It is clear by definition that each join path $\hat{P}_i$ with $u_i \in V(G)$ is paired with a_i join paths to form a_i join cycles of the form $\hat{C}_1$. Thus the Hosoya polynomials of each join path $\hat{P}_i$ are repeated a_i-1 times in the a_i join cycles of the form $\hat{C}_1$ We compute all the repetitions of $H(\hat{P}_i,z)$ from $mH(\hat{C}_1,z)$, as follows

$$\sum_{i=1}^{n}(a_i - 1) = \sum_{i=1}^{n} a_i - n = 2m - n. \quad (17)$$

Therefore the repeated number of $H(\hat{P}_i,z)$ is $[2m-n]$ $H(\hat{P},z)$. Thus we exclude $[2m-n]$ $H(\hat{P},z)$ from $mH(\hat{C}_1,z)$ to get $mH(\hat{C}_1,z)-[2m-n]$ $H(\hat{P},z)$. In $\hat{G}_q$ we consider the sum of the Hosoya polynomials of the $\binom{n}{2}$ - m join cycles of the form $\hat{C}_2$ which is $\left(\binom{n}{2} - m\right)H(\hat{C}_2,z)$. From Lemma 3.4 we know that there are $2H(\hat{P},z) + \frac{1}{q+1}H(\hat{C}_2,z)$, repetitions from $H(\hat{C}_2,z)$.

Thus from $\left(\binom{n}{2} - m\right)H(\hat{C}_2,z)$ the term $\left(\binom{n}{2} - m\right)\left(2H(\hat{P},z) + \frac{1}{q+1}H(\hat{C}_2,z)\right)+$ is a repetition. Therefore we exclude all repetitions from $\left(\binom{n}{2} - m\right)$ $H(\hat{C}_2,z)$ to get

$$\left(\binom{n}{2} - m\right)\left(H(\hat{C}_2,z) - 2H(\hat{P},z) - \frac{1}{q+1}H(\hat{C}_2,z)\right) = \left(\binom{n}{2} - m\right)\left(\frac{1}{q+1}H(\hat{C}_2,z) - 2H(\hat{P},z)\right). \quad (18)$$

We combine all the results and obtain the Hosoya polynomials of $\hat{G}_q$ as follows

$$H(\hat{G}_q,z) = mH(\hat{C}_1,z) - [2m-n]H(\hat{P},z) + \left(\binom{n}{2} - m\right)\left(\frac{1}{q+1}H(\hat{C}_2,z) - 2H(\hat{P},z)\right)$$

$$= mH(\hat{C}_1,z) + \left(\binom{n}{2} - m\right)\frac{1}{q+1}H(\hat{C}_2,z)$$

$$-(2m-n)H(\hat{P},z)$$

$$-\left(\binom{n}{2}-m\right)2H(\hat{P},z)$$

$$= mH(\hat{C}_1,z)+\left(\binom{n}{2}-m\right)\frac{1}{q+1}H(\hat{C}_2,z)-(2m-n)H(\hat{P},z)$$

$$-(n(n-1)-2m)H(\hat{P},z)$$

$$= mH(\hat{C}_1,z)+\left(\binom{n}{2}-m\right)\frac{1}{q+1}H(\hat{C}_2,z)-(n(n-1)-n)H(\hat{P},z)$$

$$= mH(\hat{C}_1,z)+\left(\binom{n}{2}-m\right)\frac{1}{q+1}H(\hat{C}_2,z)-n\,(n-2)H(\hat{P},z).$$

By applying Theorem 1.1, the Wiener indices of even cycles, odd cycles and paths are $W(C_{2q+2})=\frac{(2q+2)^3}{8}$, $W(C_{2q+1})=(2q+2)(2q+1)(2q)/8$ and $W(P_{2q+1})=\binom{q+2}{3}$ respectively. Hence applying Theorem 3.2 and Theorem 3.5 we get the following corollaries.

Corollary 3.6 Let $\hat{G}_q$ be the subdivided join of K_n. Then the Wiener index of $\hat{G}_q$ is

$$W(\hat{G}_q)=n\binom{q+1}{2}\left(\frac{4nq-n-2q+5}{6}\right). \qquad (19)$$

Corollary 3.7 Let $\hat{G}_q$ be the subdivided join of a diameter 2 graph, where G has size m and order n. Then the Wiener index of $\hat{G}_q$ is

$$W(\hat{G}_q)=\binom{q+1}{2}\left(n\frac{1-q+n+2qn}{3}-m\right). \qquad (20)$$

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كثيرات حدود هوسويا للاتصال المجزأ

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الملخص

كثيرات حدود هوسويا للقطر 1 والقطر 2 معروفة وشائعة. لقد قمنا بتوسعة مفهوم اتصال الرأس لرسم بياني إلى اتصال مجزأ. ثم قدمنا صيغة لكثيرات حدود هوسويا لاتصال مجزأ لرسم بياني كامل وكذلك صيغة كثيرات حدود هوسويا للاتصال المجزأ للقطر 2 من الرسم البياني.