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Characterizations of $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras

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ABSTRACT

In this paper, we define the concepts of $(\in, \in \vee q)$-interval valued fuzzy H-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras and investigated some of their related properties. Some characterizations of these generalized interval valued fuzzy H-ideal are derived.

Keywords: BCK-algebra; $(\in, \in \vee q)$-interval valued fuzzy H-ideal; $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal.

INTRODUCTION

The theory of BCK-algebras was introduced by (Imai & Iseki, 1966) in his pioneering paper. BCK/BCI-algebras are two important classes of logical algebras proposed by Iseki (1980); Iseki & Tanaka (1978); Meng (1994); Meng & Jun (1994). Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras.

The fundamental concept of fuzzy set was introduced by Zadeh (1965) provides a natural framework for generalizing some of the basic notions of algebra, for example, logic, set theory, groupoids, group theory, semigroup theory, ring theory, semiring theory etc. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. Since then, the concept of fuzzy set has been applied to many branches of mathematics. The concept was applied to the theory of groupoids and groups by Rosenfeld (1971), where he introduced the fuzzy subgroup of a group. Since then the literature of various algebraic structures has been fuzzified. In Xi (1991) applied fuzzy subsets in BCK-algebras and studied fuzzy BCK-algebras. He defined the concept of fuzzy ideals and fuzzy implicative ideal. The concept of fuzzy subset and various operations on it were first introduced by Zadeh (1965). Since then, fuzzy subsets have been applied to diverse fields. The study of fuzzy subsets and their application to mathematical contexts have reached to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. Since then, many mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures, including lattices and Boolean algebras. Khalid & Ahmad (1999) introduced fuzzy H-ideals in BCI-algebras and studied their properties. Zhan & Tan (2003) studied doubt fuzzy H-ideals in BCK-algebras. Zadeh (1975) made an extension of the concept of a fuzzy set by an interval valued fuzzy set. This interval valued fuzzy set is referred to as an interval valued fuzzy set. The theory was further enriched by many authors (Deschrijver, 2007; Gorzalczany, 1987; Jun, 2000, 2001; Ma *et al*., 2008, 2009). Zadeh (1975) also constructed a method of approximate inference using his interval valued fuzzy sets. Biswas (1994) defined interval valued fuzzy subgroups (i.e., interval valued fuzzy subgroups) of Rosenfeld's nature, and investigated some elementary properties. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in the paper of Bhakat & Das (1996) by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu & Liu (1980). In fact, the $(\in, \in \vee q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems with other algebraic structures. Bhakat (2000) introduced the concept of $(\in, \in \vee q)$-fuzzy normal, quasinormal and maximal subgroups and investigated related results. Jun (2004, 2005) defined the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI-algebra and investigated related results. In Davvaz (2006) applied this concept to $(\in, \in \vee q)$-fuzzy subnear-rings and obtained some useful results. Davvaz & Corsini (2007) redefined fuzzy Hv-submodule and many-valued implications. Zhan *et al*. (2008) discussed $(\in, \in \vee q)$-interval valued fuzzy hyperideals in hypernear-rings. Ma *et al*. (2008, 2009) studied $(\in, \in \vee q)$-interval valued fuzzy ideals in BCI-algebras.

In this paper, we show that an interval valued fuzzy set $\widetilde{\alpha}$ of a BCK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy H-ideal of X if and only if $\widetilde{\alpha}_{\widetilde{t}}\,(\neq \phi)$ is an H-ideal of X for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. We also prove that an interval valued fuzzy set $\widetilde{\alpha}$ of a BCK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal X if and only if $\widetilde{\alpha}_{\widetilde{t}}\,(\neq \phi)$ is an H-ideal of X for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.

PRELIMINARIES

In this section, we include some basic definitions and preliminary facts about BCK-algebras which are essential for our results. Throughout this paper, X always denotes a BCK-algebra, unless otherwise specified.

We give here only those concepts of BCK-algebras which are important for our treatment, and for details about the theory of these algebras we may refer to (Imai & Iseki, 1966; Iseki, 1980; Iseki & Tanaka, 1978).

Definition 2.1. (Meng & Jun, 1994) A BCK-algebra X is a general algebra $(X, *, 0)$ of type (2, 0) satisfying the following conditions:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$

(BCK-2) $(x * (x * y)) * y = 0$

(BCK-3) $x * x = 0$

(BCK-4) $0 * x = 0$

(BCK-5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for all $x, y, z \in X$.

We can define a partial order "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2. (Iseki & Tanaka, 1978; Meng & Jun, 1994; Mostafa, 1997). In any BCK-algebra X, the following are true:

(i) $(x * y) * z = (x * z) * y$

(ii) $(x * z) * (y * z) \leq x * y$

(iii) $(x * y) * (x * z) \leq z * y$

(iv) $x * 0 = x$

(v) $x * (x * (x * y)) = x * y$

(vi) $x * y \leq x$

for all $x, y, z \in X$.

Definition 2.3. (Meng, 1994) A non-empty subset I of a BCK-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all $x, y \in X$.

Definition 2.4. (Zhan & Tan, 2005) A non-empty subset I of a BCK-algebra X is called an H-ideal of X if it satisfies (I1) and (I3), where

(I1) $0 \in I$,

(I3) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$,

for all $x, y, z \in X$.

FUZZY H-IDEAL

In this section, we review some fuzzy logic concepts.

Definition 3.1. (Bhakat & Das, 1996; Mostafa, 1997; Zadeh, 1965)

(i) A fuzzy set α of X is a function $\alpha : X \to [0, 1]$.

(ii) For a fuzzy set α of a BCK-algebra X and $t \in (0, 1]$, the crisp set

$$\alpha_t = \{x \in X \mid \alpha(x) \geq t\}$$

is called the level subset of α.

Definition 3.2. (Mostafa, 1997) A fuzzy set α of a BCK-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

(F1) $\alpha(0) \geq \alpha(x)$,

(F2) $\alpha(x) \geq \alpha(x * y) \wedge \alpha(y)$,

for all $x, y \in X$.

Definition 3.3. (Zhan & Tan, 2005) A fuzzy set α of a BCK-algebra X is called a fuzzy H-ideal of X if it satisfies (F1) and (F3), where

(F1) $\alpha(0) \geq \alpha(x)$,

(F3) $\alpha(x * z) \geq \alpha(x * (y * z)) \wedge \alpha(y)$,

for all $x, y, z \in X$.

Theorem 3.4. (Zhan & Tan, 2005) A fuzzy set α of a BCK-algebra X is a fuzzy H-ideal of X if and only if, for all $t \in (0, 1]$, α_t is either empty or an H-ideal of X.

INTERVAL VALUED FUZZY H-IDEAL IN BCK-ALGEBRA

An interval valued fuzzy set $\widetilde{\alpha}$ defined on X is given by

$$\widetilde{\alpha} = \{(x, [\alpha^-(x), \alpha^+(x)]) \mid x \in X\},$$

for all $x \in X$ (briefy, denoted by $\widetilde{\alpha} = [\alpha^-, \alpha^+]$) where α^- and α^+ are two fuzzy sets in X such that $\alpha^-(x) \leq \alpha^+(x)$, for all $x \in X$.

Let

$$\widetilde{\alpha}(x) = [\alpha^-(x), \alpha^+(x)],$$

and let H[0, 1] denotes the family of all closed subintervals of [0, 1]. If

$$\alpha^-(x) = \alpha^+(x) = c \text{ (say)}$$

where $0 \leq c \leq 1$, then we have $\widetilde{\alpha}(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to H[0, 1]. Thus $\tilde{\alpha}(x) \in$ H[0, 1], for all $x \in X$ and therefore the interval valued fuzzy set $\tilde{\alpha}$ is given by

$$\tilde{\alpha} = \{(x, [\alpha^-(x), \alpha^+(x)])\},$$

for all $x \in X$, where $\tilde{\alpha} : X \rightarrow H[0, 1]$.

Now, let us define what is known as refined minimum (rmin) of two elements in H[0, 1]. We also define the symbols "$\geq$", "$\leq$", "=" in case of two elements in H[0, 1].

Consider two elements

$$H_1 = [a_1, b_1], H_2 = [a_2, b_2] \in H[0, 1].$$

Then

$$\text{rmin}(H_1, H_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}];$$

$$H_1 \geq H_2 \text{ if and only if } a_1 \geq a_2 \text{ and } b_1 \geq b_2$$

and $H_1 > H_2$ if and only if $a_1 > a_2$, $b_1 > b_2$ or $a_1 \geq a_2$, $b_1 > b_2$.

Similarly

$$\text{rmax}(H_1, H_2) = [\max\{a_1, a_2\}, \max\{b_1, b_2\}].$$

Then, H[0, 1] with $\leq$ is a complete lattice, with

$$\wedge = \text{rmin}, \vee = \text{rmax}, \tilde{0} = [0, 0] \text{ and } \tilde{1} = [1, 1]$$

being the least element and the greatest element, respectively.

Definition 4.1.

(i) An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is $\tilde{\alpha} : X \rightarrow H[0, 1]$, where for each $x \in X$,

$$\tilde{\alpha}(x) = [\alpha^-(x), \alpha^+(x)] \in H[0, 1].$$

(ii) Let $\tilde{\alpha}$ be an interval valued fuzzy set of a BCK-algebra X. Then, for every $[0, 0] < \tilde{t} \leq [1, 1]$, the crisp set (Ma *et al*., 2009)

$$\tilde{\alpha}_{\tilde{t}} = \{x \in X \mid \tilde{\alpha}(x) \geq \tilde{t}\}$$

is called the level subset of $\tilde{\alpha}$.

Definition 4.2. An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is called an interval valued fuzzy H-ideal of X if it satisfies (A) and (B), where

(A) $\tilde{\alpha}(0) \geq \tilde{\alpha}(x)$,

(B) $\tilde{\alpha}(x * z) \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y)$, for all x, y, z $\in$ X.

Example 4.3. Let X = {0, *a*, b, c, d} be a BCK-algebra in which $*$ is defined as follows:

$*$	0	*a*	b	c	d
0	0	0	0	0	0
a	*a*	0	*a*	0	*a*
b	b	b	0	b	0
c	c	*a*	c	0	c
d	d	d	b	d	0

Let $[0, 0] < \widetilde{s}_0, \widetilde{s}_1, \widetilde{s}_2 \leq [1, 1]$ be such that $\widetilde{s}_0 > \widetilde{s}_1 > \widetilde{s}_2$. We define an interval valued fuzzy set $\widetilde{\alpha} : X \to [0, 1]$ by $\widetilde{\alpha}(0) = \widetilde{s}_0$, $\widetilde{\alpha}(a) = \widetilde{\alpha}(c) = \widetilde{s}_1$ and $\widetilde{\alpha}(b) = \widetilde{\alpha}(d) = \widetilde{s}_2$. Simple calculations show that $\widetilde{\alpha}$ is an interval valued fuzzy H-ideal of X.

Theorem 4.4. An interval valued fuzzy set $\widetilde{\alpha}$ of *a* BCK-algebra X is an interval valued fuzzy H-ideal of X if and only if the set $\widetilde{\alpha}_{\widetilde{t}}\, (\neq \phi)$ is an H-ideal of X for all $[0, 0] < \widetilde{t} \leq [1, 1]$.

Proof. Assume that $\widetilde{\alpha}$ is an interval valued fuzzy H-ideal of X and let $\widetilde{t} \in H[0, 1]$ be such that $x \in \widetilde{\alpha}_{\widetilde{t}}$. Then

$$\widetilde{\alpha}(0) \geq \widetilde{\alpha}(x) \geq \widetilde{t},$$

we have $0 \in \widetilde{\alpha}_{\widetilde{t}}$. Let x, y, z $\in$ X be such that

$$x * (y * z) \in \widetilde{\alpha}_{\widetilde{t}} \text{ and } y \in \widetilde{\alpha}_{\widetilde{t}}.$$

Then

$$\widetilde{\alpha}(x * (y * z)) \geq \widetilde{t} \text{ and } \widetilde{\alpha}(y) \geq \widetilde{t}.$$

It follows from (B) that

$$\begin{aligned} \widetilde{\alpha}(x * z) &\geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \\ &\geq \widetilde{t} \wedge \widetilde{t} \\ &= \widetilde{t}. \end{aligned}$$

Thus $x * z \in \widetilde{\alpha}_{\widetilde{t}}$. Hence $\widetilde{\alpha}_{\widetilde{t}}$ is an H-ideal of X.

Conversely, suppose that $\widetilde{\alpha}_{\widetilde{t}}\, (\neq \phi)$ is an H-ideal of X for all $[0, 0] < \widetilde{t} \leq [1, 1]$. Assume that there exists $a \in$ X such that

$$\tilde{\alpha}(0) < \tilde{\alpha}(a).$$

Let

$$\tilde{\alpha}(0) = [\alpha^-(0), \alpha^+(0)]$$

and
$$\tilde{\alpha}(a) = [\alpha^-(a), \alpha^+(a)].$$

Then

$$\alpha^-(0) < \alpha^-(a) \text{ and } \alpha^+(0) < \alpha^+(a).$$

If we take

$$\tilde{\delta} = [\delta^-, \delta^+] = \frac{1}{2}(\tilde{\alpha}(0) + \tilde{\alpha}(a)),$$

then

$$[\delta^-, \delta^+] = [\frac{1}{2}(\alpha^-(0) + \alpha^-(a)), \frac{1}{2}(\alpha^+(0) + \alpha^+(a))].$$

Hence

$$\alpha^-(0) < \delta^- < \alpha^-(a) \text{ and } \alpha^+(0) < \delta^+ < \alpha^+(a).$$

This implies that

$$\tilde{\alpha}(0) = [\alpha^-(0), \alpha^+(0)] < [\delta^-, \delta^+] < [\alpha^-(a), \alpha^+(a)].$$

This shows that $0 \notin \tilde{\alpha}_{\tilde{\delta}}$, which leads to a contradiction. Therefore

$$\tilde{\alpha}(0) \geq \tilde{\alpha}(x),$$

for all $x \in X$. Now suppose that there are a, b, c $\in$ X such that

$$\tilde{\alpha}(a * c) < \tilde{\alpha}(a * (b * c)) \wedge \tilde{\alpha}(b).$$

Let

$$\tilde{\alpha}(a * c) = [\alpha^-(a * c), \alpha^+(a * c)],$$

$$\tilde{\alpha}(a * (b * c)) = [\alpha^-(a * (b * c)), \alpha^+(a * (b * c))]$$

and

$$\tilde{\alpha}(b) = [\alpha^-(b), \alpha^+(b)]$$

Put

$$\tilde{\beta} = [\beta^-, \beta^+] = \frac{1}{2}(\tilde{\alpha}(a * c) + (\tilde{\alpha}(a * (b * c)) \wedge \tilde{\alpha}(b))).$$

Then

$$\alpha^-(a * c) < \beta^- < \alpha^-(a * (b * c)) \wedge \alpha^-(b)$$

and
$$\alpha^+(a * c) < \beta^+ < \alpha^+(a * (b * c)) \wedge \alpha^+(b).$$

It follows that

$$\widetilde{\alpha}(a * c) = [\alpha^-(a * c), \alpha^+(a * c)]$$
$$< [\beta^-, \beta^+] < [\alpha^-(a * (b * c)) \wedge \alpha^-(b), \alpha^+(a * (b * c)) \wedge \alpha^+(b)]$$

So that

$$a * c \notin \widetilde{\alpha}_{\widetilde{\beta}}.$$

But

$$\widetilde{\alpha}(a * (b * c)) = [\alpha^-(a * (b * c)), \alpha^+(a * (b * c))] > \widetilde{\beta}$$

and $$\widetilde{\alpha}(b) = [\alpha^-(b), \alpha^+(b)] > \widetilde{\beta},$$

i.e.,

$$a * (b * c) \in \widetilde{\alpha}_{\widetilde{\beta}} \text{ and } b \in \widetilde{\alpha}_{\widetilde{\beta}}.$$

This leads to a contradiction. Hence

$$\widetilde{\alpha}(x * z) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y),$$

for all x, y, z $\in$ X. Thus $\widetilde{\alpha}$ is an interval valued fuzzy H-ideal of X.

$(\in, \in \vee q)$-INTERVAL VALUED FUZZY H-IDEAL IN BCK-ALGEBRA

Ma *et al*. (2008, 2009) extended the notion of belongingness and quasi-coincidence of a fuzzy point with a fuzzy set and defined the notions of belongingness and quasi-coincidence of an interval valued fuzzy point with an interval valued fuzzy set.

For any

$$\widetilde{\alpha}(x) = [\alpha^-(x), \alpha^+(x)] \text{ and } \widetilde{t} = [t^-, t^+].$$

We define

$$\widetilde{\alpha}(x) + \widetilde{t} = [\alpha^-(x) + t^-, \alpha^+(x) + t^+],$$

for x $\in$ X. In particular, if $t^- + \alpha^-(x) > 1$, we write as

$$\widetilde{\alpha}(x) + \widetilde{t} > [1, 1].$$

Let x $\in$ X and $\widetilde{t} \in$ H[0, 1], an interval valued fuzzy set $\widetilde{\alpha}$ of a BCK-algebra X of the form

$$\widetilde{\alpha}(y) = \begin{cases} \tilde{t}(\neq [0,0]) & \text{if } y = x \\ [0,0] & \text{if } y \neq x \end{cases}$$

is said to be an interval valued fuzzy point with support x and interval value $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. An interval valued fuzzy point $x_{\widetilde{t}}$ belong to (resp., quasi-coincident with) an interval valued fuzzy set $\widetilde{\alpha}$, written by $x_{\widetilde{t}} \in \widetilde{\alpha}$ (resp., $x_{\widetilde{t}}\, q\, \widetilde{\alpha}$) if $\widetilde{\alpha}(x) \geq \widetilde{t}$ (resp. $\widetilde{\alpha}(x) + \widetilde{t} > [1, 1]$). If $x_{\widetilde{t}} \in \widetilde{\alpha}$ or $x_{\widetilde{t}}\, q\, \widetilde{\alpha}$, then we write $x_{\widetilde{t}} \in \vee\, q\, \widetilde{\alpha}$. If $\widetilde{\alpha}(x) < \widetilde{t}$ (resp., $\widetilde{\alpha}(x) + \widetilde{t} \leq [1, 1]$), then we call $x_{\widetilde{t}} \overline{\in} \widetilde{\alpha}$ (resp., $x_{\widetilde{t}}\, \overline{q}\, \widetilde{\alpha}$). The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

An interval valued fuzzy set

$$\widetilde{\alpha}(x) = [\alpha^-(x), \alpha^+(x)]$$

of a BCK-algebra X is said to satisfy the condition (F) if the following holds:

(F) $\widetilde{\alpha}(x) \leq [0.5, 0.5]$ or $[0.5, 0.5] < \widetilde{\alpha}(x)$, for all $x \in X$.

We emphasize all interval valued fuzzy sets of a BCK-algebra X must satisfy the condition (F) and any two elements of H[0, 1] are comparable unless otherwise specified.

Next, we define the concept of $(\in, \in \vee q)$-interval valued fuzzy H-ideals in BCK-algebras and investigates some of their properties.

Definition 5.1. An interval valued fuzzy set $\widetilde{\alpha}$ of a BCK-algebra X is called an $(\in, \in \vee q)$-interval valued fuzzy H-ideal of X if for all $[0, 0] < \widetilde{t} \leq [1, 1]$, $[0, 0] < \widetilde{r} \leq [1, 1]$ and for all $x, y, z \in X$,

(G) $x_{\widetilde{t}} \in \widetilde{\alpha} \Rightarrow 0_{\widetilde{t}} \in \vee\, q\, \widetilde{\alpha}$,

(H) $(x * (y * z))_{\widetilde{t}} \in \widetilde{\alpha}$ and $y_{\widetilde{r}} \in \widetilde{\alpha} \Rightarrow (x * z)_{\widetilde{t} \wedge \widetilde{r}} \in \vee\, q\, \widetilde{\alpha}$.

Proposition 5.2. Every interval valued fuzzy H-ideal of a BCK-algebra X is an $(\in, \in \vee q)$-interval valued fuzzy H-ideal.

Proof. Straightforward.

Theorem 5.3. The conditions (G) and (H) in Definition 5.1, are equivalent to the following conditions respectively:

(I) $\widetilde{\alpha}(0) \geq \widetilde{\alpha}(x) \wedge [0.5, 0.5]$,

(J) $\widetilde{\alpha}(x * z) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5]$,

for all $x, y, z \in X$.

Proof. (G) $\Rightarrow$ (I)

Let $x \in X$. Now, we consider the following two cases:

(a) $\widetilde{\alpha}(x) \leq [0.5, 0.5]$

(b) $\widetilde{\alpha}(x) > [0.5, 0.5]$.

Case (a): If $\widetilde{\alpha}(0) < \widetilde{\alpha}(x) \wedge [0.5, 0.5]$, for some $x \in X$, then

$$\widetilde{\alpha}(0) < \widetilde{\alpha}(x).$$

This implies

$$\widetilde{\alpha}(0) < \widetilde{t} < \widetilde{\alpha}(x)$$

for some $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$, and so

$$x_{\widetilde{t}} \in \widetilde{\alpha} \text{ but } 0_{\widetilde{t}} \overline{\in} \widetilde{\alpha}.$$

Since

$$\widetilde{\alpha}(0) + \widetilde{t} < [1, 1].$$

We have $0_{\widetilde{t}}\ \overline{q}\ \widetilde{\alpha}$. It follows that

$$0_{\widetilde{t}} \overline{\in \vee q}\, \widetilde{\alpha}.$$

This is a contradiction. Hence

$$\widetilde{\alpha}(0) \geq \widetilde{\alpha}(x) \wedge [0.5, 0.5],$$

for all $x \in X$.

Case (b): If $\widetilde{\alpha}(0) < \widetilde{\alpha}(x) \wedge [0.5, 0.5]$, then

$$x_{[0.5,\, 0.5]} \in \widetilde{\alpha}, \text{ but } 0_{[0.5,\, 0.5]} \overline{\in \vee q}\, \widetilde{\alpha},$$

This contradicts (G). Hence (I) holds.

(I) $\Rightarrow$ (G)

Let $x_{\widetilde{t}} \in \widetilde{\alpha}$. Then $\widetilde{\alpha}(x) \geq \widetilde{t}$. If $0_{\widetilde{t}} \in \widetilde{\alpha}$, then (G) holds. If $0_{\widetilde{t}} \overline{\in} \widetilde{\alpha}$, then

$$\widetilde{\alpha}(0) < \widetilde{t} < \widetilde{\alpha}(x).$$

It follows from (I) that

$$\widetilde{\alpha}(0) \geq \widetilde{\alpha}(x) \wedge [0.5, 0.5],$$

and so

$\widetilde{\alpha}(0) \geq [0.5, 0.5]$. Therefore

$$\widetilde{\alpha}(0) + \widetilde{t} > [1, 1].$$

Thus (G) holds.

(H) $\Rightarrow$ (J)

Suppose that $x, y, z \in X$. Now, we consider the following cases:

(a) $\widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \leq [0.5, 0.5]$

(b) $\widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) > [0.5, 0.5]$.

Case (a): Assume that

$$\widetilde{\alpha}(x * z) < \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5]$$

for some x, y, z $\in$ X. Then, we have

$$\widetilde{\alpha}(x * z) < \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y).$$

Now, choose $\widetilde{t}$ such that

$$\widetilde{\alpha}(x * z) < \widetilde{t} < \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y).$$

Then

$$(x * (y * z))_{\widetilde{t}} \in \widetilde{\alpha} \text{ and } y_{\widetilde{t}} \in \widetilde{\alpha}, \text{but } (x * z)_{\widetilde{t}} \overline{\in \vee q}\, \widetilde{\alpha}.$$

This contradicts (H). Hence

$$\widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5] \leq \widetilde{\alpha}(x * z).$$

Case (b): Assume that

$$\widetilde{\alpha}(x * z) < [0.5, 0.5].$$

Then

$$(x * (y * z))_{[0.5,\, 0.5]} \in \widetilde{\alpha} \text{ and } y_{[0.5,\, 0.5]} \in \widetilde{\alpha}, \text{but } (x * z)_{[0.5,\, 0.5]} \overline{\in \vee q}\, \widetilde{\alpha},$$

which is a contradiction. Hence (J) holds.

(J) $\Rightarrow$ (H)

Let $(x * (y * z))_{\widetilde{t}} \in \widetilde{\alpha}$ and $y_{\widetilde{r}} \in \widetilde{\alpha}$. Then

$$\widetilde{\alpha}(x * (y * z)) \geq \widetilde{t} \text{ and } \widetilde{\alpha}(y) \geq \widetilde{r}.$$

Now, we have

$$\widetilde{\alpha}(x * z) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5]$$

$$\widetilde{\alpha}(x * z) \geq \widetilde{t} \wedge \widetilde{r} \wedge [0.5, 0.5].$$

If $\widetilde{t} \wedge \widetilde{r} > [0.5, 0.5]$, then

$$\widetilde{\alpha}(x * z) \geq [0.5, 0.5].$$

This implies

$$\widetilde{\alpha}(x * z) + \widetilde{t} \wedge \widetilde{r} > [1, 1].$$

If $\widetilde{t} \wedge \widetilde{r} \leq [0.5, 0.5]$, then

$$\widetilde{\alpha}(x * z) \geq \widetilde{t} \wedge \widetilde{r}.$$

Therefore

$$(x * z)_{\tilde{t} \wedge \tilde{r}} \in \vee q \tilde{\alpha} .$$

By using the concept of Definition 5.1 and Theorem 5.3, we obtain the following corollary:

Corollary 5.4. An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is an $(\in, \in \vee q)$ -interval valued fuzzy H-ideal of X if and only if the conditions (I) and (J) in Theorem 5.3 hold.

Example 5.5. Let X = {0, 1, 2} be a BCK-algebra with Caylay table as follow (Meng & Jun, 1994):

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Let $\tilde{\alpha}$ be an interval valued fuzzy set in X defined by $\tilde{\alpha}(0)$ = [0.8, 0.8], $\tilde{\alpha}(1) = \tilde{\alpha}$ (2) = [0.6, 0.6]. Simple calculations show that $\tilde{\alpha}$ is an $(\in, \in \vee q)$-interval valued fuzzy H-ideal of X.

Theorem 5.6. An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is an $(\in, \in \vee q)$ -interval valued fuzzy H-ideal of X if and only if $\tilde{\alpha}_{\tilde{t}}$ $(\neq \phi)$ is an H-ideal of X for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$.

Proof. Let $\tilde{\alpha}$ be an $(\in, \in \vee q)$-interval valued fuzzy H-ideal of X and $[0, 0] < \tilde{t} \leq [0.5, 0.5]$. Suppose $x \in \tilde{\alpha}_{\tilde{t}}$. Then

$$\tilde{\alpha}(0) \geq \tilde{\alpha}(x) \wedge [0.5, 0.5]$$

$$\tilde{\alpha}(0) \geq \tilde{t} \wedge [0.5, 0.5]$$

$$= \tilde{t} .$$

This implies $0 \in \tilde{\alpha}_{\tilde{t}}$. If $x * (y * z), y \in \tilde{\alpha}_{\tilde{t}}$ then

$$\tilde{\alpha}(x * (y * z)) \geq \tilde{t} \text{ and } \tilde{\alpha}(y) \geq \tilde{t} .$$

Now

$$\tilde{\alpha}(x * z) \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y) \wedge [0.5, 0.5]$$

$$\tilde{\alpha}(x * z) \geq \tilde{t} \wedge \tilde{t} \wedge [0.5, 0.5]$$

$$\geq \tilde{t} \wedge [0.5, 0.5]$$

$$= \tilde{t} .$$

This implies that

$$x * z \in \widetilde{\alpha}_{\widetilde{t}}.$$

Hence $\widetilde{\alpha}_{\widetilde{t}}$ is an H-ideal of X.

Conversely, let $\widetilde{\alpha}$ be an interval valued fuzzy set of X such that $\widetilde{\alpha}_{\widetilde{t}}\ (\neq \phi)$ is an H-ideal of X for all $[0, 0] < \widetilde{t} \leq [0.5, 0.5]$. By setting

$$\widetilde{\alpha}(x) \geq \widetilde{\alpha}(x) \wedge [0.5, 0.5] = \widetilde{u}_0.$$

Hence $x \in \widetilde{\alpha}_{\widetilde{u}_0}$. Since $0 \in \widetilde{\alpha}_{\widetilde{u}_0}$, therefore

$$\begin{aligned} \widetilde{\alpha}(0) &\geq \widetilde{u}_0 \\ &= \widetilde{\alpha}(x) \wedge [0.5, 0.5]. \end{aligned}$$

Now for every $x, y, z \in X$, we can write

$$\widetilde{\alpha}(x * (y * z)) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5] = \widetilde{t}_0$$

and
$$\widetilde{\alpha}(y) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5] = \widetilde{t}_0$$

Thus,

$$x * (y * z),\ y \in \widetilde{\alpha}_{\widetilde{t}_0} \text{ and so } x * z \in \widetilde{\alpha}_{\widetilde{t}_0}.$$

This shows that

$$\widetilde{\alpha}(x * z) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) \wedge [0.5, 0.5].$$

Therefore, $\widetilde{\alpha}$ is an $(\in, \in \vee\, q)$-interval valued fuzzy H-ideal of X.

$(\overline{\in}, \overline{\in} \vee \overline{q})$-INTERVAL VALUED FUZZY H-IDEAL IN BCK-ALGEBRAS

In this section, we define the concept of $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras and investigates some of their related properties.

Definition 6.1. An interval valued fuzzy set $\widetilde{\alpha}$ of a BCK-algebra X is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X if for all $[0, 0] < \widetilde{t} \leq [1, 1]$ and for all $x, y \in X$,

(K) $0_{\widetilde{t}}\ \overline{\in}\ \widetilde{\alpha} \Rightarrow x_{\widetilde{t}}\ \overline{\in} \vee \overline{q}\ \widetilde{\alpha}$,

(L) $x_{\widetilde{t}}\ \overline{\in}\ \widetilde{\alpha} \Rightarrow (x * y)_{\widetilde{t}}\ \overline{\in} \vee \overline{q}\ \widetilde{\alpha}$ or $y_{\widetilde{t}}\ \overline{\in} \vee \overline{q}\ \widetilde{\alpha}$.

Theorem 6.2. Let $\widetilde{\alpha}$ be an interval valued fuzzy set of a BCK-algebra X. Then (K) is equivalent to (M) and (L) is equivalent to (N), where

(M) $\widetilde{\alpha}(0) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x)$

(N) $\quad \tilde{\alpha}(x) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * y) \wedge \tilde{\alpha}(y)$.

Proof. Straightforward.

Example 6.3. Let X = {0, 1, 2, 3, 4} be a BCK-algebra with the following Cayley table (Meng & Jun, 1994):

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	2	1	0	0
4	4	3	2	1	0

Let $\tilde{\alpha}$ be an interval valued fuzzy set in X defined by $\tilde{\alpha}(0) = [0.91, 0.97]$, $\tilde{\alpha}(1) = [0.73, 0.78]$, $\tilde{\alpha}(2) = [0.5, 0.5]$, $\tilde{\alpha}(3) = [0.44, 0.49]$ and $\tilde{\alpha}(4) = [0.25, 0.28]$. By simple calculations show that $\tilde{\alpha}$ is an $(\in, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X.

Definition 6.4. An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X if for all $[0, 0] < \tilde{t} \leq [1, 1]$ and for all x, y, z $\in$ X,

(K) $\quad 0_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha} \;\Rightarrow\; x_{\tilde{t}} \,\overline{\in} \vee \overline{q}\, \tilde{\alpha}$,

(O) $\quad (x * z)_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha} \;\Rightarrow\; (x * (y * z))_{\tilde{t}} \,\overline{\in} \vee \overline{q}\, \tilde{\alpha}$ or $y_{\tilde{t}} \,\overline{\in} \vee \overline{q}\, \tilde{\alpha}$.

Theorem 6.5. Let $\tilde{\alpha}$ be an interval valued fuzzy set of a BCK-algebra X. Then (K) is equivalent to (M) and (O) is equivalent to (P), where

(M) $\quad \tilde{\alpha}(0) \vee [0.5, 0.5] \geq \tilde{\alpha}(x)$

(P) $\quad \tilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y)$.

Proof. (K) $\Rightarrow$ (M)

If there exists x $\in$ X such that

$$\tilde{\alpha}(0) \vee [0.5, 0.5] < \tilde{\alpha}(x) = \tilde{t}.$$

Then

$$[0.5, 0.5] < \tilde{t} \leq [1, 1],\ \tilde{\alpha}(0) < \tilde{t} \text{ and } x \in \tilde{\alpha}_{\tilde{t}}.$$

Thus,

$$0_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha}.$$

By (K), we have

$$x_{\tilde{t}} \,\overline{\in} \vee \overline{q}\, \tilde{\alpha},$$

that is,

$$\tilde{\alpha}(x) < \tilde{t} \quad \text{or} \quad \tilde{\alpha}(x) + \tilde{t} \le [1, 1].$$

Since $\tilde{\alpha}(x) = \tilde{t}$, therefore

$$\tilde{t} \le [0.5, 0.5].$$

This is a contraction. Hence

$$\tilde{\alpha}(0) \vee [0.5, 0.5] \ge \tilde{\alpha}(x)$$

for all x $\in$ X.

(M) $\Rightarrow$ (K)

Let $0_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha}$. Then $\tilde{\alpha}(0) < \tilde{t}$, either

$$\tilde{\alpha}(0) \ge [0.5, 0.5]$$

or

$$\tilde{\alpha}(0) < [0.5, 0.5]$$

If $\tilde{\alpha}(0) \ge [0.5, 0.5]$, then by (M)

$$\tilde{\alpha}(0) \vee [0.5, 0.5] = \tilde{\alpha}(0) \ge \tilde{\alpha}(x).$$

As

$$\begin{aligned} &\tilde{\alpha}(0) < \tilde{t} \ \Rightarrow\ \tilde{\alpha}(x) < \tilde{t} \\ &\Rightarrow\ x_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha}. \end{aligned}$$

If $\tilde{\alpha}(0) < [0.5, 0.5]$, then by (M)

$$\tilde{\alpha}(0) \vee [0.5, 0.5] = [0.5, 0.5] \ge \tilde{\alpha}(x).$$

Let $x_{\tilde{t}} \in \tilde{\alpha} \Rightarrow \tilde{\alpha}(x) \ge \tilde{t}$. Now

$$\begin{aligned} \tilde{\alpha}(x) + \tilde{t} &\le [0.5, 0.5] + [0.5, 0.5] \\ &= [1, 1] \\ &\Rightarrow x_{\tilde{t}}\, \overline{q}\, \tilde{\alpha}. \end{aligned}$$

(a) If $\tilde{\alpha}(0) \ge \tilde{\alpha}(x)$, then

$$\tilde{\alpha}(x) < \tilde{t} \ \text{ and so } x_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha}.$$

That is,

$$x_{\tilde{t}} \,\overline{\in} \vee \overline{q}\, \tilde{\alpha}.$$

(b) If $\widetilde{\alpha}(0) < \widetilde{\alpha}(x)$, by (M),

$$[0.5, 0.5] \geq \widetilde{\alpha}(x).$$

(i) If $\widetilde{\alpha}(x) < \widetilde{t}$, then $x_{\widetilde{t}} \,\overline{\in}\, \widetilde{\alpha}$ and so

$$x_{\widetilde{t}} \,\overline{\in} \vee \overline{q}\, \widetilde{\alpha}.$$

(ii) If $\widetilde{\alpha}(x) \geq \widetilde{t}$, then

$$\widetilde{t} \leq \widetilde{\alpha}(x) \leq [0.5, 0.5],$$

it follows that

$$x_{\widetilde{t}} \,\overline{q}\, \widetilde{\alpha},$$

and thus

$$x_{\widetilde{t}} \,\overline{\in} \vee \overline{q}\, \widetilde{\alpha}.$$

(O) $\Rightarrow$ (P)

If there exist x, y, z $\in$ X such that

$$\widetilde{\alpha}(x * z) \vee [0.5, 0.5] < \widetilde{t} = \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y).$$

Then

$$[0.5, 0.5] < \widetilde{t} \leq [1, 1],\ (x * z)_{\widetilde{t}} \,\overline{\in}\, \widetilde{\alpha} \text{ and } (x * (y * z))_{\widetilde{t}} \in \widetilde{\alpha} \text{ and } y_{\widetilde{t}} \in \widetilde{\alpha}.$$

By (O), we have

$$(x * (y * z))_{\widetilde{t}} \,\overline{q}\, \widetilde{\alpha} \text{ or } y_{\widetilde{t}} \,\overline{q}\, \widetilde{\alpha}.$$

Then

$$(\widetilde{t} \leq \widetilde{\alpha}(x * (y * z)) \text{ and } \widetilde{t} + \widetilde{\alpha}(x * (y * z)) \leq [1, 1])$$

or $$(\widetilde{t} \leq \widetilde{\alpha}(y) \text{ and } \widetilde{t} + \widetilde{\alpha}(y) \leq [1, 1]).$$

Thus,

$$\widetilde{t} \leq [0.5, 0.5].$$

This is a contradiction.

(P) $\Rightarrow$ (O)

Let $(x * z)_{\widetilde{t} \wedge \widetilde{r}} \,\overline{\in}\, \widetilde{\alpha}$. Then

$$\widetilde{\alpha}(x * z) < \widetilde{t} \wedge \widetilde{r}.$$

(a) If $\widetilde{\alpha}(x * z) \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y)$, then

$$\widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y) < \widetilde{t} \wedge \widetilde{r}$$

and consequently,

$$\tilde{\alpha}(x * (y * z)) < \tilde{t} \quad \text{or} \quad \tilde{\alpha}(y) < \tilde{r}\,.$$

It follows that

$$(x * (y * z))_{\tilde{t}} \,\overline{\in}\, \tilde{\alpha} \quad \text{or} \quad y_{\tilde{r}} \,\overline{\in}\, \tilde{\alpha}.$$

Thus

$$(x * (y * z))_{\tilde{t}} \,\overline{\in \vee q}\, \tilde{\alpha} \quad \text{or} \quad y_{\tilde{r}} \,\overline{\in \vee q}\, \tilde{\alpha}.$$

(b) If $\tilde{\alpha}(x * z) < \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y)$, then by (P),

$$[0.5, 0.5] \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y).$$

Putting $(x * (y * z))_{\tilde{t}} \in \tilde{\alpha}$, $y_{\tilde{r}} \in \tilde{\alpha}$, then

$$\tilde{t} \leq \tilde{\alpha}(x * (y * z)) \leq [0.5, 0.5] \quad \text{or} \quad \tilde{r} \leq \tilde{\alpha}(y) \leq [0.5, 0.5].$$

It follows that

$$(x * (y * z))_{\tilde{t}} \,\overline{q}\, \tilde{\alpha} \quad \text{or} \quad y_{\tilde{r}} \,\overline{q}\, \tilde{\alpha},$$

and thus

$$(x * (y * z))_{\tilde{t}} \,\overline{\in \vee q}\, \tilde{\alpha} \quad \text{or} \quad y_{\tilde{r}} \,\overline{\in \vee q}\, \tilde{\alpha}.$$

Corollary 6.6. An interval valued fuzzy set $\tilde{\alpha}$ of a BCK-algebra X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X if and only if it satisfies the conditions (M) and (P).

Example 6.7. Let X = {0, 1, 2, 3} be a BCK-algebra with Cayley table as follows (Meng & Jun, 1994):

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let $\tilde{\alpha}$ be an interval valued fuzzy set in X defined by $\tilde{\alpha}(0) = [0.67, 0.69]$, $\tilde{\alpha}(3) = [0.83, 0.89]$ and $\tilde{\alpha}(1) = \tilde{\alpha}(2) = [0.33, 0.39]$. By simple calculations show that $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal as well as an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

Theorem 6.8. Let $\tilde{\alpha}$ be an interval valued fuzzy set of a BCK-algebra X. Then $\tilde{\alpha}_{\tilde{t}}\ (\neq \phi)$ is an H-ideal of X for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$ if and only if $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

Proof. Suppose $\tilde{\alpha}_{\tilde{t}}\ (\neq \phi)$ is an H-ideal of X. If there exist x $\in$ X such that

$$\tilde{\alpha}(0) \vee [0.5, 0.5] < \tilde{\alpha}(x) = \tilde{t}\ .$$

Then

$$[0.5, 0.5] < \tilde{t} \leq [1, 1],\ \tilde{\alpha}(0) < \tilde{t} \text{ and } x \in \tilde{\alpha}_{\tilde{t}}\ .$$

Since $\tilde{\alpha}_{\tilde{t}}$ is an H-ideal of X, we have

$$0 \in \tilde{\alpha}_{\tilde{t}} \text{ and so } \tilde{\alpha}(0) \geq \tilde{t}\ .$$

This is a contradiction. Hence (M) holds.

Suppose that for some x, y, z $\in$ X, we have

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] < \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y) = \tilde{t}\ .$$

Then

$$[0.5, 0.5] < \tilde{t} \leq [1, 1],\ \tilde{\alpha}(x * z) < \tilde{t}\ .$$

Since x $*$ (y $*$ z), y $\in$ $\tilde{\alpha}_{\tilde{t}}$ and $\tilde{\alpha}_{\tilde{t}}$ is an H-ideal of X, so

$$x * z \in \tilde{\alpha}_{\tilde{t}}\ .$$

Thus

$$\tilde{\alpha}(x * z) \geq \tilde{t}\ .$$

This contradicts to

$$\tilde{\alpha}(x * z) < \tilde{t}\ .$$

Hence (P) holds. Thus $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

Conversely, assume that the conditions (M) and (P) hold. To prove that $\tilde{\alpha}_{\tilde{t}}$ is an H-ideal of X. Suppose that

$$[0.5, 0.5] < \tilde{t} \leq [1, 1],\ x \in \tilde{\alpha}_{\tilde{t}}\ .$$

Then

$$[0.5, 0.5] < \tilde{t} \leq \tilde{\alpha}(x) \leq \tilde{\alpha}(0) \vee [0.5, 0.5] = \tilde{\alpha}(0).$$

This implies that

$$0 \in \tilde{\alpha}_{\tilde{t}}\ .$$

Let x $*$ (y $*$ z), y $\in$ $\tilde{\alpha}_{\tilde{t}}$. Then

$$\begin{aligned}[0.5, 0.5] < \tilde{t} &\leq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y)\\ &\leq \tilde{\alpha}(x * z) \vee [0.5, 0.5] \quad \text{(By using condition (P))}\\ &= \tilde{\alpha}(x * z)\end{aligned}$$

and so

$$x * z \in \tilde{\alpha}_{\tilde{t}}.$$

Therefore, $\tilde{\alpha}_{\tilde{t}}$ is an H-ideal of X.

Remark 6.9. Let $\tilde{\alpha}$ be an interval valued fuzzy set of a BCK-algebra X. Then

(1) $\tilde{\alpha}$ is an interval valued fuzzy H-ideal of X if and only if $\tilde{\alpha}_{\tilde{t}}\ (\neq \phi)$ is an H-ideal of X for all $[0, 0] < \tilde{t} \leq [1, 1]$.

(2) $\tilde{\alpha}$ is an $(\in, \in \vee\, q)$-interval valued fuzzy H-ideal of X if and only if $\tilde{\alpha}_{\tilde{t}}\ (\neq \phi)$ is an H-ideal of X for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$.

(3) $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy H-ideal of X if and only if $\tilde{\alpha}_{\tilde{t}}\ (\neq \phi)$ is an H-ideal of X for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$.

Next we characterizations of $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras.

Lemma 6.10. Let $\tilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy ideal of a BCK-algebra X. If the inequality $x * y \leq z$ holds in X, then

$$\tilde{\alpha}(x) \vee [0.5, 0.5] \geq \tilde{\alpha}(y) \wedge \tilde{\alpha}(z)$$

for all x, y, z $\in$ X.

Proof. Suppose that the inequality $x * y \leq z$ holds in X. Then

$$\begin{aligned}\tilde{\alpha}(x * y) \vee [0.5, 0.5] &\geq \tilde{\alpha}((x * y) * z) \wedge \tilde{\alpha}(z)\\ &= \tilde{\alpha}(0) \wedge \tilde{\alpha}(z)\\ &\geq \tilde{\alpha}(z) \qquad \text{(by using condition (M))}\end{aligned}$$

It follows that

$$\tilde{\alpha}(x) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * y) \wedge \tilde{\alpha}(y)$$

$$\tilde{\alpha}(x) \vee [0.5, 0.5] \vee [0.5, 0.5] \geq (\tilde{\alpha}(x * y) \wedge \tilde{\alpha}(y)) \vee [0.5, 0.5]$$

$$\begin{aligned}\tilde{\alpha}(x) \vee [0.5, 0.5] &\geq (\tilde{\alpha}(x * y) \vee [0.5, 0.5]) \wedge (\tilde{\alpha}(y) \vee [0.5, 0.5])\\ &\geq \tilde{\alpha}(z) \wedge \tilde{\alpha}(y)\\ &\geq \tilde{\alpha}(y) \wedge \tilde{\alpha}(z).\end{aligned}$$

Theorem 6.11. Every $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy H-ideal of a BCK-algebra X is an $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy ideal of X.

Proof. Let $\tilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee\, \overline{q})$-interval valued fuzzy H-ideal of X. Then for all x, y, z $\in$ X, we have

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y).$$

Putting z = 0 in above, we get

$$\widetilde{\alpha}(x * 0) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * 0)) \wedge \widetilde{\alpha}(y)$$

$$\widetilde{\alpha}(x) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * y) \wedge \widetilde{\alpha}(y) \qquad \text{(by Proposition 2.2(iv))}$$

This means that $\widetilde{\alpha}$ satisfies (N). Combining with (M) implies that $\widetilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X.

Theorem 6.12. Let $\widetilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of a BCK-algebra X. Then the following are equivalent:

(i) $\widetilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

(ii) $\widetilde{\alpha}(x * y) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x)$, for all x, y $\in$ X.

Proof. (i) $\Rightarrow$ (ii)

Let $\widetilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X. If z = y and y = 0 in (P), we have

$$\widetilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y)$$

$$\begin{aligned}\widetilde{\alpha}(x * y) \vee [0.5, 0.5] &\geq \widetilde{\alpha}(x * (0 * y)) \wedge \widetilde{\alpha}(0) \\ &\geq \widetilde{\alpha}(x * 0) \wedge \widetilde{\alpha}(0) && \text{(by BCK-4)} \\ &= \widetilde{\alpha}(x) \wedge \widetilde{\alpha}(0) && \text{(by Proposition 2.2(iv))} \\ &\geq \widetilde{\alpha}(x) && \text{(by using condition (M))}\end{aligned}$$

Thus $\widetilde{\alpha}$ satisfies (ii).

(ii) $\Rightarrow$ (i)

Suppose that $\widetilde{\alpha}$ satisfies

$$\widetilde{\alpha}(x * y) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x),$$

for all x, y $\in$ X. Then we have

$$\widetilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * y) \qquad (1)$$

Since

$$\begin{aligned}(x * y) * (x * (y * z)) &\leq (y * z) * y && \text{(by Proposition 2.2(iii))} \\ &= (y * y) * z && \text{(by Proposition 2.2(i))} \\ &= 0 * z && \text{(by BCK-3)} \\ &= 0 && \text{(by BCK-4)}\end{aligned}$$

By using Lemma 6.10, we have

$$\begin{aligned}\widetilde{\alpha}(x * y) \vee [0.5, 0.5] &\geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(0) \\ &\geq \widetilde{\alpha}(x * (y * z)) \qquad (2) && \text{(by using condition (M))}\end{aligned}$$

It follows from (1) that

$$\tilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \vee [0.5, 0.5] \geq \tilde{\alpha}(x * y) \vee [0.5, 0.5]$$

$$\tilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * y) \vee [0.5, 0.5]$$

$$\geq \tilde{\alpha}(x * (y * z)) \quad (3) \quad \text{(by using (2))}$$

Since $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X, we have

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \tilde{\alpha}((x * z) * y) \wedge \tilde{\alpha}(y)$$

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] \vee [0.5, 0.5] = (\tilde{\alpha}((x * y) * z) \wedge \tilde{\alpha}(y)) \vee [0.5, 0.5]$$

(by Proposition 2.2(i))

$$= (\tilde{\alpha}((x * y) * z) \vee [0.5, 0.5]) \wedge (\tilde{\alpha}(y) \vee [0.5, 0.5])$$

$$\geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y) \qquad \text{(by using (3))}$$

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y).$$

This shows that $\tilde{\alpha}$ satisfies (P). Hence $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

Theorem 6.13. Let $\tilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of a BCK-algebra X. Then the following are equivalent:

(i) $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

(ii) $\tilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z))$.

(iii) $\tilde{\alpha}(x * y) \vee [0.5, 0.5] \geq \tilde{\alpha}(x)$, for all $x, y \in X$.

Proof. (i) $\Rightarrow$ (ii)

Since $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X, so we have

$$\tilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z)) \wedge \tilde{\alpha}(y).$$

Putting $x = x * y$ and $y = 0$ in above we get

$$\tilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \geq \tilde{\alpha}((x * y) * (0 * z)) \wedge \tilde{\alpha}(0)$$

$$\geq \tilde{\alpha}((x * y) * 0) \wedge \tilde{\alpha}(0) \quad \text{(BCK-4)}$$

$$= \tilde{\alpha}(x * y) \wedge \tilde{\alpha}(0) \qquad \text{(by Proposition 2.2(iv))}$$

$$\geq \tilde{\alpha}(x * y) \qquad (4) \qquad \text{(by using condition (M))}$$

On the other hand

$(x * y) * (0 * z) = (x * y) * ((y * z) * y) \leq x * (y * z)$ (by Proposition 2.2(ii))

By using Lemma 6.10, we have

$$\tilde{\alpha}((x * y) * (0 * z)) * [0.5, 0.5] \geq \tilde{\alpha}(x * (y * z))$$

$$\widetilde{\alpha}((x * y) * 0) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \qquad \text{(BCK-4)}$$

$$\widetilde{\alpha}(x * y) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \qquad \text{(by Proposition 2.2(iv))}$$

$$\widetilde{\alpha}(x * y) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \qquad (5)$$

From (4) it follows that

$$\widetilde{\alpha}((x * y) * z) \vee [0.5, 0.5] \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * y) \vee [0.5, 0.5]$$

$$\begin{aligned} \widetilde{\alpha}((x * y) * z) \vee [0.5, 0.5] &\geq \widetilde{\alpha}(x * y) \vee [0.5, 0.5] \\ &\geq \widetilde{\alpha}(x * (y * z)) \qquad \text{(by using (5))} \end{aligned}$$

(ii) $\Rightarrow$ (iii)

If z = y and y = 0 in (P), we have

$$\widetilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y)$$

$$\begin{aligned} \widetilde{\alpha}(x * y) \vee [0.5, 0.5] &\geq \widetilde{\alpha}(x * (0 * y)) \wedge \widetilde{\alpha}(0) \\ &\geq \widetilde{\alpha}(x * 0) \wedge \widetilde{\alpha}(0) && \text{(BCK-4)} \\ &\geq \widetilde{\alpha}(x) \wedge \widetilde{\alpha}(0) && \text{(by Proposition 2.2(iv))} \\ &\geq \widetilde{\alpha}(x) && \text{(by using condition (M))} \end{aligned}$$

(iii) $\Rightarrow$ (i)

The proof follows from Theorem 6.12.

Theorem 6.14. In a BCK-algebra X, every $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

Proof. Let $\widetilde{\alpha}$ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy ideal of X. It is sufficient to show that $\widetilde{\alpha}$ satisfies the condition (P). Let x, y, z $\in$ X. Then

$$\begin{aligned} ((x * z) * (x * (y * z))) * y &= ((y * z) * z) * y && \text{(by Proposition 2.2(iii))} \\ &= ((y * z) * y) * z && \text{(by Proposition 2.2(i))} \\ &= ((y * y) * z) * z && \text{(by Proposition 2.2(i))} \\ &= (0 * z) * z && \text{(by BCK-3)} \\ &= 0 * z && \text{(by BCK-4)} \\ &= 0 && \text{(by BCK-4)} \end{aligned}$$

It follows from Lemma 6.10 that

$$\widetilde{\alpha}(x * z) \vee [0.5, 0.5] \geq \widetilde{\alpha}(x * (y * z)) \wedge \widetilde{\alpha}(y).$$

Thus condition (P) holds. Therefore $\tilde{\alpha}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideal of X.

CONCLUSION

In the study of fuzzy algebraic system, we see that the interval valued fuzzy H-ideals with special properties always play a fundamental role.

In this paper, we define $(\in, \in \vee q)$-interval valued fuzzy H-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras and give several properties of interval valued fuzzy H-ideals with special properties in BCK-algebras in terms of these concept.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning further development of fuzzy BCK-algebras and their applications in other branches of algebra. In the future study of fuzzy BCK-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCK-algebras by using this concept;

(2) To apply this concept to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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وصف مثاليات H المشوشة فترية القيم في جبريات BCK

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خلاصة

نقوم في هذا البحث بتعريف مثاليات H المشوشة فترية القيم في جبريات BCK، ثم نجري دراسة الخصائص المتعلقة بتلك المثاليات. نقوم بعد ذلك بإستخراج خصائص ومميزيات هذه المثاليات المعممة.