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Extended Darboux frame curvatures of Frenet curves lying on parametric 3-surfaces

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Abstract

In this paper, we consider a Frenet curve lying on a parametric hypersurface in a 4-dimensional Euclidean space and obtain the expressions of its curvatures with respect to the hypersurface.

Keywords: Darboux frame; Frenet curve; geodesic curvature; geodesic torsion; hypersurface.

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1. Introduction

A surface curve in Euclidean 3-space not only has ordinary curvatures κ, τ with respect to the space, but also curvatures κ_n, κ_g, τ_g with respect to the surface itself. These curvatures play important roles in understanding the geometrical properties of the curve. The ordinary curvatures of a curve measure the twisting and turning of the Frenet frame, and they define the Darboux vector field $\mathbf{d} = \tau\,\mathbf{t} + \kappa\mathbf{b}$ of the Frenet frame {t, n, b} (Spivak, 1999; Struik, 1950). The curvatures of a curve with respect to the surface define the Darboux vector field $\mathbf{d} = \tau_g\mathbf{t} - \kappa_n\mathbf{v} + \kappa_g\mathbf{u}$ of the Darboux frame $\{\mathbf{t}, \mathbf{v}, \mathbf{u}\}$.

The computation of ordinary curvatures for parametric curves in 3-space is well-known. Also, if we have a curve lying on an implicit or parametric surface in 3-space, we know how we can compute the Darboux frame curvatures (do Carmo, 1976; O'Neill, 2006; Patrikalakis & Maekawa, 2002; Spivak, 1999; Struik, 1950).

In addition, the generalization of the Frenet frame into higher dimensional spaces and the computations of its curvatures are well-known in (Gluck, 1966). However, the generalization of the Darboux frame even into 4-space is new.(Dı̈ldı̈l et al., 2017) define the extended Darboux frame field along a Frenet curve lying on a hypersurface in Euclidean 4-space and give the geometrical meanings of the new curvatures of the curve with respect to the hypersurface. Also, they compute the expressions of these curvatures by considering the curve lying on an implicit hypersurface.

In this paper, we give the formulas of the extended Darboux frame field curvatures of a Frenet curve which lies on a parametric hypersurface in Euclidean 4-space.

2. Preliminaries

2.1 Vector product in $\mathbb{E}^4$ and its properties

Definition 1. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. The vector

$$\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

is called the *ternary product* (*vector product*) of the vectors $\mathrm{a} = \sum_{i=1}^{4} a_i\mathbf{e_i}$, $\mathbf{b} = \sum_{i=1}^{4} b_i\mathbf{e_i}$, and $\mathbf{c} = \sum_{i=1}^{4} c_i\mathbf{e_i}$ (Williams & Stein, 1964).

The ternary product has the following properties (Williams & Stein, 1964):

$$\mathbf{d}\otimes\mathbf{e}\otimes(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \langle\mathbf{a},\mathbf{e}\rangle & \langle\mathbf{b},\mathbf{e}\rangle & \langle\mathbf{c},\mathbf{e}\rangle \\ \langle\mathbf{a},\mathbf{d}\rangle & \langle\mathbf{b},\mathbf{d}\rangle & \langle\mathbf{c},\mathbf{d}\rangle \end{vmatrix},$$

$$\langle \mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}, \mathbf{d}\otimes\mathbf{e}\otimes\mathbf{f}\rangle = \begin{vmatrix} \langle \mathbf{a},\mathbf{d}\rangle & \langle \mathbf{a},\mathbf{e}\rangle & \langle \mathbf{a},\mathbf{f}\rangle \\ \langle \mathbf{b},\mathbf{d}\rangle & \langle \mathbf{b},\mathbf{e}\rangle & \langle \mathbf{b},\mathbf{f}\rangle \\ \langle \mathbf{c},\mathbf{d}\rangle & \langle \mathbf{c},\mathbf{e}\rangle & \langle \mathbf{c},\mathbf{f}\rangle \end{vmatrix}. \quad (1)$$

2.2 Curves on a hypersurface in $\mathbb{E}^4$

Let $M \subset E^4$ be a regular hypersurface parametrized by $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ and $\beta : I \subset \mathbb{R} \to M$ be an arbitrary curve with arc-length parametrization. Since M is regular, the partial derivatives $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ are linearly independent at every point of M, i.e. $\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3 \neq \mathbf{0}$, where $\mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial u_i}$. Thus, the unit normal vector of M is given by

$$\mathbf{N} = \frac{\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3}{||\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3||}.$$

The first and second fundamental form coefficients of M are given by, respectively,

$$g_{ij} = \langle \mathbf{R}_i, \mathbf{R}_j \rangle, \quad h_{ij} = \langle \mathbf{R}_{ij}, \mathbf{N} \rangle,$$

where $\mathbf{R}_{ij} = \frac{\partial^2 \mathbf{R}}{\partial u_j \partial u_i}$, $1 \leq i, j \leq 3$.

Besides, since the curve $\beta(s)$ lies on M, we may also write $\beta(s) = \mathbf{R}\big(u_1(s),\ u_2(s),\ u_3(s)\big)$. Then we have

$$\beta'(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i',$$

$$\beta''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i'' + \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i' u_j', \quad (2)$$

$$\beta'''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i''' + 3\sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i'' u_j' + \sum_{i,j,k=1}^{3} \mathbf{R}_{ijk} u_i' u_j' u_k',$$

where $\mathbf{R}_{ijk} = \frac{\partial^3 \mathbf{R}}{\partial u_k \partial u_j \partial u_i}$.

Definition 2. A unit speed curve $\beta : I \to \mathbb{E}^4$ of class C^4 is called a Frenet curve if the vectors $\beta'(s)$, $\beta''(s)$, $\beta'''(s)$ are linearly independent at each point along the curve.

2.3 The extended Darboux frame field in $\mathbb{E}^4$

Let M be an orientable hypersurface oriented by the unit normal vector field N in $\mathbb{E}^4$, and β be a Frenet curve of class C^4 with arc-length parameter s lying on M. Let

$$\mathsf{T}(s) = \beta'(s), \mathsf{N}(s) = N(\beta(s)).$$

The extended Darboux frame field along β is constructed in (Düldül *et al*., 2017) as follows:

Case 1. If the set $\{\mathsf{N}, \mathsf{T}, \beta''\}$ is linearly independent, then, using the Gram-Schmidt orthonormalization method gives the orthonormal set {N, T, E}, where

$$\mathsf{E} = \frac{\beta'' - \langle \beta'', \mathsf{N}\rangle \mathsf{N}}{||\beta'' - \langle \beta'', \mathsf{N}\rangle \mathsf{N}||}.$$

Case 2. If the set {N, T, β''} is linearly dependent, i.e. if β'' is in the direction of the normal vector N, applying the Gram-Schmidt orthonormalization method to {N, T, β'''} yields the orthonormal set {N, T, E}, where

$$\mathsf{E} = \frac{\beta''' - \langle \beta''', \mathsf{N}\rangle \mathsf{N} - \langle \beta''', \mathsf{T}\rangle \mathsf{T}}{||\beta''' - \langle \beta''', \mathsf{N}\rangle \mathsf{N} - \langle \beta''', \mathsf{T}\rangle \mathsf{T}||}.$$

In each case, defining D = N ⊗ T ⊗ E yields a new orthonormal frame field {T, E, D, N} along the curve β instead of its Frenet frame field. These new frame fields are called, “extended Darboux frame field of first kind” or in short, “ED-frame field of first kind” in case 1, and “extended Darboux frame field of second kind” or in short, “ED-frame field of second kind” in case 2, respectively.

The differential equations of ED-frame fields are given by (Düldül *et. al*., 2017)

Case 1:

$$\begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix},$$

Case 2:

$$\begin{bmatrix} \mathsf{T}' \\ \mathsf{E}' \\ \mathsf{D}' \\ \mathsf{N}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{T} \\ \mathsf{E} \\ \mathsf{D} \\ \mathsf{N} \end{bmatrix},$$

where κ_n denotes the normal curvature; κ_g^i and τ_g^i are called the geodesic curvature and geodesictorsion of order i of the curve β $(i = 1, 2)$, respectively.

3. The ED-frame field curvatures of Frenet curves on parametric 3-surfaces in Case 1

Proposition 1. Let M be an oriented hypersurface given by its parametric equation $\mathbf{R}=\mathbf{R}(u_1, u_2, u_3)$ and β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on M. Then the normal curvature of the curve β is given by

$$\kappa_n = \sum_{i,j=1}^{3} h_{ij} u_i' u_j'. \tag{3}$$

Proof. Let N denote the unit normal vector field of M along β. Since $\kappa_n = \langle \mathsf{T}', \mathsf{N}\rangle$, the assertion is clear from (2).

Proposition 2. Let β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on an oriented hypersurface M which is given by the parametric equation $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$. Then the geodesic curvature of order 1 of β is given by

$$\kappa_g^1 = \left\{ \sum_{i,j=1}^{3} g_{ij} u_i'' u_j'' + 2 \sum_{i,j,k=1}^{3} \langle \mathbf{R}_{ij}, \mathbf{R}_k \rangle u_i' u_j' u_k'' + \sum_{i,j,k,\ell=1}^{3} \langle \mathbf{R}_{ij}, \mathbf{R}_{k\ell} \rangle u_i' u_j' u_k' u_\ell' - \left(\sum_{i,j=1}^{3} h_{ij} u_i' u_j' \right)^2 \right\}^{\frac{1}{2}}. \tag{4}$$

Proof. We have $\kappa_g^1 = \langle \mathsf{T}', \mathsf{E}\rangle$ and

$$\mathsf{E} = \frac{\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}$$

(Dıldıl *et al.,* 2017), i.e.

$$\kappa_g^1 = \left\{ \langle \mathsf{T}', \mathsf{T}'\rangle - \langle \mathsf{T}', \mathsf{N}\rangle^2 \right\}^{\frac{1}{2}}.$$

If we substitute (2) into the last equation, we get (4).

By using the above Propositions, since $(k_1)^2 = (\kappa_g^1)^2 + (\kappa_n)^2$ in Case 1, we may give the following corollary:

Corollary 1. Let β be a Frenet curve lying on the parametric hypersurface $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$. Then the first curvature k_1 of β can be obtained by

$$(k_1)^2 = \sum_{i,j=1}^{3} g_{ij} u_i'' u_j'' + 2 \sum_{i,j,k=1}^{3} \langle \mathbf{R}_{ij}, \mathbf{R}_k \rangle u_i' u_j' u_k'' + \sum_{i,j,k,\ell=1}^{3} \langle \mathbf{R}_{ij}, \mathbf{R}_{k\ell} \rangle u_i' u_j' u_k' u_\ell'.$$

Proposition 3. Let M be an oriented hypersurface given by its parametric equation $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ and β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on M. Then the geodesic torsion of order 1 of β is given by

$$\tau_g^1 = \frac{1}{\kappa_g^1} \left(\sum_{i,j=1}^{3} h_{ij} u_i'' u_j' - \sum_{i,j=1}^{3} h_{ij}' u_i' u_j' + \sum_{i,j,k=1}^{3} \langle \mathbf{R}_{ijk}, \mathsf{N} \rangle u_i' u_j' u_k' \right), \tag{5}$$

where κ_g^1 is given by (4).

Proof. We have

$$\tau_g^1 = \langle \mathsf{E}', \mathsf{N}\rangle = \frac{-\langle \mathsf{T}', \mathsf{N}'\rangle}{||\mathsf{T}' - \langle \mathsf{T}', \mathsf{N}\rangle \mathsf{N}||}$$

(Dıldıl et al., 2017), i.e.

$$\tau_g^1 = \frac{-1}{\kappa_g^1} \langle \mathsf{T}', \mathsf{N}'\rangle. \tag{6}$$

We may write

$$\langle \mathsf{T}', \mathsf{N}'\rangle = \sum_{i=1}^{3} \langle \mathbf{R}_i, \mathsf{N}'\rangle u_i'' + \sum_{i,j=1}^{3} \langle \mathbf{R}_{ij}, \mathsf{N}'\rangle u_i' u_j'.$$

Since

$$\langle \mathbf{R}_i, \mathsf{N}'\rangle = -\langle \mathbf{R}_i', \mathsf{N}\rangle$$

and

$$\langle \mathbf{R}_{ij}, \mathsf{N}\rangle = h_{ij},$$

we obtain

$$\langle \mathbf{R}_{ij}, \mathsf{N}'\rangle = h_{ij}' - \sum_{k=1}^{3} \langle \mathbf{R}_{ijk}, \mathsf{N}\rangle u_k'.$$

Substituting the above equations into (6) yields (5).

Proposition 4. Let β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on an oriented hypersurface M which is given by the parametric equation $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$. Then the geodesic curvature of order 2 of β is given by

$$\kappa_g^2 = \frac{1}{\omega\left(\kappa_g^1\right)^2}\left\{\sum_{i,j=1}^{3} g_{ij}\sigma_i u_j' + \left(\sum_{i,j=1}^{3} h_{ij}u_i'u_j'\right)\left(\sum_{i,j=1}^{3} h_{ij}\rho_i u_j'\right)\right\}, \quad (7)$$

where $\omega = ||\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3||$,

$$\begin{aligned}\sigma_i = &\left(\sum_{\ell=1}^{3} g_{j\ell}u_\ell''' + 3\sum_{\ell,m=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u_\ell'' u_m' + \sum_{\ell,m,n=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell mn}\rangle u_\ell' u_m' u_n'\right)\\ &\times\left(\sum_{\ell=1}^{3} g_{k\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\\ &-\left(\sum_{\ell=1}^{3} g_{j\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\\ &\times\left(\sum_{\ell=1}^{3} g_{k\ell}u_\ell''' + 3\sum_{\ell,m=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u_\ell'' u_m' + \sum_{\ell,m,n=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell mn}\rangle u_\ell' u_m' u_n'\right), \quad (8)\end{aligned}$$

and

$$\begin{aligned}\rho_i = &\left(\sum_{\ell=1}^{3} g_{j\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\times\sum_{\ell=1}^{3} g_{k\ell}u_\ell'\\ &-\left(\sum_{\ell=1}^{3} g_{k\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\times\sum_{\ell=1}^{3} g_{j\ell}u_\ell',\end{aligned}$$

$i, j, k = 1, 2, 3$(cyclic).

Proof. We have (Düldül *et al.* , 2017)

$$\kappa_g^2 = \frac{1}{\left(\kappa_g^1\right)^2}\left\{\left\langle \mathsf{T}, \mathsf{T}' \otimes \mathsf{T}'' \otimes \mathsf{N}\right\rangle - \left\langle \mathsf{T}', \mathsf{N}\right\rangle\left\langle \mathsf{N}', \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{T}'\right\rangle\right\}. \quad (9)$$

We may write

$$\begin{aligned}\mathsf{T}' \otimes \mathsf{T}'' \otimes \mathsf{N} &= \frac{1}{\omega}\mathsf{T}' \otimes \mathsf{T}'' \otimes (\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3)\\ &= \frac{1}{\omega}\begin{vmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \\ \langle \mathbf{R}_1, \mathsf{T}''\rangle & \langle \mathbf{R}_2, \mathsf{T}''\rangle & \langle \mathbf{R}_3, \mathsf{T}''\rangle \\ \langle \mathbf{R}_1, \mathsf{T}'\rangle & \langle \mathbf{R}_2, \mathsf{T}'\rangle & \langle \mathbf{R}_3, \mathsf{T}'\rangle \end{vmatrix}\\ &= \frac{1}{\omega}\sum_{i=1}^{3}\sigma_i \mathbf{R}_i,\end{aligned}$$

where

$$\begin{aligned}\sigma_i &= \langle \mathbf{R}_j, \mathsf{T}''\rangle\langle \mathbf{R}_k, \mathsf{T}'\rangle - \langle \mathbf{R}_j, \mathsf{T}'\rangle\langle \mathbf{R}_k, \mathsf{T}''\rangle\\ &= \left(\sum_{\ell=1}^{3} g_{j\ell}u_\ell''' + 3\sum_{\ell,m=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u_\ell'' u_m' + \sum_{\ell,m,n=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell mn}\rangle u_\ell' u_m' u_n'\right)\\ &\times\left(\sum_{\ell=1}^{3} g_{k\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\\ &-\left(\sum_{\ell=1}^{3} g_{j\ell}u_\ell'' + \sum_{\ell,m=1}^{3}\langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u_\ell' u_m'\right)\\ &\times\left(\sum_{\ell=1}^{3} g_{k\ell}u_\ell''' + 3\sum_{\ell,m=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u_\ell'' u_m' + \sum_{\ell,m,n=1}^{3}\langle \mathbf{R}_k, \mathbf{R}_{\ell mn}\rangle u_\ell' u_m' u_n'\right),\end{aligned}$$

$i, j, k = 1, 2, 3$(cyclic).

Then we obtain

$$\left\langle \mathsf{T}, \mathsf{T}' \otimes \mathsf{T}'' \otimes \mathsf{N}\right\rangle = \frac{1}{\omega}\sum^{3} g_{ij}\sigma_i u_j'. \quad (10)$$

Similarly,

$$\begin{aligned}\mathsf{T} \otimes \mathsf{T}' \otimes \mathsf{N} &= \tfrac{1}{\omega}\mathsf{T} \otimes \mathsf{T}' \otimes (\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3)\\ &= \tfrac{1}{\omega}\sum_{i=1}^{3}\rho_i \mathbf{R}_i,\end{aligned}$$

where

$$\begin{aligned}\rho_i &= \langle \mathbf{R}_j, \mathsf{T}'\rangle\langle \mathbf{R}_k, \mathsf{T}\rangle - \langle \mathbf{R}_k, \mathsf{T}'\rangle\langle \mathbf{R}_j, \mathsf{T}\rangle \\ &= \left(\sum_{\ell=1}^{3} g_{j\ell}u''_\ell + \sum_{\ell,m=1}^{3} \langle \mathbf{R}_j, \mathbf{R}_{\ell m}\rangle u'_\ell u'_m\right) \\ &\times \sum_{\ell=1}^{3} g_{k\ell}u'_\ell \\ &- \left(\sum_{\ell=1}^{3} g_{k\ell}u''_\ell + \sum_{\ell,m=1}^{3} \langle \mathbf{R}_k, \mathbf{R}_{\ell m}\rangle u'_\ell u'_m\right) \\ &\times \sum_{\ell=1}^{3} g_{j\ell}u'_\ell,\end{aligned}$$

$i, j, k = 1, 2, 3$(cyclic).

Let $\mathsf{Z} = \mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3$. Then

$$\mathsf{N} = \frac{1}{\omega}\mathsf{Z}, \quad \mathsf{N}' = \frac{1}{\omega}\mathsf{Z}' - \frac{1}{\omega^3}\langle \mathsf{Z}, \mathsf{Z}'\rangle \mathsf{Z}. \quad (11)$$

Thus, we have

$$\begin{aligned}&\left\langle \mathsf{N}', \mathsf{T} \otimes \mathsf{T}' \otimes \mathsf{N}\right\rangle = \\ &= \frac{1}{\omega^2}\left\langle (\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3)', \sum_{i=1}^{3} \rho_i \mathbf{R}_i\right\rangle \\ &= \frac{-1}{\omega}\sum_{i=1}^{3} \rho_i \langle \mathbf{R}'_i, \mathsf{N}\rangle \\ &= \frac{-1}{\omega}\sum_{i=1}^{3} \rho_i \left\langle \sum_{j=1}^{3} \mathbf{R}_{ij}u'_j, \mathsf{N}\right\rangle \\ &= \frac{-1}{\omega}\sum_{i,j=1}^{3} h_{ij}\rho_i u'_j. \qquad (12)\end{aligned}$$

By substituting (10) and (12) into (9), we obtain the expression of the geodesic curvature of order 2 as given in (7).

Proposition 5. Let β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on an oriented hypersurface $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$. Then the geodesic torsion of order 2 of β is obtained by

$$\tau_g^2 = \frac{1}{\omega\kappa_g^1}\sum_{i,j=1}^{3} h_{ij}\rho_i u'_j. \qquad (13)$$

Proof. Since we have $\tau_g^2 = \langle \mathsf{D}', \mathsf{N}\rangle = -\frac{1}{\kappa_g^1}\langle \mathsf{N}', \mathsf{T} \otimes \mathsf{T}' \otimes \mathsf{N}\rangle$ (Düldül *et al.*, 2017), the assertion is clear from (12).

Corollary 2. The relation between the invariants of ED-frame field of first kind is given by

$$\left(\kappa_g^1\right)^2 \kappa_g^2 - \kappa_n \kappa_g^1 \tau_g^2 = \frac{1}{\omega}\sum_{i,j=1}^{3} g_{ij}\sigma_i u'_j,$$

where σ_i is given by (8).

4. The ED-frame field curvatures of Frenet curves on parametric 3-surfaces in Case 2

The normal curvature $\kappa_n = \langle \mathsf{T}', \mathsf{N}\rangle$ is obtained by (3).

Proposition 6. Let β be a Frenet curve of class C^n $(n \geq 4)$ with arc-length parameter s lying on an oriented hypersurface $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$. Then the geodesic torsion of order 1 of β is obtained by

$$\begin{aligned}\tau_g^1 = -\Bigg\{ & -\frac{1}{\omega^4}\left(\sum_{\ell=1}^{3}\begin{vmatrix} \langle \mathbf{R}_1, \mathbf{R}'_\ell\rangle & g_{1m} & g_{1n} \\ \langle \mathbf{R}_2, \mathbf{R}'_\ell\rangle & g_{2m} & g_{2n} \\ \langle \mathbf{R}_3, \mathbf{R}'_\ell\rangle & g_{3m} & g_{3n}\end{vmatrix}\right)^2 \\ & + \frac{1}{\omega^2}\sum_{i,\ell=1}^{3}\begin{vmatrix} \langle \mathbf{R}'_i, \mathbf{R}'_\ell\rangle & \langle \mathbf{R}'_i, \mathbf{R}_m\rangle & \langle \mathbf{R}'_i, \mathbf{R}_n\rangle \\ \langle \mathbf{R}_j, \mathbf{R}'_\ell\rangle & g_{jm} & g_{jn} \\ \langle \mathbf{R}_k, \mathbf{R}'_\ell\rangle & g_{km} & g_{kn}\end{vmatrix} \\ & - \left(\sum_{i,j=1}^{3} h_{ij}u'_i u'_j\right)^2\Bigg\}^{\frac{1}{2}},\end{aligned}$$

where $i, j, k = 1, 2, 3$ (cyclic), $\ell, m, n = 1, 2, 3$ (cyclic), and

$$\left\langle \mathbf{R}'_i, \mathbf{R}'_\ell\right\rangle = \sum_{r,q=1}^{3}\left\langle \mathbf{R}_{ir}, \mathbf{R}_{\ell q}\right\rangle u'_r u'_q,$$

$$\left\langle \mathbf{R}'_i, \mathbf{R}_m\right\rangle = \sum_{r=1}^{3}\left\langle \mathbf{R}_{ir}, \mathbf{R}_m\right\rangle u'_r.$$

Proof. The geodesic torsion of order 1 of β is obtained by (Düldül *et. al.,* 2017)

$$\tau_g^1 = \langle \mathsf{E}', \mathsf{N}\rangle = -\left\{\left\langle \mathsf{N}', \mathsf{N}'\right\rangle - \left\langle \mathsf{N}', \mathsf{T}\right\rangle^2\right\}^{\frac{1}{2}}.$$

Since $\langle \mathsf{N}', \mathsf{N}'\rangle =$

$$= \frac{1}{\omega^2}\left\langle \left(\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3\right)', \left(\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3\right)'\right\rangle$$

$$-\frac{1}{\omega^4}\left\langle \mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3, \left(\mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3\right)'\right\rangle^2$$

$$=\frac{1}{\omega^2}\sum_{i,\ell=1}^{3}\left\langle \mathbf{R}_i'\otimes\mathbf{R}_j\otimes\mathbf{R}_k, \mathbf{R}_\ell'\otimes\mathbf{R}_m\otimes\mathbf{R}_n\right\rangle$$

$$-\frac{1}{\omega^4}\left(\sum_{\ell=1}^{3}\left\langle \mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3, \mathbf{R}_\ell'\otimes\mathbf{R}_m\otimes\mathbf{R}_n\right\rangle\right)^2,$$

if we use (1), we obtain the expression of the geodesic torsion of order 1 as desired.

Proposition 7. Let β be a Frenet curve of class $C^n(n\geq 4)$ with arc-length parameter s lying on an oriented hypersurface $\mathbf{R}=\mathbf{R}(u_1,\ u_2,\ u_3)$. Then the geodesic curvature of order 2 of β is obtained by

$$\kappa_g^2=\frac{1}{\left(\tau_g^1\right)^2\omega^3}$$
$$\times\sum_{i=1}^{3}\lambda_i\left\{2\sum_{r,q=1}^{3}\det\{\mathbf{R}_i,\mathbf{R}_{ir},\mathbf{R}_{jq},\mathbf{R}_k\}u_r'u_q'\right.$$
$$+2\sum_{r,q=1}^{3}\det\{\mathbf{R}_i,\mathbf{R}_{ir},\mathbf{R}_j,\mathbf{R}_{kq}\}u_r'u_q'$$
$$\left.-\omega\left(\sum_{r,q=1}^{3}\left\langle\mathbf{R}_{irq},\mathsf{N}\right\rangle u_r'u_q'+\sum_{r=1}^{3}h_{ir}u_r''\right)\right\}, \tag{14}$$

where
$$\lambda_i=\omega\sum_{\ell,m=1}^{3}\left(g_{jm}h_{k\ell}-g_{km}h_{j\ell}\right)u_\ell'u_m',$$

$i,j,k=1,2,3$(cyclic).

Proof. We have (Düldül *et al.* , 2017)

$$\kappa_g^2=\left\langle\mathsf{E}',\mathsf{D}\right\rangle=\frac{-1}{\left(\tau_g^1\right)^2}\left\langle\mathsf{N}',\mathsf{N}\otimes\mathsf{T}\otimes\mathsf{N}''\right\rangle. \tag{15}$$

If we substitute (11) and

$$\mathsf{N}''=\frac{1}{\omega}\mathsf{Z}''-\frac{2}{\omega^3}\left\langle\mathsf{Z},\mathsf{Z}'\right\rangle\mathsf{Z}'-\frac{1}{\omega^3}||\mathsf{Z}'||^2\mathsf{Z}$$
$$-\frac{1}{\omega^3}\left\langle\mathsf{Z},\mathsf{Z}''\right\rangle\mathsf{Z}+\frac{3}{\omega^5}\left\langle\mathsf{Z},\mathsf{Z}'\right\rangle^2\mathsf{Z}$$

into (15), we obtain

$$\kappa_g^2=\frac{1}{\left(\tau_g^1\right)^2\omega^3}\left\langle\mathsf{T}\otimes\mathsf{Z}'\otimes\mathsf{Z},\mathsf{Z}''\right\rangle. \tag{16}$$

On the other hand,

$$\mathsf{T}\otimes\mathsf{Z}'\otimes\mathsf{Z}=\mathsf{T}\otimes\mathsf{Z}'\otimes\left(\mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3\right)$$
$$=\sum_{i=1}^{3}\lambda_i\mathbf{R}_i, \tag{17}$$

where

$$\lambda_i=\langle\mathbf{R}_j,\mathsf{Z}'\rangle\langle\mathbf{R}_k,\mathsf{T}\rangle-\langle\mathbf{R}_j,\mathsf{T}\rangle\langle\mathbf{R}_k,\mathsf{Z}'\rangle$$
$$=\omega\sum_{\ell,m=1}^{3}\left(g_{jm}h_{k\ell}-g_{km}h_{j\ell}\right)u_\ell'u_m',$$

$i,j,k=1,2,3$(cyclic). Since

$$\mathsf{Z}''=\mathbf{R}_1''\otimes\mathbf{R}_2\otimes\mathbf{R}_3+\mathbf{R}_1\otimes\mathbf{R}_2''\otimes\mathbf{R}_3$$
$$+\mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3''+2\Big(\mathbf{R}_1'\otimes\mathbf{R}_2'\otimes\mathbf{R}_3$$
$$+\mathbf{R}_1'\otimes\mathbf{R}_2\otimes\mathbf{R}_3'+\mathbf{R}_1\otimes\mathbf{R}_2'\otimes\mathbf{R}_3'\Big), \tag{18}$$

if we use (17) and (18), we may write

$$\left\langle\mathsf{T}\otimes\mathsf{Z}'\otimes\mathsf{Z},\mathsf{Z}''\right\rangle=\sum_{i=1}^{3}\lambda_i\left\langle\mathbf{R}_i,\mathsf{Z}''\right\rangle$$
$$=\sum_{i=1}^{3}\lambda_i\left(2\left\langle\mathbf{R}_i,\mathbf{R}_i'\otimes\mathbf{R}_j'\otimes\mathbf{R}_k\right\rangle\right.$$
$$+2\left\langle\mathbf{R}_i,\mathbf{R}_i'\otimes\mathbf{R}_j\otimes\mathbf{R}_k'\right\rangle$$
$$\left.-\omega\left\langle\mathbf{R}_i'',\mathsf{N}\right\rangle\right). \tag{19}$$

Also, we get

$$\left\langle\mathbf{R}_i,\mathbf{R}_i'\otimes\mathbf{R}_j'\otimes\mathbf{R}_k\right\rangle=$$
$$=\sum_{r,q=1}^{3}\det\{\mathbf{R}_i,\mathbf{R}_{ir},\mathbf{R}_{jq},\mathbf{R}_k\}u_r'u_q', \tag{20}$$

$$\left\langle\mathbf{R}_i,\mathbf{R}_i'\otimes\mathbf{R}_j\otimes\mathbf{R}_k'\right\rangle=$$
$$=\sum_{r,q=1}^{3}\det\{\mathbf{R}_i,\mathbf{R}_{ir},\mathbf{R}_j,\mathbf{R}_{kq}\}u_r'u_q', \tag{21}$$

and
$$\left\langle\mathbf{R}_i'',\mathsf{N}\right\rangle=\sum_{r,q=1}^{3}\left\langle\mathbf{R}_{irq},\mathsf{N}\right\rangle u_r'u_q'$$
$$+\sum_{r=1}^{3}h_{ir}u_r''. \tag{22}$$

Thus, substituting (20)-(22) into (19), the geodesic curvature of order 2 is obtained from (16) as given in (14).

Remark 1. If the hypersurface is totally geodesic in $\mathbb{E}^4$, then the expressions of the extended Darboux frame curvatures become simple due to the vanishing second fundamental form coefficients, i.e. $\kappa_n = \tau_g^1 = \tau_g^2 = 0$,

$$\kappa_g^1 = \left(\sum_{i,j=1}^{3} g_{ij} u_i'' u_j''\right)^{\frac{1}{2}},$$

$$\kappa_g^2 = \frac{1}{\omega\left(\kappa_g^1\right)^2} \sum_{i,j=1}^{3} g_{ij}\sigma_i u_j'.$$

Remark 2. If the hyper-surface is totally umbilical but not totally geodesic in $\mathbb{E}^4$, then the geodesic torsions of order i of the curve vanish.

5. Examples

In this section, we use our results to obtain the extended Darboux frame curvatures of two Frenet curves lying on a parametric hypercylinder.

Example 1. Let us consider the parametric hypercylinder C given by $\mathbf{R}(u_1, u_2, u_3) = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2, u_3)$ and the curve $\beta(s) = \mathbf{R}\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}\right)$. It is easy to see that β is a unit speed Frenet curve on C. Also, it is easy to verify that case 1 is valid along *β*. Thus, applying the method defined in (Düldül *et al.*, 2017), we obtain the ED-frame of first kind at the point *β*(0) = (1, 0, 0, 1) as

$$\mathsf{T}(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),$$

$$\mathsf{E}(0) = \left(0, 0, 0, -1\right),$$

$$\mathsf{D}(0) = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right),$$

$$\mathsf{N}(0) = \left(1, 0, 0, 0\right).$$

We obtain the non-vanishing first and second fundamental form coefficients of the hypercylinder at $\beta(0)$ as $g_{11} = g_{22} = g_{33} = 1$ and $h_{11} = h_{22} = -1$, respectively. Thus, since $u_1' = u_2' = \frac{1}{\sqrt{2}}, u_3' = 0,\ u_1'' = u_2'' = 0,\ u_3'' = -\frac{1}{2}$, if we use (3) and (4), we find the normal curvature and the geodesic curvature of order 1 at $\beta(0)$ as $\kappa_n = -1$ and $\kappa_g^1 = \frac{1}{2}$, respectively.

We also obtain $h_{ij}' = 0, 1 \leq i, j \leq 3$, $\omega = 1$, $\sigma_1 = \frac{\sqrt{2}}{8}$, $\sigma_2 = -\frac{1}{\sqrt{2}}$, $\sigma_3 = 0$, $\rho_1 = \frac{\sqrt{2}}{4}$, $\rho_2 = -\frac{\sqrt{2}}{4}$, $\rho_3 = 0$. Substituting these results into (5), (7) and (13), we obtain the geodesic torsion of order 1, geodesic curvature of order 2 and geodesic torsion of order 2 as $\tau_g^1 = 0$, $\kappa_g^2 = -\frac{3}{2}$, $\tau_g^2 = 0$, respectively.

Example 2. Let us consider the parametric hypercylinder C given in Example 1 again. It is easy to see that $\gamma(s) = \mathbf{R}\left(\frac{\pi}{6}, \frac{s}{2}, \frac{\sqrt{3}}{2}s\right)$ is a unit speed Frenet curve on C. Also, it is easy to verify that case 2 is valid along γ. Thus, applying the method defined in (Düldül *et. al.*, 2017) 2017), we obtain the ED-frame of second kind at the point $\gamma(0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, 0\right)$ as

$$\mathsf{T}(0) = \left(0, 0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

$$\mathsf{E}(0) = \left(0, 0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$

$$\mathsf{D}(0) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, 0\right),$$

$$\mathsf{N}(0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, 0\right).$$

The non-vanishing first and second fundamental form coefficients of the hypercylinder at $\gamma(0)$ are $g_{11} = g_{22} = g_{33} = 1$ and $h_{11} = h_{22} = -1$, respectively. Furthermore, since

$$u_1' = 0, u_2' = \frac{1}{2}, u_3' = \frac{\sqrt{3}}{2},$$

$$\mathbf{R}_{12} = \mathbf{R}_{21} = \mathbf{R}_{13} = \mathbf{R}_{31} = \mathbf{R}_{23} = \mathbf{0},$$

$$\mathbf{R}_{32} = \mathbf{R}_{33} = \mathbf{0}$$

at $\gamma(0)$, we obtain $\langle \mathbf{R}_i', \mathbf{R}_m \rangle = 0$ for all $i, m \in \{1, 2, 3\}$ and $\langle \mathbf{R}_i', \mathbf{R}_\ell' \rangle = 0$ except $i = \ell = 2$. We also have $u_1'' = u_2'' = u_3'' = 0,\ \omega = 1$.

Therefore, if we use equation (3), Proposition 6 and Proposition 7, we obtain the normal curvature, the geodesic torsion of order 1, geodesic curvature of order 2 as $\kappa_n = -\frac{1}{4}$, $\tau_g^1 = -\frac{\sqrt{3}}{4}$, $\kappa_g^2 = 0$, respectively.

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تقوسات هيكل Darboux الممتدة من منحنيات Frenet التي تقع على أسطح معلمية ثلاثية الأبعاد

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الملخص

في هذا البحث، قمنا بدراسة منحنى Frenet الواقع على سطح فوقي معلمي في فضاء إقليدي (Euclidean) رباعي الأبعاد، وحصلنا على صيغ تقوساته بالنسبة للسطح الفوقي.