Kuwait J. Sci. **45 (4)** pp 1-8, 2018

A new series space $\left|\bar{N}_p^{\theta}\right|(\mu)$ and matrix operators with applications

Fadime Gökçe*, Mehmet A. Sarıgöl
Dept. of Mathematics, University of Pamukkale, TR-20070 Denizli, Turkey
Corresponding author: fgokce@pau.edu.tr

Abstract

The space $\left|\bar{N}_p^{\theta}\right|_k$ of all series summable by the absolute weighted mean method has recently been introduced and studied in several publications. In the present paper, we define a new notion of generalized absolute summability, which includes several well-known summability methods, and construct a series space $\left|\bar{N}_p^{\theta}\right|(\mu)$ corresponding to it. Further, we obtain several properties of the new space and characterize certain matrix transformations on that space. We also deduce some important results as special cases.

Keywords: Absolute weighted summability; $BK-AK$ spaces; bounded linear operators; matrix transformations; sequence spaces.

Mathematics Subject Classification: 40C05, 40D25, 40F05, 46A45, 46B03, 46B28.

1. Introduction

Let X, Y be any two subsets of ω, the set of all sequences of complex numbers, and $A=(a_{nv})$ be an infinite matrix of complex numbers for $n, v \geq 0$. By $A(x) = (A_n(x))$, we denote the A-transform of a sequence $x=(x_v)$, i.e.,

$$A_n(x)=\sum_{v=0}^{\infty} a_{nv}x_v \qquad (1)$$

provided that the series are convergent for $n \geq 0$. If $A(x) \in Y$, whenever $x \in X$, then we say that A defines a matrix mapping from X into Y and denote it by $A: X \to Y$. By (X, Y), we mean the class of all infinite matrices A such that $A: X \to Y$, and also the matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A=\{x=(x_n)\in\omega : A(x)\in X\}. \qquad (2)$$

A subspace X is called an FK space if it is a Frechet space, that is, a complete locally convex linear metric space, with continuous coordinates $R_n : X \to \mathbb{C}$ $(n=0,1,2,\ldots)$, where $R_n(x)=x_n$ for all $x \in X$; an FK space whose metric is given by a norm is said to be a BK space. An FK space $X \supset \phi$, the set of all finite sequences, is said to have the AK property if

$$\lim_{m\to\infty} x^{[m]}=\lim_{m\to\infty}\sum_{n=0}^{m} x_n e^{(n)}=x,$$

for every sequence $x \in X$, where $e^{(n)}$ is a sequence whose only non-zero term is one in n-th place for $n \geq 0$. For example, it is well known that Maddox's space

$$l(\mu)=\left\{x=(x_n):\sum_{n=1}^{\infty}|x_n|^{\mu_n}<\infty\right\}$$

is an FK space with AK with respect to its natural paranorm

$$g(x)=\left(\sum_{n=0}^{\infty}|x_n|^{\mu_n}\right)^{1\backslash M}$$

where $M=\max\{1;\sup_n \mu_n\}$; it is even a BK space if $\mu_n \geq 1$ for all $n \in \mathbb{N}$ with respect to the norm

$$\|x\|=\inf\left\{\delta>0:\sum_{n=0}^{\infty}|x_n/\delta|^{\mu_n}\leq 1\right\},$$

(Maddox 1969; 1968; 1967; Nakano 1951).

Research on absolute summability factors and the comparison of summability methods plays an important role in Fourier Analysis and Approximation Theory and has been pursued by many authors (see, for example, Altay & Basar 2006; Bor et al. 2015; Bor 1985; Borwein & Cass 1968; Bosanquet & Chow 1957; Bosanquet 1950; Bosanquet 1945; Hazar & Sarıgöl 2018; Das 1970; Flett 1957; Kalaivani & Youvaraj 2013; Mazhar 1971; McFadden 1942; Mehdi 1960; Orhan & Sarıgöl 1993; Sarıgöl 2016a; Sarıgöl 2016b; Sarıgöl 2015; Sarıgöl 2011; Sarıgöl 2010; Sarıgöl 1993; Sarıgöl 1991a; Sarıgöl 1991b; Sulaiman 1992; Tanaka 1978).

Here, we note that these problems correspond to the special matrix transformations such as identity matrix and diagonal matrix. Concerning these topics, some sequence spaces have been generated and examined by several authors (see Altay & Basar 2006; Choudhary & Mishra 1993; Grosse-Erdmann 1993; Maddox 1968; Maddox 1969; Malkowsky & Rakocevic 2007; Mohapatra & Sarıgöl 2018; Mursaleen & Noman 2011; Mursaleen & Noman 2010; Nakano 1951).

The space $\left|\bar{N}_p^{\theta}\right|_k$ has recently been derived by Sarıgöl (2011) using Sulaiman's (1992) summability method $|\bar{N}, p_n, \theta_n|_k$, and studied by Mohapatra & Sarıgöl (2018), Sarıgöl (2016b) and Ozarslan & Ozgen (2015). The purpose of the present paper is to generalize this space to a new space $\left|\bar{N}_p^{\theta}\right|(\mu)$, show that it is a BK-space with AK and characterize certain matrix transformations on that space. In doing so, we also deduce some important results of Bosanquet (1950), Mohapatra & Sarıgöl (2018), Sarıgöl (2011), Orhan & Sarıgöl (1993) and Sunouchi (1949) as special cases.

First, to define the space $\left|\bar{N}_p^{\theta}\right|(\mu)$, we need a new notion of the generalized absolute summability method that also includes some well-known summability methods. Let $\sum a_v$ be a given infinite series with s_n as its

n-th partial sum, A is an infinite matrix of complex numbers and (θ_n) is any positive sequence. Let (μ_n) be any bounded sequence of positive real numbers. Then we say that the series $\sum a_\nu$ is summable $|A,\theta|(\mu)$, if

$$\sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} |A_n(s) - A_{n-1}(s)|^{\mu_n} < \infty. \tag{3}$$

Then it should be noted that the summability $|A,\theta|(\mu)$ includes the following well-known summability methods for special cases of μ, θ and $k \geq 1$:

(a) If $\mu_n = k$ for all n, then $|A,\theta|(\mu)$ is reduced to the summability $|A,\theta|_k$ (Sarıgöl, 2010).

(b) If $\mu_n = k$ and $\theta_n = n$ or $\theta_n = 1/|a_{nn}|$ for all n, then $|A,\theta|(\mu)$ is reduced to the summability $|A|_k$ (Sarıgöl, 1991b).

(c) If $\mu_n = k,\ \theta_n = n$ for all n and $A = (C,\alpha)$, Cesáro means of order $\alpha > -1$, then $|A,\theta|(\mu)$ is reduced to the summability $|C,\alpha|_k$ (Flett, 1957).

(d) If $\mu_n = k,\ \theta_n = \alpha_n^{1/(k-1)}$ for all n and A = (C, α), then $|A,\theta|(\mu)$ is reduced to the summability $|C,\alpha,\alpha_n|_k$ (Bor *et al*., 2015).

(e) If $\mu_n = k,\ \theta_n = n$ for all n, and $A = (C,\alpha,\beta)$, Cesáro means of order (α,β), $\alpha + \beta \neq -1,-2,...$, then $|A,\theta|(\mu)$ is reduced to the summability $|C,\alpha,\beta|_k$ (Das, 1970).

(f) If $\mu_n = k,\ \theta_n = n$ for all n, and $A = (R,p_n)$, Riesz means, then $|A,\theta|(\mu)$ is reduced to the summability $|R,p_n|_k$ (Sarıgöl, 1993).

(g) If $\mu_n = k$ for all n, and $A = (\bar{N},p_n)$, the weighted means, then $|A,\theta|(\mu)$ is reduced to the summability $|\bar{N},p_n,\theta_n|_k$ (Sulaiman, 1992).

(h) If $\mu_n = k,\ \theta_n = P_n/p_n$ for all n, and $A = (\bar{N},p_n)$, the weighted means, then $|A,\theta|(\mu)$ is reduced to the summability $|\bar{N},p_n|_k$ (Bor, 1985).

(i) If $\mu_n = k,\ \theta_n = n$ for all n and $A = (N,p_n)$, Nörlund means, then $|A,\theta|(\mu)$ is reduced to the summability $|N,p_n|_k$ (Borwein & Cass, 1968).

(j) If $\mu_n = k$ for all n and A is the generalized Nörlund means, then $|A,\theta|(\mu)$ is reduced to the summability $|N,p_n,q_n|_k$. In particular, for $k = 1$, it is reduced the summability $|N,p_n,q_n|$ (Tanaka, 1978).

(k) If $\mu_n = k$ for all n and $\theta_n = \gamma(n)n^{1/k^*}$, where $\gamma: [1,\infty) \to [1,\infty)$ is a nondecreasing function, then $|A,\theta|(\mu)$ is reduced to the summability $|A,\gamma|_k$ (Kalaivani & Youvaraj, 2013).

Definition. Let (p_n) be a positive sequence with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, $(P_{-1} = p_{-1} = 0)$. We define a space $\left|\bar{N}_p^\theta\right|(\mu)$ as the set of all series summable by the absolute summability $|A,\theta|(\mu)$, where A is the weighted mean matrix:

$$a_{n\nu} = \begin{cases} p_\nu/P_n, & 0 \leq \nu \leq n \\ 0, & \nu > n. \end{cases}$$

Then, it can be written from (1) that

$$A_n(s) - A_{n-1}(s) = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=0}^{n} P_{\nu-1} a_\nu .$$

which implies $A_0(s) = a_0$, and for $n \geq 1$,

$$A_n(s) - A_{n-1}(s) = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu .$$

To understand the space $\left|\bar{N}_p^\theta\right|(\mu)$ better, it is useful to state it in terms of the series $\sum a_\nu$. In fact, it is clear by (3) that the space $\left|\bar{N}_p^\theta\right|(\mu)$ can be written as

$$\left|\bar{N}_p^\theta\right|(\mu) = \left\{ a : \sum_{n=1}^{\infty} \theta_n^{\mu_{n-1}} |\chi_n(a)|^{\mu_n} < \infty \right\}$$

where

$$\chi_n(a) = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu .$$

It is also trivial that, in the special case $\mu_n = k$ for all $n \geq 0$, the series space $\left|\bar{N}_p^\theta\right|(\mu)$ is reduced to the space $\left|\bar{N}_p^\theta\right|_k$(Sarıgöl, 2011) and the space $\left|R_p\right|_k$ with $\theta_n = n$ (Orhan & Sarıgöl, 1993). Further, with the notation (2), it can be redefined by $\left|\bar{N}_p^\theta\right|(\mu) = \left(l(\mu)\right)_{T(\theta,\mu,p)}$, where the matrix $T(\theta,\mu,p)$ is given by

$$t_{n\nu}(\theta,\mu,p) = \begin{cases} 1, & n = 0, \nu = 0 \\ \theta_n^{1/\mu_n^*} \frac{p_n P_{\nu-1}}{P_n P_{n-1}}, & 1 \leq \nu \leq n \\ 0, & \nu > n, \end{cases} \tag{4}$$

to which the inverse is $S(\theta,\mu,p)$

$$s_{00}(\theta,\mu,p) = 1,$$

$$s_{n\nu}(\theta,\mu,p) = \begin{cases} -\theta_{n-1}^{-1/\mu_{n-1}^*} \frac{P_{n-2}}{p_{n-1}}, & \nu = n-1 \\ \theta_n^{-1/\mu_n^*} \frac{P_n}{p_n}, & \nu = n \\ 0, & \nu \neq n-1, n \end{cases} \tag{5}$$

where μ_n^* is the conjugate of μ_n, i.e. $1/\mu_n + 1/\mu_n^* = 1$, $\mu_n > 1$, and $1/\mu_n^* = 0$ for $\mu_n = 1$.

In addition, for simplicity of presentation we take for all $n, \nu \geq 0$,

$$\hat{a}_{n\nu} = \frac{P_\nu}{\theta_\nu^{1/\mu_\nu^*} p_\nu} \left(a_{n\nu} - \frac{P_{\nu-1}}{P_\nu} a_{n,\nu+1} \right).$$

With these notations, we establish the following theorems.

Theorem 1.1. Let (θ_n) be a sequence of positive numbers and (μ_n) be a bounded sequence of positive numbers. Then the set $\left|\bar{N}_p^\theta\right|(\mu)$ becomes a linear space with the coordinate-wise addition and scalar multiplication. It is also an FK-space with AK in respect to the paranorm $h(x) = g(T(x))$ with

$$g(T(x)) = \left(\sum_{n=0}^{\infty} \theta_n^{\mu_{n-1}} |T_n(x)|^{\mu_n} \right)^{1/M}$$

where $\theta_0 = 1$ and $M = \max\{1, \sup_n \mu_n\}$.

Theorem 1.2. Let $A = (a_{n\nu})$ be an infinite matrix of complex numbers and (θ_n) be a sequence of positive numbers. If (μ_n) is an arbitrary bounded sequence of positive numbers such that $\mu_n > 1$ for all n, then

$A\in\left(\left|\bar{N}_p^{\theta}\right|(\mu),\left|\bar{N}_q\right|\right)$ if and only if there exists an integer $M>1$ such that, for $n=0,1,\ldots$,

$$\sup_m\left|\frac{M^{-1}P_m a_{nm}}{\theta_m^{1/\mu_m^*}p_m}\right|^{\mu_m^*}<\infty, \qquad (6)$$

$$\sum_{\nu=0}^{\infty}\left|M^{-1}\hat{a}_{n\nu}\right|^{\mu_\nu^*}<\infty, \qquad (7)$$

$$\sum_{\nu=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{M^{-1}q_n}{Q_nQ_{n-1}}\left|\sum_{j=1}^{n}Q_{j-1}\,\hat{a}_{j\nu}\right|\right)^{\mu_\nu^*}<\infty \qquad (8)$$

where (q_n) is a positive sequence with $Q_n=q_0+q_1+\cdots+q_n\to\infty$ as $n\to\infty$, $(Q_{-1}=q_{-1}=0)$.

Theorem 1.3. Let $A=(a_{n\nu})$ be an infinite matrix of complex numbers, (θ_n) and (ψ_n) be sequences of positive numbers. If (μ_n) and (λ_n) are arbitrary bounded sequences of positive numbers and $\mu_n\le 1$ and $\lambda_n\ge 1$ for all n, then, $A\in\left(\left|\bar{N}_p^{\theta}\right|(\mu),\left|\bar{N}_q^{\psi}\right|(\lambda)\right)$ if and only if there exists an integer $M>1$ such that, for $n=0,1,\ldots$,

$$\sup_\nu\left|\hat{a}_{n\nu}\right|^{\mu_\nu}<\infty, \qquad (9)$$

$$\sup_m\left|\frac{P_m a_{nm}}{\theta_m^{1/\mu_m^*}p_m}\right|<\infty, \qquad (10)$$

and

$$\sup_\nu\sum_{n=1}^{\infty}\left|\frac{\psi_n^{1/\lambda_n^*}q_nM^{-1/\mu_\nu}}{Q_nQ_{n-1}}\sum_{j=1}^{n}Q_{j-1}\,\hat{a}_{j\nu}\right|^{\lambda_n}<\infty. \qquad (11)$$

2. Needed Lemmas

We require the following lemmas for the proof of our theorems.

Lemma 2.1. (Stieglitz & Tietz, 1977) $A\in(l,c)$ if and only if

(i) $\lim_n a_{n\nu}$ exists for each ν,

(ii) $\sup_{n,\nu}|a_{n\nu}|<\infty$. (12)

Lemma 2.2. (Grosse-Erdmann, 1993) Let (μ_n) and (λ_n) be any two bounded sequences of strictly positive numbers.

(i) If $\mu_n\le 1$, then, $A\in(l(\mu),c)$ if and only if

$$\begin{aligned}&(i)'\ \lim_n a_{n\nu}\text{ exists for each }\nu,\\&(ii)'\ \sup_n\sup_\nu|a_{n\nu}|^{\mu_\nu}<\infty.\end{aligned} \qquad (13)$$

(ii) If $\mu_\nu>1$ for all ν, then $(l(\mu),c)$ iff

$(i)'$ (13) $(i)'$ holds

$(ii)'$ There exists an integer $M>1$ such that

$$\sup_n\sum_{\nu=0}^{\infty}\left|a_{n\nu}M^{-1}\right|^{\mu_\nu^*}<\infty.$$

(iii) If $\mu_\nu>1$ for all ν, then $A\in(l(\mu),l)$ if and only if there exists an integer $M>1$ such that

$$\sup\left\{\sum_{\nu=0}^{\infty}\left|\sum_{n\in N}a_{n\nu}M^{-1}\right|^{\mu_\nu^*}:N\subset\mathbb{N}\text{ finite}\right\}<\infty. \qquad (14)$$

(iv) If $\mu_\nu\le 1$ and $\lambda_n\ge 1$ for all $\nu\in\mathbb{N}$ $A\in(l(\mu),l(\lambda))$ if and only if there exists some M such that

$$\sup_\nu\sum_{n=0}^{\infty}\left|a_{n\nu}M^{-1/\mu_\nu}\right|^{\lambda_n}<\infty.$$

It may be noticed that the condition (14) exposes a rather difficult condition in applications. Thus, the following lemma, which derives a condition to be equivalent to (14), is more useful in many cases and also provides great convenience in computations.

Lemma 2.3. (Sarıgöl, 2013) Let $A=(a_{n\nu})$ be an infinite matrix with complex numbers, (μ_n) be a bounded sequence of positive numbers,

$$U_\mu[A]=\sum_{\nu=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{n\nu}|\right)^{\mu_\nu}$$

and

$$L_\mu[A]=\sup\left\{\sum_{\nu=0}^{\infty}\left|\sum_{n\in N}a_{n\nu}\right|^{\mu_\nu}:N\subset\mathbb{N}\text{ finite}\right\}.$$

If $U_\mu[A]<\infty$ or $L_\mu[A]<\infty$, then

$$(2C)^{-2}U_\mu[A]\le L_\mu[A]\le U_\mu[A],$$

where $C=\max\{1,2^{H-1}\}$, $H=\sup_\nu\mu_\nu$.

Lemma 2.4. (Malkowsky & Rakocevic, 2007) Let X be an FK space with AK, T be a triangle matrix, S be its inverse and Y be an arbitrary subset of ω. Then, we have $A\in(X_T,Y)$ if and only if $\hat{A}\in(X,Y)$ and $V^{(n)}\in(X,c)$ for all n, where

$$\hat{a}_{n\nu}=\sum_{j=\nu}^{\infty}a_{nj}s_{j\nu};\ \ n,\nu=0,1,\ldots, \qquad (15)$$

and

$$v_{m\nu}^{(n)}=\begin{cases}\sum_{j=\nu}^{m}a_{nj}s_{j\nu}, & 0\le\nu\le m\\ 0, & \nu>m.\end{cases} \qquad (16)$$

3. Proofs of Theorems

In this section, we only give the proofs of our theorems, making use of lemmas.

Proof of Theorem 1.1. The first part is a routine verification, so it is omitted. Let us consider the matrix T defined by (4). Then T defines a matrix map from ω into ω since it is a triangle matrix. Furthermore, since ω and $l(\mu)$ are FK spaces and $\left|\bar{N}_p^{\theta}\right|(\mu)=\left(l(\mu)\right)_T$, then T is a continuous linear map. Thus, $\left|\bar{N}_p^{\theta}\right|(\mu)$ is an FK-space by Corollary 7.3.7 and Theorem 7.3.14 of Boos & Cass (2000). Finally, to show that $\left|\bar{N}_p^{\theta}\right|(\mu)$ is a space with AK, let us consider the base $\left(e^{(n)}\right)$ of $l(\mu)$ where $e^{(n)}$ is a sequence whose only non-zero term is one in n-th place for $n \geq 1$. Let $r_n = T^{-1}(e^{(n)})$, $x \in |\bar{N}_p^\theta|(\mu)$ and $y = T(x)$. Then, since $y \in l(\mu)$, there exists only a unique sequence of scalars (λ_i) such that $g\left(y - \sum_{i=0}^m \lambda_i e^{(i)}\right) \to 0$. Thus, it is clear that

$$h\left(x - \sum_{i=0}^m \lambda_i r_i\right) = g\left(y - \sum_{i=0}^m \lambda_i e^{(i)}\right),$$

which gives the desired conclusion.

Proof of Theorem 1.2. Note that taking $\theta_0 = 1$ does not disrupt generality. Let $\mu_n > 1$ for all n, $T = T(\theta, \mu, p)$ and $T^{(1)} = T(1,1,q)$ defined by (4). We can denote the inverse of the matrix T by S defined by (5). Then, it is clear that $|\bar{N}_p^\theta|(\mu) = (l(\mu))_T$ and $|\bar{N}_q| = (l)_{T^{(1)}}$. So, by Lemma 2.4, we have $A \in (|\bar{N}_p^\theta|(\mu), |\bar{N}_q|)$ if and only if $\hat{A} \in (l(\mu), |\bar{N}_q|)$ and $V^{(n)} \in (l(\mu), c)$, where $\hat{A}$ and $V^{(n)}$ are given by (15) and (16), respectively. Besides, if $\hat{B} = T^{(1)}\hat{A}$ then, it is easily seen that $\hat{A} \in (l(\mu), |\bar{N}_q|)$ iff $\hat{B} \in (l(\mu), l)$ because, if $\hat{A}(x) \in |\bar{N}_q|$ for all $x \in l(\mu)$, then $T^{(1)}\left(\hat{A}(x)\right) \in l$, i.e. $\hat{B}(x) \in l$. Further, a few calculations reveal that for all $n, \nu \geq 0$,

$$\hat{a}_{n\nu} = \frac{P_\nu}{\theta_\nu^{1/\mu_\nu^*} p_\nu}\left(a_{n\nu} - \frac{P_{\nu-1}}{P_\nu} a_{n,\nu+1}\right) \tag{17}$$

and

$$v_{m\nu}^{(n)} = \begin{cases} \hat{a}_{n\nu}, 0 \leq \nu \leq m-1 \\ \dfrac{P_m a_{nm}}{\theta_m^{1/\mu_m^*} p_m}, \quad \nu = m, m \geq 1 \\ 0, \qquad \nu > m. \end{cases} \tag{18}$$

Also, since the matrix $\hat{B}$ is defined by

$$\hat{b}_{n\nu} = \sum_{j=0}^\infty t_{nj}^{(1)} \hat{a}_{j\nu},$$

we have for all $\nu \geq 0$,

$$\hat{b}_{n\nu} = \begin{cases} \hat{a}_{0\nu}, \quad n = 0 \\ \dfrac{q_n}{Q_n Q_{n-1}} \displaystyle\sum_{j=1}^n Q_{j-1}\, \hat{a}_{j\nu}, \quad n \geq 1. \end{cases}$$

Now, applying Lemma 2.2 (ii) with the matrix $V^{(n)}$, since (13) $(i)'$ holds, it follows that $V^{(n)} \in (l(\mu), c)$ iff there exists an integer $M > 1$ such that

$$\sup_m \left\{\sum_{\nu=0}^{m-1} \left|v_{m\nu}^{(n)} M^{-1}\right|^{\mu_\nu^*} + \left|v_{mm}^{(n)} M^{-1}\right|^{\mu_m^*}\right\} < \infty,$$

which is satisfied iff the conditions (6) and (7) hold. Again, if we apply Lemma 2.2 (iii) with the matrix $\hat{B}$, then we have $\hat{B} \in (l(\mu), l)$ iff there exists an integer $M > 1$ such that (14) holds, equivalently, by Lemma 2.3,

$$\sum_{\nu=0}^\infty \left(\sum_{n=0}^\infty \left|M^{-1}\hat{b}_{n\nu}\right|\right)^{\mu_\nu^*} < \infty. \tag{19}$$

On the other hand, it is easily seen that (19) is satisfied iff

(8) and the condition, which is satisfied by (7),

$$\sum_{\nu=0}^\infty (M^{-1}|\hat{a}_{0\nu}|)^{\mu_\nu^*} < \infty$$

hold. Thus the proof is completed.

Proof of Theorem 1.3. Let $\mu_\nu \leq 1$ and $\lambda_\nu \geq 1$ for all ν, $T = T(\theta, \mu, p)$ and $T^{(1)} = T(\psi, \lambda, q)$. Then, $|\bar{N}_p^\theta|(\mu) = (l(\mu))_{T(\theta,\mu,p)}$ and $|\bar{N}_q^\psi|(\lambda) = (l(\lambda))_{T(\psi,\lambda,q)}$. So, as in the above Theorem, $A \in \left(|\bar{N}_p^\theta|(\mu), |\bar{N}_q^\psi|(\lambda)\right)$ if and only if $\hat{B} = T^{(1)}\hat{A} \in (l(\mu), l(\lambda))$ and $V^{(n)} \in (l(\mu), c)$, where the matrices $\hat{A}$ and $V^{(n)}$ are defined by (17) and (18), respectively. Now considering that

$$\hat{b}_{n\nu} = \sum_{j=0}^\infty t_{nj}^{(1)} \hat{a}_{j\nu},$$

we get the matrix $\hat{B}$ as for all $\nu \geq 0$,

$$\hat{b}_{n\nu} = \begin{cases} \hat{a}_{0\nu}, \quad n = 0 \\ \dfrac{\psi_n^{1/\lambda_n^*} q_n}{Q_n Q_{n-1}} \displaystyle\sum_{j=1}^n Q_{j-1}\, \hat{a}_{j\nu}, \quad n \geq 1. \end{cases}$$

Now, applying Lemma 2.2 (i) and (iv) with the matrices $V^{(n)}$ and $\hat{B}$, it follows that $V^{(n)} \in (l(\mu), c)$ iff, for $n = 0, 1, \ldots$, the conditions (9) and (10) hold, and that $\hat{B} \in (l(\mu), l(\lambda))$ iff there exists an integer M such that

$$\sup_\nu \sum_{n=0}^\infty \left|\hat{b}_{n\nu} M^{-1/\mu_\nu}\right|^{\lambda_n} < \infty, \tag{20}$$

which is satisfied if and only if the condition (11) and the following condition hold:

$$\sup_{\nu \geq 1} \left|\hat{a}_{0\nu} M^{-1/\mu_\nu}\right| < \infty. \tag{21}$$

Note that condition (9) includes condition (21). In fact, if (9) holds, then there exists a number H such that $|\xi_\nu| \leq H^{1/\mu_\nu}$ for all ν, which implies

$$\left|M^{-1/\mu_\nu}\xi_\nu\right| \leq \left(\frac{H}{M}\right)^{1/\mu_\nu},$$

where $\xi_\nu = \hat{a}_{0\nu}$. This completes the proof.

4. Applications

Our theorems have several consequences depending on sequences $\lambda, \mu, \theta, \psi$ and a matrix A as parameters. For example, if A is chosen as a diagonal matrix W such as $w_{n\nu} = \varepsilon_n$ for $\nu = n$, and zero otherwise, then $W \in \left(|\bar{N}_p^\theta|(\mu), |\bar{N}_q^\psi|(\lambda)\right)$ leads to the conclusion that $\sum \varepsilon_\nu x_\nu$ is summable $|\bar{N}, q_n, \psi_n|(\lambda)$ when $\sum x_\nu$ is summable $|\bar{N}, p_n, \theta_n|(\mu)$. Hence, if $I \in \left(|\bar{N}_p^\theta|(\mu), |\bar{N}_q^\psi|(\lambda)\right)$, where I is the identity matrix, leads to the comparisons of these methods, i.e., $|\bar{N}_p^\theta|(\mu) \subset |\bar{N}_q^\psi|(\lambda)$. Now one can easily obtain the following results.

Corollary 4.1. Let (θ_n) be a sequence of positive numbers. If (μ_n) is any bounded sequence of positive numbers such that $\mu_n > 1$ for all n, then $|\bar{N}_p^\theta|(\mu) \subset |\bar{N}_q|$ if and only if there exists an integer $M > 1$ such that

$$\sum_{\nu=0}^\infty \frac{M^{-\mu_\nu^*}}{\theta_\nu}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu} + \left|1 - \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|\right)^{\mu_\nu^*} < \infty. \tag{22}$$

Proof. Take $A = I$ in Theorem 1.2. Then (6) and (7) are directly satisfied, and (8) is reduced to (22). In fact, since for $\nu \geq 0$,

$$\sum_{j=\nu}^{n} Q_{j-1}\hat{a}_{j\nu} = \begin{cases} \theta_\nu^{\frac{-1}{\mu_\nu^*}} Q_{\nu-1}P_\nu/p_\nu\,, & n = \nu \\ \theta_\nu^{\frac{-1}{\mu_\nu^*}} \left(Q_\nu - \frac{q_\nu P_\nu}{p_\nu}\right), & n > \nu \end{cases}$$

and

$$\sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} = \frac{1}{Q_\nu},$$

we get

$$\sum_{n=\nu}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left|\sum_{j=\nu}^{n} Q_{j-1}\hat{a}_{j\nu}\right| = \frac{q_\nu}{Q_\nu Q_{\nu-1}} |Q_{\nu-1}\hat{a}_{\nu\nu}| + \sum_{n=\nu+1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left|\sum_{j=\nu}^{n} Q_{j-1}\hat{a}_{j\nu}\right| = \theta_\nu^{-1/\mu_\nu^*}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu} + \left|Q_\nu - \frac{P_\nu q_\nu}{p_\nu}\right| \frac{1}{Q_\nu}\right),$$

and so (8) is the same as

$$\sum_{\nu=0}^{\infty} \frac{M^{-\mu_\nu^*}}{\theta_\nu}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu} + \left|1 - \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|\right)^{\mu_\nu^*} < \infty.$$

This completes the proof.

Furthermore, taking $\theta_\nu = P_\nu/p_\nu$ and $\mu_\nu = k > 1$ for all ν in Corollary 4.1, (22) is reduced to

$$\sum_{\nu=0}^{\infty} \frac{p_\nu}{P_\nu}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu} + \left|1 - \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|\right)^{k^*} < \infty.$$

But this is impossible, since

$$\frac{p_\nu}{P_\nu}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu} + \left|1 - \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|\right)^{k^*} \geq \frac{p_\nu}{P_\nu}$$

for all ν and $\sum \frac{p_\nu}{P_\nu}$ is divergent by Abel-Dini Theorem. So we have the following result.

Corollary 4.2. If $\theta_\nu = P_\nu/p_\nu$ for all $\nu \geq 0$ then $\left|\bar{N}_p^\theta\right|(\mu) \not\subseteq \left|\bar{N}_q\right|$ for all sequences (p_n) and (q_n), i.e. there is a series $\sum a_n$ summable by $|\bar{N}, p_n, \theta_n|_k$ but not summable by $|\bar{N}, q_n|$.

Also, choosing $\mu_\nu = k > 1$ for all $\nu \geq 0$ and A is a triangle matrix, then Theorem 1.2 is reduced to the following main result given by Sarıgöl (2011).

Corollary 4.3. Let $A = (a_{n\nu})$ be an infinite triangle matrix of complex numbers and (θ_n) be a sequence of positive numbers. Then $A \in \left(\left|\bar{N}_p^\theta\right|_k, \left|\bar{N}_q\right|\right)$ if and only if

$$\sum_{\nu=1}^{\infty}\left(\sum_{n=\nu}^{\infty} \frac{q_n}{Q_n Q_{n-1}}\left|\sum_{j=\nu}^{n} Q_{j-1}\hat{a}_{j\nu}\right|\right)^{\mu_\nu^*} < \infty.$$

Corollary 4.4. Let (θ_n) and (ψ_n) be sequences of positive numbers. If (μ_n) and (λ_n) are arbitrary bounded sequences of positive numbers such that $\inf \mu_n > 0$, $\mu_n \leq 1$ and $\lambda_n \geq 1$ for all n, then $\left|\bar{N}_p^\theta\right|(\mu) \subset \left|\bar{N}_q^\psi\right|(\lambda)$ if and only if there exists an integer $M > 1$ such that

$$\sup_\nu \left|\frac{M^{-1/\mu_\nu}\psi_\nu^{1/\lambda_\nu^*} q_\nu P_\nu}{\theta_\nu^{1/\mu_\nu^*} Q_\nu p_\nu}\right|^{\lambda_\nu} < \infty$$

and

$$\sup_\nu \sum_{n=\nu+1}^{\infty}\left|\frac{M^{\frac{-1}{\mu_\nu}}\psi_n^{\frac{1}{\lambda_n^*}} q_n}{Q_n Q_{n-1}\theta_\nu^{\frac{1}{\mu_\nu^*}}}\left(Q_\nu - \frac{q_\nu P_\nu}{p_\nu}\right)\right|^{\lambda_n} < \infty.$$

To obtain this result, it is sufficient to take $A = I$ in Theorem 1.3.

We remark that for the case $\mu_n = \lambda_n = 1$ and $A = I$, Corollary 4.4 gives the well known result of Bosanquet (1950) and Sunouchi (1949), as follows.

Corollary 4.5. $\left|\bar{N}_p\right| \subset \left|\bar{N}_q\right|$ if and only if the following condition is satisfied:

$$\sup_\nu \frac{q_\nu P_\nu}{Q_\nu p_\nu} < \infty.$$

Corollary 4.6. Let A be a triangle matrix and (θ_n) be a sequence of positive numbers. Then, $A \in \left(\left|\bar{N}_p\right|, \left|\bar{N}_q^\theta\right|_k\right)$ if and only if the following conditions are satisfied:

$$\sup_\nu \left|\frac{\theta_\nu^{1/k^*} q_\nu P_\nu}{Q_\nu p_\nu} a_{\nu\nu}\right| < \infty, \tag{23}$$

$$\sup_\nu \left(\frac{P_\nu}{p_\nu}\right)^k \sum_{n=\nu+1}^{\infty} \left|\sigma_{n\nu} - \sigma_{n,\nu+1}\right|^k < \infty \tag{24}$$

$$\sup_\nu \sum_{n=\nu+1}^{\infty} \left|\sigma_{n,\nu+1}\right|^k < \infty \tag{25}$$

where

$$\sigma_{n\nu} = \frac{\theta_n^{1/k^*} q_n}{Q_n Q_{n-1}} \sum_{j=\nu}^{n} Q_{j-1} a_{j\nu}.$$

Proof. If we take $\mu_n = 1$, $\lambda_n = k \geq 1$ for all $n \geq 0$, $\psi = \theta$ and A is a triangle matrix in Theorem 1.3, then the conditions (9) and (10) directly hold, and (11) is also reduced to

$$\sup_\nu \left(\frac{P_\nu}{p_\nu}\right)^k \sum_{n=\nu}^{\infty} \left|\sigma_{n\nu} - \frac{P_{\nu-1}}{P_\nu}\sigma_{n,\nu+1}\right|^k < \infty \tag{26}$$

Note that the condition (26) is equivalent to the conditions (23), (24) and (25). In fact, we can write (26) as

$$\sup_\nu \left\{\left|\frac{\theta_\nu^{1/k^*} q_\nu P_\nu}{Q_\nu p_\nu} a_{\nu\nu}\right|^k + \Gamma_\nu\right\} < \infty, \tag{27}$$

where

$$\Gamma_\nu = \left(\frac{P_\nu}{p_\nu}\right)^k \sum_{n=\nu+1}^{\infty} \left|\sigma_{n\nu} - \frac{P_{\nu-1}}{P_\nu}\sigma_{n,\nu+1}\right|^k.$$

So it is easily seen from (27) that (23), (24) and (25) imply (26).

Conversely, if (26) is satisfied, then $A: \left|\bar{N}_p\right|_k \to \left|\bar{N}_q\right|$ is continuous linear mapping, so there exists a number M such that

$$\|A(x)\| \le M\|x\| \text{ for all } x \in \left|\bar{N}_p\right|_k. \tag{28}$$

Taking any $\nu \ge 0$, we apply (28) with $x_{\nu+1} = 1$, $x_m = 0, m \ne \nu + 1$. Hence, it can be obtained that for $\nu = 0,1, \ldots,$

$$\sum_{n=\nu+1}^{\infty} \left|\sigma_{n,\nu+1}\right|^k \le M^k. \tag{29}$$

Therefore, it follows from (29) that (26) implies (23), (24) and (25). This result was given by Sarıgöl (2011).

Furthermore, by taking $\theta_n = \psi_n = n$, $\mu_n = 1$, $\lambda_n = k > 1$ and $A = I$ in Theorem 1.3, we can deduce the following result according to Orhan & Sarıgöl (1993).

Corollary 4.7. Let $k \ge 1$. Then, $\left|R_p\right| \subset \left|R_q\right|_k$ if and only if the following conditions are satisfied:

(i) $\sup_\nu \left|\nu^{1/k^*} \frac{P_\nu q_\nu}{Q_\nu p_\nu}\right| < \infty$,

(ii) $\sup_\nu \frac{P_\nu q_\nu}{p_\nu} W_\nu < \infty$,

(iii) $\sup_\nu Q_\nu W_\nu < \infty$,

where

$$W_\nu = \left\{\sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k\right\}^{\frac{1}{k}}.$$

ACKNOWLEDGMENTS

The authors would like to sincerely thank the referees for their valuable time and effort to improve the manuscript.

References

Altay, B. & Basar, F. (2006). On the paranormed Riesz sequence spaces of non-absolute type derived by weighted mean. Journal of Mathematical Analysis and Applications, **319**: 494-508.

Boos, J. & Cass, P. (2000). Classical and modern methods in summability. Oxford University Press, New York.

Bor, H., Yu, D. & Zhou, P. (2015). Some new factor theorems for generalized absolute Cesáro summability. Positivity **19**: 111-120.

Bor, H. (1985). On $|\bar{N}, p_n|_k$ summability factors of infinite series. Tamkang J. Math. **16**(1): 13-20.

Borwein, D & Cass, F.P. (1968). Strong Nörlund summability. Mathematische Zeitschrift, **103**: 94-111.

Bosanquet, L.S. & Chow, H.C. (1957). Some remarks on convergence and summability factors. Journal of the London Mathematical Society, **32**: 73-82.

Bosanquet, L.S. (1950). Review on G. Sunouchi's paper "Notes on Fourier analysis, XVIII. Absolute summability of series with constant terms." Tohoku Mathematical Journal, **1**: 57-65, Mathematical Reviews, **11**: 654.

Bosanquet, L.S. (1945). Note on convergence and summability factors. Journal of the London Mathematical Society, **20**: 39-48.

Choudhary, B. & Mishra, S.K. (1993). On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations. Indian Journal of Pure and Applied Mathematics, **24**: 291-301.

Das, G. (1970). A Tauberian theorem for absolute summability. Proceedings of the Cambridge Philosophical Society, **67**: 321-326.

Flett, T.M. (1957). On an extension of absolute summability and some theorems of Littlewood and Paley. Proceedings of the London Mathematical Society, **7**: 113-141.

Grosse-Erdmann, K.G. (1993). Matrix transformations between the sequence spaces of Maddox. Journal of Mathematical Analysis and Applications, **180**: 223-238.

Hazar, G.C. & Sarıgöl, M.A. (2018). Compact and matrix operators on the space $|C, -1|_k$. Journal of Computational Analysis and Applications, **25**: 1014-1024.

Kalaivani, K. & Youvaraj, G.P. (2013). Generalized absolute Hausdorff summability of orthogonal series. Acta Mathematica Hungarica, **140**: 169-186.

Maddox, I.J. (1969). Some properties of paranormed sequence spaces. Journal of the London Mathematical Society, **1**: 316-322.

Maddox, I.J. (1968). Paranormed sequence spaces generated by infinite matrices. Mathematical Proceedings of the Cambridge Philosophical Society, **64**: 335-340.

Maddox, I.J. (1967). Spaces of strongly summable sequences. The Quarterly Journal of Mathematics **18**: 345-355.

Malkowsky, E. & Rakocevic, V. (2007). On matrix domains of triangles. Applied Mathematics and Computation, **189**(2): 1146-1163.

Mazhar, S.M. (1971). On the absolute summability factors of infinite series. Tohoku Mathematical Journal, **23**: 433-451.

McFadden, L. (1942). Absolute Nörlund summability. Duke Mathematical Journal, **9**: 168-207.

Mehdi, M.R. (1960). Summability factors for generalized absolute summability I. Proceedings of the London Mathematical Society, **10**(3): 180-199.

Mohapatra, R.N. & Sarıgöl, M.A. On matrix operators on the series space $\left|\bar{N}_p^\theta\right|_k$. Ukrainian Mathematical Journal (69(11):1772-1783.).

Mursaleen, M. & Noman, A. K. (2011). On generalized means and some related sequence spaces. Computers Mathematics with Applications, **61**: 988-999.

Mursaleen, M. & Noman, A. K. (2010). On some new difference sequence spaces of non-absolute type I. Mathematical and Computer Modelling, **52**: 603- 617.

Nakano, H. (1951). Modulared sequence space. Proceedings of the Japan Academy, Ser. A Mathematical Sciences, **27**: 508-512.

Orhan, C. & Sarıgöl, M.A. (1993). On absolute weighted mean summability. Rocky Mountain Journal of Mathematics, **23**(3): 1091-1097.

Ozarslan, H.S. Ozgen, H.N. (2015). Necessary conditions for absolute matrix summability methods. Bollettino dell' Unione Matematica Italiana, **8**(3): 223-228.

Sarıgöl, M.A. (2016a). Spaces of series summable by absolute Cesáro and matrix operators. Communications in Mathematics and Applications, **7**: 11-22.

Sarıgöl, M.A. (2016b). Norms and compactness of operators on abolute weighted mean summable series, Kuwait Journal of Science, **43**(4): 68-74.

Sarıgöl, M.A. (2015). Extension of Mazhar's theorem on summability factors, Kuwait Journal of Science, **42**: 1-8.

Sarıgöl, M.A. (2013). An inequality for matrix operators and its applications. Journal of Classical Analysis, **2**: 145-150.

Sarıgöl, M.A. (2011). Matrix transformatins on fields of absolute weighted mean summability. Studia Scientiarum Mathematicarum Hungarica, **48**(3): 331 341.

Sarıgöl, M.A. (2010). On local properties of factored Fourier series. Applied Mathematics and Computation**, 216**: 3386-3390.

Sarıgöl, M.A. (1993). On two absolute summability Riesz summability of infinite series. Proceedings of the American Mathematical Society, **118**: 485-488.

Sarıgöl, M.A. (1991a). Necessary and sufficient conditions for the equivalence of the summability methods $|\bar{N}, p_n|_k$ and $|C,-1|_k$. Indian Journal of Pure and Applied Mathematics, **22**(6): 483-489.

Sarıgöl, M.A. (1991b). On absolute summability factors. Commentationes Mathematicae, **31**:157-163.

Sulaiman, W.T. (1992). On summability factors of infinite series. Proceedings of the American Mathematical Society, **115**: 313-317.

Sunouchi, G. (1949). Notes on Fourier Analysis, 18, absolute summability of a series with constant terms. Tohoku Mathematical Journal, **1**:57-65.

Stieglitz, M. & Titez, H. (1977). Matrixtransformationen von Folgenraumen Eine Ergebnisüberischt. Mathematische Zeitschrift, **154**: 1-16.

Tanaka, M. (1978). On generalized Nörlund methods of summability. Bulletin of the Australian Mathematical Society, **19**:381-402.

Submission : 18/07/2017
Revision : 29/10/2017
Acceptance : 04/12/2017

فضاء $\left|\bar{N}_p^{\theta}\right|(\mu)$ جديد متسلسل ومشغلي المصفوفة مع التطبيقات

فاديم جوكشي*، محمد ساريجول

قسم الرياضيات، جامعة باموكالي، TR-20070 دنيزلي، تركيا

*fgokce@pau.edu.tr

الملخص

تم حديثاً في عدد من الأبحاث استحداث ودراسة فضاء$\left|\bar{N}_p^{\theta}\right|(\mu)$ لكل المتسلسلات القابلة للجمع باستخدام طريقة المتوسط المرجح المُطلق. في هذا البحث، نقوم بتعريف نوع جديد لقابلية الجمع المطلقة والذي يتضمن عدد من طرق الجمع المعروفة ونقوم ببناء الفضاء المتسلسل$\left|\bar{N}_p^{\theta}\right|(\mu)$ المقابل له. وبالإضافة إلى ذلك، نحصل على عدد من خواص الفضاء الجديد ونصف مصفوفات للتحويلات على هذا الفضاء ونحصل أيضاً على بعض النتائج الهامة كحالات خاصة.