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On the solutions of a three-dimensional system of difference equations

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Abstract

In this paper, we obtain the explicit solutions of a three-dimensional system of difference equations with multiplicative terms, extending some results in literature. Also, by using explicit forms of the solutions, we study the asymptotic behaviour of well-defined solutions of the system.

Keywords: Asymptotic behaviour; difference equations; explicit form solution; forbidden set; system of difference equations.

AMS Classification: 39A10, 39A11.

1. Introduction

Difference equations and their systems have been argued in the literature for several decades (Kulenović & Nurkanović, 2005; Papaschinopoulos & Schinas, 1998; Diamandescu, 2009; Papaschinopoulos & Stefanidou, 2010; Elabbasy *et al*., 2011; Taskara *et al*., 2011; Tollu *et al*., 2013; Yazlik, 2014 and references therein). The dominant trend in the theory of difference equations is actually to obtain the solutions of difference equation systems in the meaning of explicit or closed form. The solution forms are both an interesting and an elegant approach to study the existence and asymptotic properties of solutions of these systems (Yalcinkaya *et al*., 2008; Yazlik *et al*., 2014). Sedaghat (2009) determined the global behaviours of all solutions of the rational difference equations

$$x_{n+1} = \frac{ax_{n-1}}{x_n x_{n-1} + b}, \ x_{n+1} = \frac{ax_n x_{n-1}}{x_n + bx_{n-2}}, \ a, b > 0. \tag{1}$$

Stević (2004) gave a theoretical explanation for the formula of solutions of the difference equation

$$x_{n+1}=\frac{x_{n-1}}{1+x_n x_{n-1}}, \quad n\in \mathrm{N}_0. \tag{2}$$

Later, the author showed that the following two-dimensional system of difference equations

$$x_{n+1}=\frac{a x_{n-1}}{b y_n x_{n-1}+c},\ y_{n+1}=\frac{\alpha y_{n-1}}{\beta x_n y_{n-1}+\gamma}, \quad n\in \mathrm{N}_0, \tag{3}$$

can be solved (Stević, 2011). Stević (2012) studied the three-dimensional system of difference equations

$$x_{n+1}=\frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2}+c_1},\ y_{n+1}=\frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2}+c_2},\ z_{n+1}=\frac{a_3 z_{n-2}}{b_3 x_n y_{n-1} z_{n-2}+c_3}, \quad n\in \mathrm{N}_0, \tag{4}$$

and showed that the system in (4) can be solved as the two-dimensional system in (3) (see also Stević *et al*., 2012). Then, El-Metwally (2013) obtined the solutions form for the following systems of rational difference equations:

$$x_{n+1}=\frac{y_n x_{n-1}}{\pm x_{n-1}\pm y_{n-2}},\ y_{n+1}=\frac{x_n y_{n-1}}{\pm y_{n-1}\pm x_{n-2}}, \quad n\in \mathrm{N}_0. \tag{5}$$

Stević *et al*. (2014) solved in closed form the system of difference equations

$$x_n=\frac{x_{n-k} y_{n-l}}{b_n x_{n-k}+a_n y_{n-l-k}},\ y_n=\frac{y_{n-k} x_{n-l}}{d_n y_{n-k}+c_n x_{n-l-k}}, \quad n\in \mathrm{N}_0, \tag{6}$$

by generalizing systems in (5), and so considerably extended the results of El-Metwally's paper.

They used formulas in the investigation of the asymptotic behaviour of the well-defined solutions when the sequences $(a_n)_{n\in \mathrm{N}_0}, (b_n)_{n\in \mathrm{N}_0}, (c_n)_{n\in \mathrm{N}_0}$ and $(d_n)_{n\in \mathrm{N}_0}$ are all constant and $k=2l$ in (6). They presented the domain of undefinable solutions of the system.

Remark 1. While system (3) is an extension of the first equation in (1), the system in (4) is a three-dimensional extension of the system in (3). Similarly, the system in (6) is an extension of both the second equation in (1) and the system in (5).

Another extension of the second equation in (1) is the following three-dimensional system of difference equations:

$$x_{n+1}=\frac{x_n y_{n-1}}{a_0 x_n+b_0 y_{n-2}},\ y_{n+1}=\frac{y_n z_{n-1}}{a_1 y_n+b_1 z_{n-2}},\ z_{n+1}=\frac{z_n x_{n-1}}{a_2 z_n+b_2 x_{n-2}}, \quad n\in \mathrm{N}_0, \tag{7}$$

where the parameters a_i, b_i and the initial values x_{-i}, y_{-i}, z_{-i} $(i = 0,1,2)$ are real numbers.

Note that the system in (7) can be written in the form

$$\frac{y_{n-1}}{x_{n+1}} = b_0 \frac{y_{n-2}}{x_n} + a_0, \quad \frac{z_{n-1}}{y_{n+1}} = b_1 \frac{z_{n-2}}{y_n} + a_1, \quad \frac{x_{n-1}}{z_{n+1}} = b_2 \frac{x_{n-2}}{z_n} + a_2, \qquad n \in \mathrm{N}_0. \tag{8}$$

Therefore, the system in (7) reduces to first-order linear equations and so is solvable in explicit form. Using this approach, in this paper we get explicit solutions of the system in (7) and determine the forbidden set of the initial values x_{-i}, y_{-i}, z_{-i} $(i = 0,1,2)$ and also study asymptotic behavior of the solutions using their explicit forms.

2. Explicit solutions of the system

In this section we show that system (7) is solvable in explicit form. Here eight possible cases rise according to parameters a_i and b_i:

Case 1: $b_0 = 1,\ b_1 \neq 1$ and $b_2 \neq 1$

In this case, we obtain the system

$$\frac{y_{n-1}}{x_{n+1}} = \frac{y_{n-2}}{x_n} + a_0, \quad \frac{z_{n-1}}{y_{n+1}} = b_1 \frac{z_{n-2}}{y_n} + a_1, \quad \frac{x_{n-1}}{z_{n+1}} = b_2 \frac{x_{n-2}}{z_n} + a_2, \quad n \in \mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} + a_0 n, \quad \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} b_1^n + a_1 \frac{1-b_1^n}{1-b_1}, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} b_2^n + a_2 \frac{1-b_2^n}{1-b_2}. \tag{9}$$

From (9), we have

$$\begin{aligned} x_n &= \frac{1}{\left(\frac{y_{-2}}{x_0} + a_0 n\right)} y_{n-2} = \frac{1}{\left(\frac{y_{-2}}{x_0} + a_0 n\right)\left(q_1 b_1^{n-2} + \frac{a_1}{1-b_1}\right)} z_{n-4} \\ &= \frac{1}{\left(\frac{y_{-2}}{x_0} + a_0 n\right)\left(q_1 b_1^{n-2} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{n-4} + \frac{a_2}{1-b_2}\right)} x_{n-6}, \end{aligned} \tag{10}$$

$$\begin{aligned} y_n &= \frac{1}{\left(q_1 b_1^n + \frac{a_1}{1-b_1}\right)} z_{n-2} = \frac{1}{\left(q_1 b_1^n + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{n-2} + \frac{a_2}{1-b_2}\right)} x_{n-4} \\ &= \frac{1}{\left(q_1 b_1^n + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{n-2} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0 (n-4)\right)} y_{n-6} \end{aligned} \tag{11}$$

and

$$z_n = \frac{1}{\left(q_2 b_2^n + \frac{a_2}{1-b_2}\right)} x_{n-2} = \frac{1}{\left(q_2 b_2^n + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(n-2\right)\right)} y_{n-4}$$
$$= \frac{1}{\left(q_2 b_2^n + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(n-2\right)\right)\left(q_1 b_1^{n-4} + \frac{a_1}{1-b_1}\right)} z_{n-6}, \quad (12)$$

where $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$ and $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$. By decomposing (10), (11) and (12), we get the following non autonomous equations

$$x_{6(n+1)-j} = \frac{1}{\left(\frac{y_{-2}}{x_0} + a_0\left(6\left(n+1\right)-j\right)\right)\left(q_1 b_1^{6n+4-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6n+2-j} + \frac{a_2}{1-b_2}\right)} x_{6n-j}, \quad (13)$$

$$y_{6(n+1)-j} = \frac{1}{\left(q_1 b_1^{6(n+1)-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6n+4-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6n+2-j\right)\right)} y_{6n-j}, \quad (14)$$

and

$$z_{6(n+1)-j} = \frac{1}{\left(q_2 b_2^{6(n+1)-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6n+4-j\right)\right)\left(q_1 b_1^{6n+2-j} + \frac{a_1}{1-b_1}\right)} z_{6n-j}, \quad (15)$$

where $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$, $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$, $j \in \{-3,-2,-1,0,1,2\}$ and $n \in \mathrm{N}_0$. Equations (13), (14) and (15) easily can be solved as the following

$$x_{6n-j} = \frac{x_{-j}}{\prod_{i=0}^{n-1}\left(\frac{y_{-2}}{x_0} + a_0\left(6\left(i+1\right)-j\right)\right)\left(q_1 b_1^{6i+4-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6i+2-j} + \frac{a_2}{1-b_2}\right)},$$
$$y_{6n-j} = \frac{y_{-j}}{\prod_{i=0}^{n-1}\left(q_1 b_1^{6(i+1)-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6i+4-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6i+2-j\right)\right)}, \quad (16)$$
$$z_{6n-j} = \frac{z_{-j}}{\prod_{i=0}^{n-1}\left(q_2 b_2^{6(i+1)-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6i+4-j\right)\right)\left(q_1 b_1^{6i+2-j} + \frac{a_1}{1-b_1}\right)},$$

where $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$, $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$, $j \in \{-3,-2,-1,0,1,2\}$ and $n \in \mathrm{N}_0$.

Case 2: $b_1 = 1$, $b_0 \neq 1$ and $b_2 \neq 1$

In this case, the system becomes

$$\frac{y_{n-1}}{x_{n+1}} = b_0 \frac{y_{n-2}}{x_n} + a_0, \ \frac{z_{n-1}}{y_{n+1}} = \frac{z_{n-2}}{y_n} + a_1, \ \frac{x_{n-1}}{z_{n+1}} = b_2 \frac{x_{n-2}}{z_n} + a_2, \ n \in \mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} b_0^n + a_0 \frac{1-b_0^n}{1-b_0}, \ \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} + a_1 n, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} b_2^n + a_2 \frac{1-b_2^n}{1-b_2}. \quad (17)$$

From (17), the general solution follows as

$$x_{6n-j} = \frac{x_{-j}}{\prod_{i=0}^{n-1} \left(q_0 b_0^{6(i+1)-j} + \frac{a_0}{1-b_0} \right) \left(\frac{z_{-2}}{y_0} + a_1 (6i+4-j) \right) \left(q_2 b_2^{6i+2-j} + \frac{a_2}{1-b_2} \right)},$$

$$y_{6n-j} = \frac{y_{-j}}{\prod_{i=0}^{n-1} \left(\frac{z_{-2}}{y_0} + a_1 \left(6(i+1)-j \right) \right) \left(q_2 b_2^{6i+4-j} + \frac{a_2}{1-b_2} \right) \left(q_0 b_0^{6i+2-j} + \frac{a_0}{1-b_0} \right)}, \quad (18)$$

$$z_{6n-j} = \frac{z_{-j}}{\prod_{i=0}^{n-1} \left(q_2 b_2^{6(i+1)-j} + \frac{a_2}{1-b_2} \right) \left(q_0 b_0^{6i+4-j} + \frac{a_0}{1-b_0} \right) \left(\frac{z_{-2}}{y_0} + a_1 (6i+2-j) \right)},$$

where $q_0 = \frac{y_{-2}}{x_0} - \frac{a_0}{1-b_0}$, $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$, $j \in \{-3,-2,-1,0,1,2\}$ and $n \in \mathrm{N}_0$.

Case 3: $b_2 = 1$, $b_0 \neq 1$ and $b_1 \neq 1$

In this case, the system is

$$\frac{y_{n-1}}{x_{n+1}} = b_0 \frac{y_{n-2}}{x_n} + a_0, \ \frac{z_{n-1}}{y_{n+1}} = b_1 \frac{z_{n-2}}{y_n} + a_1, \ \frac{x_{n-1}}{z_{n+1}} = \frac{x_{n-2}}{z_n} + a_2, \ n \in \mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} b_0^n + a_0 \frac{1-b_0^n}{1-b_0}, \ \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} b_1^n + a_1 \frac{1-b_1^n}{1-b_1}, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} + a_2 n. \quad (19)$$

The solution can be obtained from (19) as

$$x_{6n-j}=\frac{x_{-j}}{\prod_{i=0}^{n-1}\left(q_0b_0^{6(i+1)-j}+\frac{a_0}{1-b_0}\right)\left(q_1b_1^{6i+4-j}+\frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+2-j\right)\right)},$$
$$y_{6n-j}=\frac{y_{-j}}{\prod_{i=0}^{n-1}\left(q_1b_1^{6(i+1)-j}+\frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+4-j\right)\right)\left(q_0b_0^{6i+2-j}+\frac{a_0}{1-b_0}\right)}, \quad (20)$$
$$z_{6n-j}=\frac{z_{-j}}{\prod_{i=0}^{n-1}\left(\frac{x_{-2}}{z_0}+a_2\left(6\left(i+1\right)-j\right)\right)\left(q_0b_0^{6i+4-j}+\frac{a_0}{1-b_0}\right)\left(q_1b_1^{6i+2-j}+\frac{a_1}{1-b_1}\right)},$$

where $q_0=\frac{y_{-2}}{x_0}-\frac{a_0}{1-b_0}$, $q_1=\frac{z_{-2}}{y_0}-\frac{a_1}{1-b_1}$, $j\in\{-3,-2,-1,0,1,2\}$ and $n\in\mathrm{N}_0$.

Case 4: $b_0\neq 1$ and $b_1=b_2=1$

In this case, we get the following system

$$\frac{y_{n-1}}{x_{n+1}}=b_0\frac{y_{n-2}}{x_n}+a_0,\ \frac{z_{n-1}}{y_{n+1}}=\frac{z_{n-2}}{y_n}+a_1,\ \frac{x_{n-1}}{z_{n+1}}=\frac{x_{n-2}}{z_n}+a_2,\ n\in\mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n}=\frac{y_{-2}}{x_0}b_0^n+a_0\frac{1-b_0^n}{1-b_0},\ \frac{z_{n-2}}{y_n}=\frac{z_{-2}}{y_0}+a_1n,\frac{x_{n-2}}{z_n}=\frac{x_{-2}}{z_0}+a_2n. \quad (21)$$

From (21), the solution of system (7) takes the form

$$x_{6n-j}=\frac{x_{-j}}{\prod_{i=0}^{n-1}\left(q_0b_0^{6(i+1)-j}+\frac{a_0}{1-b_0}\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+4-j\right)\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+2-j\right)\right)},$$
$$y_{6n-j}=\frac{y_{-j}}{\prod_{i=0}^{n-1}\left(\frac{z_{-2}}{y_0}+a_1\left(6\left(i+1\right)-j\right)\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+4-j\right)\right)\left(q_0b_0^{6i+2-j}+\frac{a_0}{1-b_0}\right)}, \quad (22)$$
$$z_{6n-j}=\frac{z_{-j}}{\prod_{i=0}^{n-1}\left(\frac{x_{-2}}{z_0}+a_2\left(6\left(i+1\right)-j\right)\right)\left(q_0b_0^{6i+4-j}+\frac{a_0}{1-b_0}\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+2-j\right)\right)},$$

where $q_0=\frac{y_{-2}}{x_0}-\frac{a_0}{1-b_0}$, $j\in\{-3,-2,-1,0,1,2\}$ and $n\in\mathrm{N}_0$.

Case 5: $b_1 \neq 1$ and $b_0 = b_2 = 1$

In this case, the system is expressed as

$$\frac{y_{n-1}}{x_{n+1}} = \frac{y_{n-2}}{x_n} + a_0, \ \frac{z_{n-1}}{y_{n+1}} = b_1 \frac{z_{n-2}}{y_n} + a_1, \ \frac{x_{n-1}}{z_{n+1}} = \frac{x_{n-2}}{z_n} + a_2, \ n \in \mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} + a_0 n, \ \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} b_1^n + a_1 \frac{1-b_1^n}{1-b_1}, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} + a_2 n. \tag{23}$$

From (23), we obtain the solution of system (7) as follows

$$\begin{aligned}
x_{6n-j} &= \frac{x_{-j}}{\prod_{i=0}^{n-1}\left(\frac{y_{-2}}{x_0} + a_0\left(6(i+1)-j\right)\right)\left(q_1 b_1^{6i+4-j} + \frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6i+2-j\right)\right)}, \\
y_{6n-j} &= \frac{y_{-j}}{\prod_{i=0}^{n-1}\left(q_1 b_1^{6(i+1)-j} + \frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6i+4-j\right)\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6i+2-j\right)\right)}, \\
z_{6n-j} &= \frac{z_{-j}}{\prod_{i=0}^{n-1}\left(\frac{x_{-2}}{z_0} + a_2\left(6(i+1)-j\right)\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6i+4-j\right)\right)\left(q_1 b_1^{6i+2-j} + \frac{a_1}{1-b_1}\right)},
\end{aligned} \tag{24}$$

where $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$, $j \in \{-3,-2,-1,0,1,2\}$ and $n \in \mathrm{N}_0$.

Case 6: $b_2 \neq 1$ and $b_0 = b_1 = 1$

The case yields the following system

$$\frac{y_{n-1}}{x_{n+1}} = \frac{y_{n-2}}{x_n} + a_0, \ \frac{z_{n-1}}{y_{n+1}} = \frac{z_{n-2}}{y_n} + a_1, \ \frac{x_{n-1}}{z_{n+1}} = b_2 \frac{x_{n-2}}{z_n} + a_2, \ n \in \mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} + a_0 n, \ \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} + a_1 n, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} b_1^n + a_2 \frac{1-b_2^n}{1-b_2}. \tag{25}$$

The solution can be obtained from (25) as

$$x_{6n-j}=\frac{x_{-j}}{\prod_{i=0}^{n-1}\left(\frac{y_{-2}}{x_0}+a_0\left(6\left(i+1\right)-j\right)\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+4-j\right)\right)\left(q_2b_2^{6i+2-j}+\frac{a_2}{1-b_2}\right)},$$

$$y_{6n-j}=\frac{y_{-j}}{\prod_{i=0}^{n-1}\left(\frac{z_{-2}}{y_0}+a_1\left(6\left(i+1\right)-j\right)\right)\left(q_2b_2^{6i+4-j}+\frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0}+a_0\left(6i+2-j\right)\right)}, \quad (26)$$

$$z_{6n-j}=\frac{z_{-j}}{\prod_{i=0}^{n-1}\left(q_2b_2^{6(i+1)-j}+\frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0}+a_0\left(6i+4-j\right)\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+2-j\right)\right)},$$

where $q_2=\frac{x_{-2}}{z_0}-\frac{a_2}{1-b_2}$, $j\in\{-3,-2,-1,0,1,2\}$ and $n\in\mathrm{N}_0$.

Case 7: $b_0=b_1=b_2=1$

In this case, the system is

$$\frac{y_{n-1}}{x_{n+1}}=\frac{y_{n-2}}{x_n}+a_0,\ \frac{z_{n-1}}{y_{n+1}}=\frac{z_{n-2}}{y_n}+a_1,\ \frac{x_{n-1}}{z_{n+1}}=\frac{x_{n-2}}{z_n}+a_2,\ n\in\mathrm{N}_0,$$

from which it follows that

$$\frac{y_{n-2}}{x_n}=\frac{y_{-2}}{x_0}+a_0n,\ \frac{z_{n-2}}{y_n}=\frac{z_{-2}}{y_0}+a_1n,\frac{x_{n-2}}{z_n}=\frac{x_{-2}}{z_0}+a_2n. \quad (27)$$

From (27), the solution of system (7) takes the form

$$x_{6n-j}=\frac{x_{-j}}{\prod_{i=0}^{n-1}\left(\frac{y_{-2}}{x_0}+a_0\left(6\left(i+1\right)-j\right)\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+4-j\right)\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+2-j\right)\right)},$$

$$y_{6n-j}=\frac{y_{-j}}{\prod_{i=0}^{n-1}\left(\frac{z_{-2}}{y_0}+a_1\left(6\left(i+1\right)-j\right)\right)\left(\frac{x_{-2}}{z_0}+a_2\left(6i+4-j\right)\right)\left(\frac{y_{-2}}{x_0}+a_0\left(6i+2-j\right)\right)}, \quad (28)$$

$$z_{6n-j}=\frac{z_{-j}}{\prod_{i=0}^{n-1}\left(\frac{x_{-2}}{z_0}+a_2\left(6\left(i+1\right)-j\right)\right)\left(\frac{y_{-2}}{x_0}+a_0\left(6i+4-j\right)\right)\left(\frac{z_{-2}}{y_0}+a_1\left(6i+2-j\right)\right)},$$

where $j\in\{-3,-2,-1,0,1,2\}$ and $n\in\mathrm{N}_0$.

Case 8: $b_1 \neq 1,\ b_2 \neq 1$ and $b_3 \neq 1$

In this case, the solution of each equation in system (8) is as follows

$$\frac{y_{n-2}}{x_n} = \frac{y_{-2}}{x_0} b_0^n + a_0 \frac{1-b_0^n}{1-b_0},\ \frac{z_{n-2}}{y_n} = \frac{z_{-2}}{y_0} b_1^n + a_1 \frac{1-b_1^n}{1-b_1}, \frac{x_{n-2}}{z_n} = \frac{x_{-2}}{z_0} b_2^n + a_2 \frac{1-b_2^n}{1-b_2},\ n \in \mathrm{N}_0. \quad (29)$$

From (29), we get the solution of the system (7) as

$$\begin{aligned} x_{6n-j} &= \frac{x_{-j}}{\prod_{i=0}^{n-1}\left(q_0 b_0^{6(i+1)-j} + \frac{a_0}{1-b_0}\right)\left(q_1 b_1^{6i+4-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6i+2-j} + \frac{a_2}{1-b_2}\right)}, \\ y_{6n-j} &= \frac{y_{-j}}{\prod_{i=0}^{n-1}\left(q_1 b_1^{6(i+1)-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6i+4-j} + \frac{a_2}{1-b_2}\right)\left(q_0 b_0^{6i+2-j} + \frac{a_0}{1-b_0}\right)}, \quad (30) \\ z_{6n-j} &= \frac{z_{-j}}{\prod_{i=0}^{n-1}\left(q_2 b_2^{6(i+1)-j} + \frac{a_2}{1-b_2}\right)\left(q_0 b_0^{6i+4-j} + \frac{a_0}{1-b_0}\right)\left(q_1 b_1^{6i+2-j} + \frac{a_1}{1-b_1}\right)}, \end{aligned}$$

where $q_0 = \frac{y_{-2}}{x_0} - \frac{a_0}{1-b_0}$, $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$, $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$, $j \in \{-3,-2,-1,0,1,2\}$ and $n \in \mathrm{N}_0$.

Corollary 2. Assume that $a_i + b_i = 1\ (i = 0,1,2)$ and $y_{-2} = x_0,\ z_{-2} = y_0,\ x_{-2} = z_0$. Then every solution of system (7) is six-periodic.

The next theorem establishes the forbidden set of the initial values x_{-i}, y_{-i}, z_{-i} $(i = 0,1,2)$ of (7).

Theorem 3. *Let* $\vec{\theta} = (x_0, x_{-1}, x_{-2}, y_0, y_{-1}, y_{-2}, z_0, z_{-1}, z_{-2})$. Then the forbidden set $\mathcal{F}$ of (7) is given by

$$\begin{aligned} \mathcal{F} = &\bigcup_{i=0}^{1}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ x_{-i} = 0\right\} \bigcup \bigcup_{i=0}^{1}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ y_{-i} = 0\right\} \\ &\bigcup \bigcup_{i=0}^{1}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ z_{-i} = 0\right\} \bigcup \bigcup_{n=0}^{\infty}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ \frac{y_{-2}}{x_0} = \gamma_n^{(0)}\right\} \\ &\bigcup \bigcup_{n=0}^{\infty}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ \frac{z_{-2}}{y_0} = \gamma_n^{(1)}\right\} \bigcup \bigcup_{n=0}^{\infty}\left\{\vec{\theta} \in \mathrm{R}^9 \ :\ \frac{x_{-2}}{z_0} = \gamma_n^{(2)}\right\}, \end{aligned}$$

where

$$\gamma_n^{(i)} = \begin{cases} \frac{a_i\left(1-b_i^{-n}\right)}{1-b_i} &, \ \text{if } b_i \neq 1 \\ -a_i(n+1) &, \ \text{if } b_i = 1 \end{cases},\ i \in \{0,1,2\}.$$

Proof. We observe that if $x_{-i}y_{-i}z_{-i} \neq 0$, $i \in \{0,1\}$, and $x_{-2} = 0$ or $y_{-2} = 0$ or $z_{-2} = 0$, then the solution $\{x_n, y_n, z_n\}_{n=-2}^{\infty}$ can be determined for some $n \in \mathrm{N}_0$, while the solution can not be determined for the case $x_{-i}y_{-i}z_{-i} = 0$, $i \in \{0,1\}$. Thus we can incorporate the case $x_{-i}y_{-i}z_{-i} = 0$, $i \in \{0,1\}$, into the forbidden set. If $x_{-i}y_{-i}z_{-i} \neq 0$, $i \in \{0,1\}$ then we define new variables $u_n^{(0)} = \frac{y_{n-2}}{x_n}$, $u_n^{(1)} = \frac{z_{n-2}}{y_n}$ and $u_n^{(2)} = \frac{x_{n-2}}{z_n}$. In this case, (8) can be written in the form of the linear first order difference equations

$$u_{n+1}^{(i)} = b_i u_n^{(i)} + a_i,\ i \in \{0,1,2\},\ n \in \mathrm{N}_0, \tag{31}$$

which is independent of each other. Now, we indicate that the solutions of system (7) are not defined if and only if

$$a_0 x_n + b_0 y_{n-2} = 0 \text{ or } a_1 y_n + b_1 z_{n-2} = 0 \text{ or } a_2 z_n + b_2 x_{n-2} = 0.$$

That is, the terms x_n, y_n and z_n cannot be calculated for some $n \in \mathrm{N}$, after finite number of terms are calculated. So we can establish our proof on the fact that the solutions of the system are not well-defined in the cases $x_n y_n z_n = 0$ for some $n \in \mathrm{N}$. Let

$$f_i(u) = b_i u + a_i,\ i \in \{0,1,2\}.$$

Then we can write equation (31) as the following

$$u_{n+1}^{(i)} = f_i(u_n^{(i)}),\ i \in \{0,1,2\},\ n \in \mathrm{N}_0,$$

which have the solutions

$$u_n^{(i)} = f_i^n(u_0^{(i)}),\ i \in \{0,1,2\},\ n \in \mathrm{N}_0.$$

Suppose that

$$f_i^{n_0}(u_0^{(i)}) = 0,\ i \in \{0,1,2\},\ n_0 \in \mathrm{N}_0,$$

which implies

$$f_i^{-n_0}(0) = u_0. \tag{32}$$

The inverses of the functions f_i can be calculated as follows

$$f_i^{-1}(v) = \frac{v - a_i}{b_i},\ i \in \{0,1,2\}.$$

Now note that difference equations associated with inverse functions f_i^{-1} are

$$v_{n+1}^{(i)} = \frac{v_n^{(i)} - a_i}{b_i},\ i \in \{0,1,2\},\ n \in \mathrm{N}_0. \tag{33}$$

From (32) and (33), it follows that

$$u_0^{(i)} = f_i^{-n}(0) = \frac{a_i\left(1-b_i^{-n}\right)}{1-b_i},\ i\in\{0,1,2\},\ n\in \mathrm{N},$$

for $b_i \neq 1$. If $b_i = 1$, then from (32) and (33), we have

$$u_0^{(i)} = -a\left(n+1\right),\ i\in\{0,1,2\},\ n\in \mathrm{N}_0.$$

Consequently, we have the eight possible cases in the theorem. So the proof is complete.

From the above theorem, it can be said that if initial values $x_{-i}, y_{-i}, z_{-i} \notin \mathcal{F}$, $\{i = 0,1,2\}$, then every solution of system (7) is well-defined.

Theorem 4. Assume that $b_i \neq 1, (i = 0,1,2)$, and $(x_n, y_n, z_n)_{n\geq -2}$ is a well-defined solution of system (7). Then the following statements hold.

a) If $|b_0| > 1$ and $q_0 \neq 0$ or $|b_1| > 1$ and $q_1 \neq 0$ or $|b_2| > 1$ and $q_2 \neq 0$, then $x_{6n-j} \to 0$, $y_{6n-j} \to 0$ and $z_{6n-j} \to 0$ as $n \to \infty$,

b) If $|b_0| < 1$, $|b_1| < 1$, $|b_2| < 1$ and $\left|\left(\frac{a_0}{1-b_0}\right)\left(\frac{a_1}{1-b_1}\right)\left(\frac{a_2}{1-b_2}\right)\right| < 1$, then $|x_{6n-j}| \to \infty$, $|y_{6n-j}| \to \infty$ and $|z_{6n-j}| \to \infty$ as $n \to \infty$,

c) If $|b_0| < 1$, $|b_1| < 1$, $|b_2| < 1$ and $\left|\left(\frac{a_0}{1-b_0}\right)\left(\frac{a_1}{1-b_1}\right)\left(\frac{a_2}{1-b_2}\right)\right| > 1$, then $|x_{6n-j}| \to 0$, $|y_{6n-j}| \to 0$ and $|z_{6n-j}| \to 0$ as $n \to \infty$,

d) If $|b_0| < 1$, $|b_1| < 1$, $|b_2| < 1$ and $\left(\frac{a_0}{1-b_0}\right)\left(\frac{a_1}{1-b_1}\right)\left(\frac{a_2}{1-b_2}\right) = 1$, then $(x_{6n-j})_{n\in\mathrm{N}_0}$, $(y_{6n-j})_{n\in\mathrm{N}_0}$ and $(z_{6n-j})_{n\in\mathrm{N}_0}$ are convergent,

e) If $|b_0| < 1$, $|b_1| < 1$, $|b_2| < 1$ and $\left(\frac{a_0}{1-b_0}\right)\left(\frac{a_1}{1-b_1}\right)\left(\frac{a_2}{1-b_2}\right) = -1$, then $(x_{12n-j})_{n\in\mathrm{N}_0}$, $(x_{12n+6-j})_{n\in\mathrm{N}_0}$, $(y_{12n-j})_{n\in\mathrm{N}_0}$, $(y_{12n+6-j})_{n\in\mathrm{N}_0}$, $(z_{12n-j})_{n\in\mathrm{N}_0}$ *and* $(z_{12n+6-j})_{n\in\mathrm{N}_0}$ are convergent,

where $q_0 = \frac{y_{-2}}{x_0} - \frac{a_o}{1-b_0}$, $q_1 = \frac{z_{-2}}{y_0} - \frac{a_1}{1-b_1}$ and $q_2 = \frac{x_{-2}}{z_0} - \frac{a_2}{1-b_2}$, $j\in\{-3,-2,-1,0,1,2\}$, $n\in \mathrm{N}_0$.

Proof. We will present the proof for each cases seperately. For the proofs of (a) and (b): Let

$$X_m^{(1)} := \left(q_0 b_0^{6m+6-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+4-j} + \frac{a_1}{1-b_1} \right)\left(q_2 b_2^{6m+2-j} + \frac{a_2}{1-b_2} \right),$$

$$Y_m^{(1)} := \left(q_1 b_1^{6m+6-j} + \frac{a_1}{1-b_1} \right)\left(q_2 b_2^{6m+4-j} + \frac{a_2}{1-b_2} \right)\left(q_0 b_0^{6m+2-j} + \frac{a_0}{1-b_0} \right),$$

$$Z_m^{(1)} := \left(q_2 b_2^{6m+6-j} + \frac{a_2}{1-b_2} \right)\left(q_0 b_0^{6m+4-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+2-j} + \frac{a_1}{1-b_1} \right).$$

After that the result follows from the asumptions in (a). Thus, we obtain

$$\lim_{n\to\infty}\left|x_{6n-j}\right| = \lim_{n\to\infty}\left|y_{6n-j}\right| = \lim_{n\to\infty}\left|z_{6n-j}\right| = 0.$$

As a similar approximation, by the facts in (b) and using formula (30), we have

$$\lim_{n\to\infty}\left|x_{6n-j}\right| = \lim_{n\to\infty}\left|y_{6n-j}\right| = \lim_{n\to\infty}\left|z_{6n-j}\right| = \infty.$$

For the proof of (c): By reconsidering the assumptions in the beginig of the proof, a simple calculation

$$\lim_{n\to\infty}\left|X_m^{(1)}\right| = \lim_{n\to\infty}\left|Y_m^{(1)}\right| = \lim_{n\to\infty}\left|Z_m^{(1)}\right| = \frac{\left|a_0 a_1 a_2\right|}{\left|(1-b_0)(1-b_1)(1-b_2)\right|},$$

gives the result. For the proofs (d) and (e): In fact it will be given only the proof of (d) since (e) can be obtained with the same manner. Again reconsidering the assumptions, it is seen that

$$\begin{aligned}
X_m^{(1)} &= \left(q_0 b_0^{6m+6-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+4-j} + \frac{a_1}{1-b_1} \right)\left(q_2 b_2^{6m+2-j} + \frac{a_2}{1-b_2} \right) \\
&= 1 + \frac{(1-b_0)q_0}{a_0} b_0^{6m+6-j} + \frac{(1-b_1)q_1}{a_1} b_1^{6m+4-j} + \frac{(1-b_2)q_2}{a_2} b_2^{6m+2-j} \\
&\quad + \frac{(1-b_0)(1-b_1)q_0 q_1}{a_0 a_1} b_0^{6m+6-j} b_1^{6m+4-j} + \frac{(1-b_0)(1-b_2)q_0 q_2}{a_0 a_2} b_0^{6m+6-j} b_2^{6m+2-j} \\
&\quad + \frac{(1-b_1)(1-b_2)q_1 q_2}{a_1 a_2} b_1^{6m+4-j} b_2^{6m+2-j} + O(b_0^{6m} b_1^{6m} b_2^{6m}),
\end{aligned} \tag{34}$$

$$
\begin{aligned}
Y_m^{(1)} &= \left(q_1 b_1^{6m+6-j} + \frac{a_1}{1-b_1} \right)\left(q_2 b_2^{6m+4-j} + \frac{a_2}{1-b_2} \right)\left(q_0 b_0^{6m+2-j} + \frac{a_0}{1-b_0} \right) \\
&= 1 + \frac{(1-b_1)q_1}{a_1} b_1^{6m+6-j} + \frac{(1-b_2)q_2}{a_2} b_2^{6m+4-j} + \frac{(1-b_0)q_0}{a_0} b_0^{6m+2-j} \\
&+ \frac{(1-b_1)(1-b_2)q_1 q_2}{a_1 a_2} b_1^{6m+6-j} b_2^{6m+4-j} + \frac{(1-b_0)(1-b_1)q_0 q_1}{a_1 a_0} b_0^{6m+2-j} b_1^{6m+6-j} \\
&+ \frac{(1-b_0)(1-b_2)q_0 q_2}{a_0 a_2} b_0^{6m+2-j} b_2^{6m+4-j} + O(b_0^{6m} b_1^{6m} b_2^{6m}),
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
Z_m^{(1)} &= \left(q_2 b_2^{6m+6-j} + \frac{a_2}{1-b_2} \right)\left(q_0 b_0^{6m+4-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+2-j} + \frac{a_1}{1-b_1} \right) \\
&= 1 + \frac{(1-b_2)q_2}{a_2} b_2^{6m+6-j} + \frac{(1-b_0)q_0}{a_0} b_0^{6m+4-j} + \frac{(1-b_1)q_1}{a_1} b_1^{6m+2-j} \\
&+ \frac{(1-b_0)(1-b_2)q_0 q_2}{a_0 a_2} b_2^{6m+6-j} b_0^{6m+4-j} + \frac{(1-b_0)(1-b_1)q_0 q_1}{a_0 a_1} b_0^{6m+4-j} b_1^{6m+2-j} \\
&+ \frac{(1-b_1)(1-b_2)q_1 q_2}{a_1 a_2} b_1^{6m+2-j} b_2^{6m+6-j} + O(b_0^{6m} b_1^{6m} b_2^{6m}),
\end{aligned}
\tag{36}
$$

for every $j \in \{-3,-2,-1,0,1,2\}$ and sufficiently large m. From (34), (35), (36), the assumption $|b_0|<1,\ |b_1|<1,\ |b_2|<1$, and the proof is completed by a known result on the convergence of products.

Theorem 5. Let at least one of parameters $b_i, (i=0,1,2)$, be one and $(x_n, y_n, z_n)_{n\ge -2}$ be a well-defined solution of system (7). Then $x_{6n-j} \to 0,\ y_{6n-j} \to 0,\ z_{6n-j} \to 0.$

Proof. Let

$$
\begin{aligned}
X_m^{(2)} &:= \left(\frac{y_{-2}}{x_0} + a_0 (6m+6-j) \right)\left(\frac{z_{-2}}{y_0} + a_1 (6m+4-j) \right)\left(\frac{x_{-2}}{z_0} + a_2 (6m+2-j) \right), \\
Y_m^{(2)} &:= \left(\frac{z_{-2}}{y_0} + a_1 (6m+6-j) \right)\left(\frac{x_{-2}}{z_0} + a_2 (6m+4-j) \right)\left(\frac{y_{-2}}{x_0} + a_0 (6m+2-j) \right), \\
Z_m^{(2)} &:= \left(\frac{x_{-2}}{z_0} + a_2 (6m+6-j) \right)\left(\frac{y_{-2}}{x_0} + a_0 (6m+4-j) \right)\left(\frac{z_{-2}}{y_0} + a_1 (6m+2-j) \right),
\end{aligned}
$$

$$X_m^{(3)} := \left(q_0 b_0^{6m+6-j} + \frac{a_0}{1-b_0}\right)\left(\frac{z_{-2}}{y_0} + a_1\left(6m+4-j\right)\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6m+2-j\right)\right),$$

$$Y_m^{(3)} := \left(\frac{z_{-2}}{y_0} + a_1\left(6m+6-j\right)\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6m+4-j\right)\right)\left(q_0 b_0^{6m+2-j} + \frac{a_0}{1-b_0}\right),$$

$$Z_m^{(3)} := \left(\frac{x_{-2}}{z_0} + a_2\left(6m+6-j\right)\right)\left(q_0 b_0^{6m+4-j} + \frac{a_0}{1-b_0}\right)\left(\frac{z_{-2}}{y_0} + a_1\left(6m+2-j\right)\right),$$

$$X_m^{(4)} := \left(\frac{y_{-2}}{x_0} + a_0\left(6m+6-j\right)\right)\left(q_1 b_1^{6m+4-j} + \frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6m+2-j\right)\right),$$

$$Y_m^{(4)} := \left(q_1 b_1^{6m+6-j} + \frac{a_1}{1-b_1}\right)\left(\frac{x_{-2}}{z_0} + a_2\left(6m+4-j\right)\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+2-j\right)\right),$$

$$Z_m^{(4)} := \left(\frac{x_{-2}}{z_0} + a_2\left(6m+6-j\right)\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+4-j\right)\right)\left(q_1 b_1^{6m+2-j} + \frac{a_1}{1-b_1}\right),$$

$$X_m^{(5)} := \left(\frac{y_{-2}}{x_0} + a_0\left(6m+6-j\right)\right)\left(\frac{z_{-2}}{y_0} + a_1\left(6m+4-j\right)\right)\left(q_2 b_2^{6m+2-j} + \frac{a_2}{1-b_2}\right),$$

$$Y_m^{(5)} := \left(\frac{z_{-2}}{y_0} + a_1\left(6m+6-j\right)\right)\left(q_2 b_2^{6m+4-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+2-j\right)\right),$$

$$Z_m^{(5)} := \left(q_2 b_2^{6m+6-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+4-j\right)\right)\left(\frac{z_{-2}}{y_0} + a_1\left(6m+2-j\right)\right),$$

$$X_m^{(6)} := \left(\frac{y_{-2}}{x_0} + a_0\left(6m+6-j\right)\right)\left(q_1 b_1^{6m+4-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6m+2-j} + \frac{a_2}{1-b_2}\right),$$

$$Y_m^{(6)} := \left(q_1 b_1^{6m+6-j} + \frac{a_1}{1-b_1}\right)\left(q_2 b_2^{6m+4-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+2-j\right)\right),$$

$$Z_m^{(6)} := \left(q_2 b_2^{6m+6-j} + \frac{a_2}{1-b_2}\right)\left(\frac{y_{-2}}{x_0} + a_0\left(6m+4-j\right)\right)\left(q_1 b_1^{6m+2-j} + \frac{a_1}{1-b_1}\right),$$

$$X_m^{(7)} := \left(q_0 b_0^{6m+6-j} + \frac{a_0}{1-b_0}\right)\left(\frac{z_{-2}}{y_0} + a_1\left(6m+4-j\right)\right)\left(q_2 b_2^{6m+2-j} + \frac{a_2}{1-b_2}\right),$$

$$Y_m^{(7)} := \left(\frac{z_{-2}}{y_0} + a_1\left(6m+6-j\right)\right)\left(q_2 b_2^{6m+4-j} + \frac{a_2}{1-b_2}\right)\left(q_0 b_0^{6m+2-j} + \frac{a_0}{1-b_0}\right),$$

$$Z_m^{(7)} := \left(q_2 b_2^{6m+6-j} + \frac{a_2}{1-b_2} \right)\left(q_0 b_0^{6m+4-j} + \frac{a_0}{1-b_0} \right)\left(\frac{z_{-2}}{y_0} + a_1 \left(6m+2-j\right) \right).$$

and

$$X_m^{(8)} := \left(q_0 b_0^{6m+6-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+4-j} + \frac{a_1}{1-b_1} \right)\left(\frac{x_{-2}}{z_0} + a_2 \left(6m+2-j\right) \right),$$

$$Y_m^{(8)} := \left(q_1 b_1^{6m+6-j} + \frac{a_1}{1-b_1} \right)\left(\frac{x_{-2}}{z_0} + a_2 \left(6m+4-j\right) \right)\left(q_0 b_0^{6m+2-j} + \frac{a_0}{1-b_0} \right),$$

$$Z_m^{(8)} := \left(\frac{x_{-2}}{z_0} + a_2 \left(6m+6-j\right) \right)\left(q_0 b_0^{6m+4-j} + \frac{a_0}{1-b_0} \right)\left(q_1 b_1^{6m+2-j} + \frac{a_1}{1-b_1} \right).$$

Since $\lim_{n\to\infty}\left|X_m^{(k)}\right| = \lim_{n\to\infty}\left|Y_m^{(k)}\right| = \lim_{n\to\infty}\left|Z_m^{(k)}\right| = \infty, \left(k = 2,3,\ldots,8\right)$, and by (14), (18), (20), (22), (24), (26) and (28), this completes the proof.

3. Conclusion

In this paper, we investigate an extension of the second equation in (1), that is, the system given in (7). After that we reduce this system to the first-order linear equations and then we obtain explicit solutions of the related system. Additionally, we determine the forbidden set of the initial values x_{-i}, y_{-i}, z_{-i} $\left(i = 0,1,2\right)$ and also study asymptotic behavior of the solutions using their explicit forms. Thus, we extend some recent results in the literature.

In the future studies on systems of difference equations, we except that the following topics will bring new insight:

1) The system in (7) can be extended to higher-dimensional systems;
2) The system in (7) can be extended to higher-order systems;
3) and finally, one can investigate behaviors and solubility of these extended systems.

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خلاصة

نحصل في هذا البحث على حلول صريحة لنظام معادلات فرقية ثلاثي الأبعاد له حدود ضربية. و تقوم حلولنا بتوسيع بعض النتائج المتوفرة في المنشورات . كما نقوم باستخدام الأشكال الصريحة للحلول لدراسة السلوك المقارب لحلول النظام حسنة التعريف.