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Controllability of affine control systems on graded Lie groups

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Abstract

This paper is concerned with an affine control system on a manifold which is equivalent by diffeomorphism to an invariant system on a free nilpotent Lie group, if and only if, the vector fields of the system generate graded Lie algebra and the vector fields of the induced invariant system generate free nilpotent Lie algebra.

Keywords: Affine control systems; free nilpotent Lie algebra; graded Lie algebra; invariant control systems.

1. Introduction

An affine control system is a control system of the form

$$\dot{x} = F^0(x) + \sum_{j=1}^{d} u_j F^j(x), \tag{1}$$

where x is a point on an n-dimensional manifold M, the class of admissible controls $u = (u_1, \ldots, u_d)$ to the set of piecewise constant functions with values on $\mathbb{R}^d$, and $F^j 0 \leq j \leq d$ are complete real analytic or smooth vector fields on the manifold M. Let L denote Lie algebra spanned by all iterated brackets of the vector fields $F^0, F^1, \ldots, F^d$. If $F^0, F^1, \ldots, F^d$ are a transitive family on a manifold M and invariant vector fields of the system generate free nilpotent Lie algebra G, then L is diffeomorphic to a connected, simply-connected Lie group G, where the connected, simply-connected Lie group L corresponding to L whose Lie algebra is isomorphic to G and there is a unique subalgebra G of L generated by $\{X^0, X^1, \ldots, X^d\}$ under bracketing and corresponding to this is a closed Lie subgroup G of L. By this diffeomorphism L is related to the Lie algebra of invariant vector fields on G, and all vector fields belonging to L.

The motivation for this work originated in control problems for the system (1). We are interested in the induced system related to the affine control system and constructed when the system is equivalent to a free nilpotent system, i.e., a system of the form (1) described by invariant vector fields which generate a free nilpotent Lie algebra. Therefore, we study the equivalent problem of replacing or approximating (ref. invariant) by (ref. affine). The advantage (ref. invariant) over (ref. affine) is that state space is a free nilpotent Lie group and each X^j defines an invariant vector field on this group.

Affine control systems have nice properties which lead to a very rich theory Kule (2015) for further motivation for the study of such systems. There is variety of results on controllability of affine systems. For example, Jurdjevic & Sallet (1984) have studied controllability of affine control systems on Euclidean spaces, Kara & San Martin (2006) on generalized Heisenberg Lie groups, Kara & Kule (2010) on Carnot groups, Jouan (2010) on homogeneous spaces. In a recent paper, Kule (2015) proves controllability of affine control systems on several classes of Lie groups, including nilpotent, semisimple, reductive, and simply connected ones. These papers give sufficient conditions that the affine control system be equivalent to a linear control system or an invariant control system or a bilinear control system. Also, for a complete review of research on invariant control systems we refer to Sachkov (2000).

On the other hand, Crouch (1990) and Goodman (1976) have studied the theory of graded vector spaces and associated graded nilpotent Lie algebras in the systems theoretic content. Rockland (1987) established the lifting and approximation process in an intrinsic setting. This is accomplished by considering certain filtrations of the Lie algebra generated by vector fields of the system.

The organization of this paper is as follows: we first review, in Section 2, some basic facts regarding affine control systems and associated structures. In Section 3, the classification problem of the affine control system on manifold.

2. Preliminaries

In describing the main results for affine control systems, we shall use the terminology of Ayala & Tirao (1999); Jouan (2010) as far as possible.

On a connected manifold M, consider the system

$$\dot{p} = F(p,u)$$

where u belongs to a subset U of $\mathbb{R}^d$, the mapping F is continuous on $M \times U$ and $F_u = (.,u)$ is for every $u \in U$ a C^∞ vector field on M.

Let L denote a connected Lie group with Lie algebra L. Let us denote by $X(L)$ the set of smooth vector fields on L. It is a Lie algebra for the Lie bracket of vector fields and the normalizer of L in $X(L)$ is by definition

$$\mathrm{N} = \mathrm{norm}_{X(L)}\mathrm{L}\{F \in X(L) | \forall Y \in \mathrm{L} \quad [F,Y] \in \mathrm{L}\}.$$

Thus N is the set of affine vector fields of L. From the Jacobi identity, it clearly is a Lie subalgebra of $X(L)$. We denote by *Aut* (L) the Lie group of all L-automorphisms. Let X be an infinitesimal automorphism of the Lie group L, that is, the flow $(X_t)_{t\in\mathbb{R}}$ induced by the vector field X is a one-parameter subgroup of *Aut* (L). Then, X induces a derivation D= $-ad_X$ on L for D∈*Der* L. This condition on *ad* means

$$DY= -[X,Y]$$

for ∀ $Y \in$ L and verifies X (e)=0, where e is the identity element of L. To be specific, the following theorem hold from Ayala & Tirao (1999).

Theorem 1**.** Let L be a connected Lie group, and L its Lie algebra. Then, N≅$aut(L)$ $\times_s$ L.

Here, $aut(L)$ means the Lie algebra of Aut(L), the Lie group of all automorphisms of L, and $\times_s$ denotes the semidirect product of Lie algebras.

Let X be an infinitesimal automorphism of the Lie group L, that is, the flow $(X_t)_{t\in\mathbb{R}}$ induced by the vector field X is a one-parameter subgroup of Aut(L) and verifies X (e)=0.

The kernel of the mapping $F\rightarrow ad(F)$, from N into Aut(L) is the set of left invariant vector fields. An affine vector field F can be uniquely decomposed into a sum

$$F\text{=X}+Y$$

where X is an infinitesimal automorphism of the Lie group L and Y right invariant. This property is no longer true in the general case, but remains true whenever L is connected.

3. Affine control system

Throughout this section, we assume that the system (1) to be transitive. In this statement, the transitivity assumption means that the evaluation at every point of the Lie algebra generated by the F^j are the full tangent space at the point.

The free nilpotent Lie algebra G on M generators of rank r is the quotient of the free Lie algebra by its $(r+1)$st commutator ideal G_{r+1} generated as follows: G_i= the span of i-1 iterated Lie brackets of basis elements. Free nilpotent Lie algebras add the relations that any iterated Lie bracket of more than r elements vanishes. We say vector fields of G_r have weight r.

Now we shall denote the induced system by

$$\dot{g} = X^0(g) + \sum_{j=1}^{d} u_j X^j(g),\ g \in M, \tag{2}$$

where M is a finite dimensional manifold. The requirements of an induced system, the existence of a transformation $\Psi: M \rightarrow G$ locally carrying flow of (1) onto corresponding flow of (2), are that

$$F^j = X^j + X^j \; 0 \leq j \leq d$$

where X^j are invariant vector fields of F^j such that the Lie algebra G generated by iterated Lie brackets on X^j, $0 \leq j \leq d$ is free nilpotent, and transitive on G. Furthermore, letting G denote the simply connected Lie group associated to G.

We can consider the restriction of the system (1) to its orbit through the initial state, to obtain the induced system on the free nilpotent Lie group. Thus, we are able to obtain the system (1) transitive on the state space M by restricting the system (2) to the maximal connected integral submanifold G, through a given point defined by G, because G is comprised of invariant vector fields.

In what follows, we show how to make an intrinsic construction of a free nilpotent Lie algebra G, as an invariant attached to a filtered Lie algebra L.

Lemma 2. L is graded on the manifold M with gradation defined by the weight of commutators.

Proof. Let L^j denote the subspace of L spanned by commutators of weight j, for $1 \leq j \leq k$. Suppose that L is not graded by the weight of commutators is equivalent to the existence of an element $F \in L^l$ such that F is a linear combination of elements in subspaces L^j, $j \neq l$. We define in that case $L^j(e)$ as the subspace of T_eM spanned by elements of L^j and notice that $L^j(e) \cap L^i(e) = \{0\}$ for $j \neq i$, where e stands for the identity of M.

We define a filtration on L. Suppose L_0 acts as a subalgebra of derivations of L which vanishes at e, and define L_{-1} by

$$L_{-1} = \{X \in L_0 \mid [X,L] \in L_0\}$$

and inductively define L_{-i} by

$$L_{-i} = \{X \in L_{-i+1} \mid [\ldots[X,L]\ldots L] \in L_0\}.$$

Then, each L_{-i} is a subalgebra of L by the Jacobi identity, and by the finite dimensionality of L that the sequence is finite

$$L_{-r-1} \subset L_{-r} \subset \ldots \subset L_{-1} \subset L_0 \subset L.$$

The sequence may end in either $L_{-r-1} = \{0\}$, or $[L_{-r-1}, L] \subset L_{-r-1}$ so that $L_{-r-2} = L_{-r-1}$. In the second case L_{-r-1} is an ideal in L contained in L_0, and so L_{-r-1} consists of zero vector fields on L. We thus shall suppose that $L_{-r-1} = \{0\}$. If $L_0 = L_{-1}$ then L_0 is an ideal in L. Note also that $[L, L_{-i}] \subset L_{-i+1}$, for $i = 1, \ldots, r$.

We let $\tilde{L}^l$ denote the subspace of L spanned by L^j for $j \neq l$. Now we have a sequence of subspaces for each l

$$\tilde{L}^l \cap L_{-r} \subset \tilde{L}^l \cap L_{-r+1} \subset \ldots \subset \tilde{L}^l \cap L_0 \subset \tilde{L}^l.$$

Let X_i^r, $i = 1,...,d_r$ be a basis of $\tilde{L}^l \cap L_{-r}$, and this with elements X_i^{r-1}, $i = 1,...,d_{r-1}$ to a basis for $\tilde{L}^l \cap L_{-r+1}$, and proceed by induction to define a basis for $\tilde{L}^l \cap L_0$ by a basis for $\tilde{L}^l \cap L_{-j}$ to a basis for $\tilde{L}^l \cap L_{-j+1}$ with elements X_i^{j-1}, $i = 1,...,d_{j-1}$. Finally, this basis to a basis for $\tilde{L}^l$ by elements F_i, $i = 1,...,d$.

Let $F = X + X \in L^l$ be written as

$$F = \sum_i \alpha_i X_i + \sum_i \alpha_i^0 X_i^0 + \cdots + \sum_i \alpha_i^r X_i^r \tag{3}$$

for some scalars α_i^j and α_i. Equality (3) evaluated at e becomes $F(e) = \sum_i \alpha_i X_i(e)$, since all other terms vanish at e. If $F(e) \neq 0$, we obtain a contradiction

$$F(e) = \sum_i \alpha_i X_i(e) \in \tilde{L}^l(e), \text{ for } F(e) \in L^l(e).$$

Thus $F(e) = 0$, and since X_i are basis in $\tilde{L}^l \setminus \tilde{L}^l \cap L_0$ we deduce $\alpha_i = 0$, $i = 1,..., d$. It remains to apply the Lie bracket of F with an invariant vector field $Y \in L^m$, to obtain

$$[F,Y] = \sum_i \alpha_i^0 [X_i^0, Y] + \cdots + \sum_i \alpha_i^r [X_i^r, Y].$$

and to evaluate e we get

$$[F,Y](e) = \sum_i \alpha_i^0 [X_i^0, Y](e).$$

Similarly if $[F,Y](e) \neq 0$, then we have a contradiction

$$[F,Y](e) = \sum_i \alpha_i^0 [X_i^0, Y](e) \in \tilde{L}^{l+m}(e) \text{ for } [F,Y](e) \in L^{l+m}(e).$$

Thus $\sum_i \alpha_i^0 X_i^0 \in \tilde{L}^l \cap L_{-1}$ since L is spanned by $F \in L^m$ for $m = 1,\ldots, k$. However, X_i^0 are basis in $\tilde{L}^l \cap L_0 \setminus \tilde{L}^l \cap L_{-1}$ so $\alpha_i^0 = 0$, $i = 1,..., n_0$. Therefore

$$F = \sum_i \alpha_i^1 X_i^1 + \cdots + \sum_i \alpha_i^r X_i^r.$$

If F is supposed to be a sum of terms of weight $\geq k$, then $F = 0$ by above the induction argument. Thus, L is graded by the weight of commutators.

Now, the Lie algebra L has graded Lie algebra structure. Moreover, both L and G should have compatible graded structures, so that the gradation of L

$$L = L_1 \oplus \cdots \oplus L_k$$

is defined by L_1= all linear combinations of $F^0, F^1, \ldots, F^d$ $L_2 = L_1 + [L_1, L_1], \ldots, L_{i+1} = L_i + [L_1, L_i]$. Thus, vector fields of the system (1) are a basis of L_1 and L_1 generates L.

Theorem 3 Given a smooth (analytic) system (1) with commutators of weight $\leq k$ in $F^0, F^1, \ldots, F^d$ spanning T_eM, then there exists a graded lie algebra $L = \sum_{i=1}^{k} \oplus L_i$ generated by vector fields of the system is isomorphic to the free nilpotent Lie algebra G generated by invariant vector fields of the system. Also, a smooth (analytic) submersion $\Psi : W \to U$ where W and U are neighborhoods of e and e_G in M and G respectively, is a diffeomorphism satisfying $\Psi(e) = e_G$. Here, e and e_G are the identity elements in manifold M and simply-connected, connected Lie group G the corresponding G, respectively.

Proof. Firstly, considering the maps γ_i: $L_i \to G_i$ defined by setting $\gamma_i(F_i) = X_i$. Note that due to the graded structure of G this projections are well-defined, and, in particular, independent of any choice of basis for G. Select a graded basis of L expressed as

$$X_1, \ldots, X_{d_1}, \mathrm{X}_{d_1+1}, \ldots, \mathrm{X}_{s_1}, X_{s_1+1}, \ldots, X_{d_k}, \mathrm{X}_{d_k+1}, \ldots, \mathrm{X}_{s_k}.$$

Let $X_{s_{k-1}+1}, \ldots, X_{d_k}, \mathrm{X}_{d_k+1}, \ldots, \mathrm{X}_{s_k}$ be a basis for L^k which any basis, $\mathrm{X}_{d_k+1}, \ldots, \mathrm{X}_{s_k}$ of $L^k \cap L_0$, and let $X_{s_{j-1}+1}, \ldots, X_{d_j}, \mathrm{X}_{d_j+1}, \ldots, \mathrm{X}_{s_j}$ be a basis for L^j, which any basis $\mathrm{X}_{d_j+1}, \ldots, \mathrm{X}_{s_j}$ of $L^j \cap L_0$ for $1 \leq j \leq k$, s_0=0. By Lemma 2, the set of elements constructed in this way gives a basis for L; since $L_0 = \sum_{j=1}^{k} \oplus (L^j \cap L_0)$. Note further that all elements $X_{s_{j-1}+1}, \ldots, \mathrm{X}_{s_j}$ have weight j, $1 \leq j \leq$ k. If $F \in L_i$ then $F = \sum^{d_k}_{i=1} \tau_i X_i + \sum^{s_k}_{i=d_k+1} \tau_i \mathrm{X}_i$ where $\tau_i \in C^\infty(M)$ and $X = \sum^{s_k}_{i=1} \tau_i(e) \gamma_i(F_i)$. Thus γ_i is onto G_i and γ_i is an isomorphism.

Next, let $(t, x) \to (F_i)_t(x)$ be the flow of the (complete) vector fields F_i, and define a map $\Psi : W \to U$ by

$$\psi(e) = (F_1)_t \circ \cdots \circ (F_{s_k})_t (e) = e_G,$$

where W and U are neighborhoods of e and e_G in M and G respectively. Hence Ψ defines a diffeomorphism of M onto G.

Theorem 4. Under the conditions of theorem 3, if the induced invariant system on G is controllable, then the affine system is controllable.

Proof. Let the system (2) be controllable on G and let ϕ be a smooth (analytic) action of G on M, i.e.,

$$\phi : G \times M \to M, \qquad (g, x) \to \phi(g)x.$$

It follows that $g \to \phi(g)e = \phi'(g)$ such that $\phi'(e_G) = e$, thus, the tangent mapping ϕ'_* at e satisfies

$$\phi'_* X = F \circ \phi'.$$

Thus, the vector fields may be any graded basis of L, because $L_0=\{0\}$, i.e., all the vector fields in L_0 vanish at e. This implies that the affine system (1) is under ϕ'_* equivalent to the induced invariant system on G. Hence, the system is controllable.

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تحكيمه أنظمة تحكم تآلفية على زمر لي المدرجة

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خلاصة

يدرس هذا البحث نظام التحكم التآلفي على منطوي، حيث يكافئ هذا النظام، بواسطة تماثل تفاضلي، وذلك إذا وفقط إذا قامت حقول المتجهات للنظام بتوليد جبرية لي مدرجة و قامت حقول المتجهات للنظام اللامتغير المحدث بتوليد جبرية لي متلاشية.