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NSE characterization of projective special linear group $L_3(7)$

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ABSTRACT

Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let nse $(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al and Liu's works, the groups $L_3(2)$, $L_3(4)$ and $L_3(5)$ are unique determined by nse (G). In this paper, we prove that if G is a group such that nse $(G)= nse(L_3(7))$, then $G \cong L_3(7)$.

Keywords: Element order; projective special linear group; Thompson's problem; number of elements of the same order; simple group.

INTRODUCTION

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (Shi, 1989).

Thompson's Problem. Let $T(G) = \{(n, s_n) | n \in \omega(G) \text{ and } s_n \in nse(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

A finite group G is called a simple K_n -group, if G is a simple group with $|\pi(G)| = n$.

It was proved that: Let G be a group and M some simple K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $| = |M|$ and nse $(G) = nse(M)$ (Shao *et al.,* 2009; Shao *et al.,* 2008). And also the group A_{12} is characterizable by order and nse (Liu & Zhang, 2012). Recently, all sporadic simple groups have been proved to be characterizable by nse and order (Asboei *et al.,* 2013).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson's Problem, in other words, it remains only nse (G), whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set nse (G) (Khatami *et al.,* 2011; Shen *et al.,* 2010). The author has proved that the groups $L_3(4)$ and $L_3(5)$ are characterizable by nse (Liu 2013). In this paper, it is shown that the group $L_3(7)$ also can be characterized by nse.

Here we introduce some notations which will be used. Let $a.b$ denote the product of an integer a by an integer b. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let nse $(G) = \{s_k | k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime p such that G contains an element of order p. $L_n(q)$ denotes the projective special linear group of degree n over finite fields of order q. $U_n(q)$ denotes the projective special unitary group of degree n over finite fields of order q. The other notations are standard (Conway *et al.,* 1985).

SOME LEMMAS

Lemma 1. (Frobenius, 1895) Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m || L_m(G)|$.

Lemma 2. (Miller, 1904) Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.

Lemma 3. (Shen *et al.,* 2010) Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.

Lemma 4. (Hall, 1959) Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, (m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:

(1) $q_i^{\beta_i} \equiv 1(\text{mod } p_j)$ for some p_j.

(2) The order of some chief factor of G is divided by $q_i^{\beta_i}$.

Lemma 5. (Shao & Jiang, 2010) Let G be a finite group, $P \in_p^{Syl} (G)$, where $p \in \pi(G)$. Suppose that G has a normal series $K \lhd L \lhd G$ and $p || K|$, then the following statements hold:

(1) $N_{G/K}(PK/K) = N_G(P)K/K$.

(2) If $P \leq L$, then $|G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(G) = n_p(L)$.

(3) If $P \leq L$, then $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, namely, $n_p(L/K)t = n_p(G) = n_p(L)$, for some integer t. In particular, $|N_K(P)|t = |K|$.

To prove $G \cong L_3(7)$, we need the structure of simple K_4-groups.

Lemma 6. (Shi, 1991) Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:

(1) A_7, A_8, A_9 or A_{10}.

(2) M_{11}, M_{12} or J_2.

(3) One of the following:

(a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1, b \geq 1, c \geq 1$, and v is a prime greater than 3.

(b) $L_2(2^m)$, where $2^m - 1 = u, 2^m + 1 = 3t^b$ with $m \geq 2, u$, t are primes, $t > 3, b \geq 1$.

(c) $L_2(3^m)$, where $3^m + 1 = 4t, 3^{m-1} = 2u^c$ or $3^m + 1 = 4t^b, 3^{m-1} = 2u$, with $m \geq 2, u, t$ are odd primes, $b \geq 1, c \geq 1$.

(4) One of the following 28 simple groups: $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), L_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8)$, $U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^2D_4(2)$ or $2F_4(2)'$.

Lemma 7. Let G be a simple K_4-group and $\{19\} \subseteq \pi(G) \subseteq \{2, 3, 7, 19\}$. Then $G \cong L_3(7)$.

Proof. From Lemma 6(1)(2), order consideration rules out this case.

So we consider Lemma 6(3). We will deal with this with the following cases.

Case 1. $G \cong L_2(r)$, where $r \in \{3, 7, 19\}$.

Let $r = 3$. Then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.

Let $r = 7$, then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.

Let $r = 19$, then $|\pi(r^2 - 1)| = 3$. Hence $G \cong L_2(19)$, but $5 \mid |G|$, a contradiction.

Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 7, 19\}$.

Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in N, a contradiction.

Let $u = 7$, then $m = 3$, and $2^3 + 1 = 3t^b$. Thus $t = 3$ and $b = 1$. But $t > 3$, a contradiction.

Let $u = 19$, then $2^m - 1 = 19$. But the equation has no solution in N.

Case 3. $G \cong L_2(3^m)$.

We will consider this case by the following two cases.

Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.

We can suppose that $t \in \{3, 7, 19\}$.

Let $t = 3, 19$, the equation $3^m + 1 = 4t$ has no solution. So we rule out the case.

Let $t = 7$, then $m = 3$ and so $3^3 - 1 = 2.11$, which means $11 \mid |G|$, a contradiction.

Subcase 3.2. $3^m + 1 = 4t^b 3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.

We can suppose that $u \in \{3, 7, 19\}$.

Let $u = 3, 7, 19$, then the equation $3^m - 1 = 2u$ has no solution in N, a contradiction.

In review of Lemma 6(4), order consideration, $G \cong L_3(7)$.

This completes the proof of the Lemma.

MAIN THEOREM AND ITS PROOF

Let G be a group such that nse (G) = nse $(L_3(7))$, and $s_n$be the number of elements of order n By Lemma 3, we have that G is finite. We note that $s_n = k\varphi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\varphi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have

$$\begin{aligned} &\varphi(m) \mid s_m \\ &m \mid \sum_{d \mid m} s_d \end{aligned} \tag{1}$$

Theorem 1. Let G be a group with nse (G) = nse $L_3(7))$ = 1,2793, 52136,117306, 117648, 134064, 156408, 234612, 469224, 59270, where $L_3(7)$ is the projective special linear group of degree 3 over finite field of order 7. Then $G \cong L_3(7)$.

Proof. We prove the theorem by first proving that $\pi(G) \subseteq \{2, 3, 7, 19\}$, secondly showing that $|G| = |L_3(7)|$,and finally conclude that $G \cong L_3(7)$.

By (1), $\pi(G)\{2, 3, 5, 7, 17, 19, 117307, 234613\}$ If $m > 2$, then $\varphi(m)$ is even, then $s_2 = 2793, 2 \in \pi(G)$.

In the following, we prove that $17 \notin \pi(G)$ .If $17 \in \pi(G)$, then by (1), $s_{17} = 592704$. If $2.17 \in \omega(G)$, then by Lemma 1, $2.17 \mid 1 + s_2 + s_{17} + s'_{2.17} s_{2.17} \notin nse(G)$. Therefore $2.17 \notin \omega(G)$. It follows that the Sylow 17-subgroup P_{17} of G acts fixed point freely on the set of elements of order 2 and $|P_{17}| \mid s_2$, a contradiction. Similarly by (1), we can prove that the primes $117307, 234613 \notin \pi(G)$.

Hence we have $\pi(G)\{2, 3, 5, 7, 19\}$. Furthermore, by (1) $s_3 = 52136, s_5 = 134064, 469224$ *or* 592704, $s_7 = 117648$ and $s_{19} = 592704$.

If $2^a \in \omega(G)$, then $\varphi(2^a) = 2^{a-1} \mid s_{2^a}$ and so $0 \leq a \leq 7$.

By Lemma 1, $|P_2| \mid 1 + s_2 + s_{2^2} + ... + s_{2^7}$ and so $|P_2| \mid 2^7$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 4$.

Let exp $(P_3) = 3$. Then by Lemma 1, $|P_3| \mid 1 + s_3$ and $|P_3| \mid 3^3$.

Let exp $(P_3) = 3^2$. Then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2}$ and $|P_3| \mid 3^4$ (when $s_9 = 592704$).

Let exp $(P_3) = 3^3$. Then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3}$ and $|P_3| \mid 3^6$ (when $s_9 = 117306, s_{27} = 134064$).

Let exp $(P_3) = 3^4$. Then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3} + s_{3^4}$ and $|P_3| 3 \wedge 4$ (when $s_9 = 117648$, $s_{27} = 469224$ and $s_{81} = 592704$).

Therefore $|P_3| \mid 3^6$.

If $2^2.3 \in \omega(G)$, then by (Shao & Jiang, 2014), $s_{2^2.3} = 2.s_3.t$ for some integer t. But the equation has no solution since $s_{2^2.3} \in$ nse (G). Therefore $2^2.3 \notin \omega(G)$. Similarly $2.3^3 \notin \omega(G)$.

If $5^a \in \omega(G)$, then $a = 1$.

If $2.5 \in \omega(G)$, then by (Shao & Jiang, 2014), $s_{2.5} = s_5.t$ for some integer t and so $s_{2.5} = s_5$. But by Lemma 1, $2.5 \mid 1 + s_2 + s_5 + s_{2.5}$ (270922, 941242, 1188202), a contradiction. Therefore $2.5 \notin \omega(G)$. Similarly $3.5 \notin \omega(G)$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 4$.

Let exp $(P_7) = 7$. Then by Lemma 1, $|P_7| \mid 1 + s_7$ and $|P_7| \mid 7^6$.

Let exp $(P_7) = 7^2$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2}$ and $|P_7| \mid 7^3$ (when s_{7^2}\{117306, 156408, 234612, 469224,592704\}).

Let exp $(P_7) = 7^3$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ and $|P_7| \mid 7^3$ (when $s_{7^2} \in$ \{ 117306, 156408, 234612, 469224592704\}).

Let exp $(P_7) = 7^4$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3} + s_{7^4}$ and $|P_7| \mid 7^4$.

Therefore $|P_7| \mid 7^6$.

If $3.7 \in \omega(G)$, then by (Shao & Jiang 2014), $s_{3.7} = 2.s_7.t$ for some integer t. But the equation has no solution since $s_{3.7} \in nse(G)$. Therefore $3.7 \notin \omega(G)$. Similarly $4.7 \notin \omega(G)$.

If $19^a \in \omega(G)$, then $1 \leq a \leq 2$. Since $s_{19^2} \notin nse(G)$, then $a = 1$. By Lemma 1, $|P_{19}| \mid 1 + s_{19}$ and $|P_{19}| \mid 19$.

If $2.19 \in \omega(G)$, then by (Shao & Jiang 2014), $s_{2.19} = s_{19}$. By Lemma 1, $2.19 \mid 1 + s_2 + s_{19} + s_{2.19}$, a contradiction. Hence $2.19 \notin \omega(G)$. Similarly $3.19, 5.19, 7.19 \notin \omega(G)$.

To remove the prime 5, we assume that $7 \in \pi(G)$.

If $3, 5, 19 \notin \pi(G)$. Then G is a 2-group. Since $\omega(G) = 8$ and there are exactly ten numbers in nse $(L_3(7))$ and consequently, we have a contradiction.

Let $19 \in \pi(G)$. Then since $|P_{19}| = 19, n_{19} = s_{19}/\varphi(19) = 2^5.3.7^3$ and $7 \in \pi(G)$, a contradiction.

Let $5 \in \pi(G)$. We know that s_5 = 134064, 469224, 592704.

Let $s_5 = 134064$. Then by Lemma 1, $|P_5| \mid 1 + s_5$ and $|P_5| = 5$ Since $n_5 = s_5/\varphi(5) = 2^2.3^2.7^2.19$, a contradiction.

Let $s_5 = 469224$. Then $|P_5| = 5^2$.

If $|P_5| = 5$, then $n_5 = 2.3^2.7^3.19, 7, 19 \in \pi(G)$, a contradiction.

If $|P_5| = 5^2$, then we can assume that $\{2, 3, 5\} \subseteq \pi(G)$. Therefore $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^b.5^2$ where k_1, k_2 ,..., $k_8 a, b$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 2$, then the equation has no solution in N.

Let $s_5 = 592704$. Then $|P_5| = 5$. Since $n_5 = s_5/\varphi(5) = 2^4.3^3.7^3, 7 \in \pi(G)$, a contradiction.

Let $3 \in \pi(G)$.

Let exp $(P_3) = 3$. Then $|P_3| \mid 3^3$.

If $|P_3| = 3$, then since $n_3 = s_3/\varphi(3) = 2^2.7^3.19, 7, 19 \in \pi(G)$, a contradiction.

If $|P_3| = 3^2$, then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^2$ where $k_1 k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 0$, then the equation has no solution in N.

Similarly we can rule out the other case $|P_3| = 3^3$.

Let exp $(P_3) = 3^2$. Then $|P_3| \mid 3^4$ (when $s_9 = 592704$).

f $|P_3| = 3^2$, then 7 or $19 \in \pi(G)$, a contradiction.

If $|P_3| = 3^3$, then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^3$, where $k_1, k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 1$, then the equation has no solution in N.

Similarly we can rule out the case $|P_3| = 3^4$.

Let exp $(P_3) = 3^3$. Then $|P_3| \mid 3^6$ (when $s_9 = 117306, s_{27} = 134064$).

If $|P_3| = 3^3$. Then since $n_3 = s_{3^3}/\varphi(3^3)$, 7 or $19 \in \pi(G)$, a contradiction.

If $|P_3| > 3^3$, we also can rule out.

Let exp $(P_3) = 3^4$.

If $|P_3| = 3^4$, then since $s_{81} = 592704), n_3 = s_{3^4}/\varphi(3^4) = 2^5.7^3, 7 \in \pi(G)$, a contradiction.

If $|P_3| > 3^4$, then by Lemma 2, $s_{3^4} = 3^4.t$ for some integer t. But the equation has no solution since $s_{3^4} \in nse(G)$

Therefore $7 \in \pi(G)$.

If $5.7 \in \omega(G)$, then by Lemma 1, $5.7 \mid 1 + s_5 + s_7 + s_{5.7}$. But $s_{5.7} \notin nse(G)$. So $5.7 \notin \omega(G)$. It follows that the Sylow 5-subgroup of G acts fixed point freely on the set of elements of order 7 and $|P_5| \mid s_7$, a contradiction. So $5 \notin \pi(G)$.

If $19 \in \pi(G)$, then since $n_{19} = s_{19}/\varphi(19) = 2^5.3.7^3, 3, 7 \in \pi(G)$, then we only have to consider two proper sets $\{2, 7\}, \{2, 3, 7\}$ and finally the whole set $\{2, 3, 7, 19\}$.

Case a. $\pi(G) = \{2, 7\}$.

We know that exp $(P_7) = 7, 7^2, 7^3, 7^4$.

Let exp $(P_7) = 7$. Then $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^6$.

Let $|P_7| = 7$. Then since $n_7 = s_7/\varphi(7) = 117648/6 = 2^3.3.19.43, 19 \in \pi(G)$, a contradiction.

Let $|P_7| = 7^2$. Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.49$, where $k_1, k_2, ..., k_8, a$ are non-negative integers and $0 \leq \sum_{k=0}^{8} k_i \leq 0$. So the equation has no solution in N.

If $|P_7| > 7^2$, similarly we can rule out as the case "$\exp(P_7) = 7$ and $|P_7| = 7^2$".

Let exp $(P_7) = 7^2$. Then $|P_7| \mid 7^3$.

Let $|P_7| = 7^2$. Then $s_{7^2} \in \{117306, 134064, 156408, 234612, 469224, 592704\}$. Since $n_7 = s_{7^2}/\varphi(7^2)$, 3 or $19 \in \pi(G)$.

If $|P_7| = 7^3$ Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.7^3$

where $k_1, k_2, ..., k_8, a,$ are non-negative integers and $0 \le \sum_{k=0}^{8} k_i \le 1$. So the equation has no solution in N.

Let exp $(P_7) = 7^3$. Then $|P_7| = 7^3$. Since $s_{7^3} \in \{117306, 134064, 156408, 234612, 469224, 592704\}$ and $n_7 = s_{7^3}/\varphi(7^3)$, 3 or 19 $\in \pi(G)$.

Let exp $(P_7) = 7^4$. Then $|P_7| = 7^4$. Since $s_{7^4} \in \{117306, 156408, 234612, 469224, 592704\}$ and $n_7 = s_{7^3}/\varphi(7^3)$, 3 or 19 $\in \pi(G)$.

Case b. $\pi(G) = \{2, 3, 7\}$.

Let exp $(P_7) = 7$. Then $|P_7| \mid 1 + s_7$ and so $|P_7| \mid 7^6$.

Let $|P_7| = 7$. Then since $n_7 = s_7/\varphi(7) = 117648/6 = 2^3.3.19.43$, $19 \in \pi(G)$,a contradiction.

Let $|P_7| = 7^2$ Then $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^b.49$ where $k_1, k_2, ..., k_8, a, b$ are non-negative integers and $0 \le \sum_{k=0}^{8} k_i \le 5$. Since $1876896 \le |G| = 2^a.3^b.7^2 \le 1876896 + 5.592704$, the equation has no solution in N.

If $|P_7| > 7^2$, similarly we can rule out as above.

Let exp $(P_7) = 7^2$. Then $|P_7| \mid 7^3$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^b.49$ where $k_1, k_2, ..., k_8, a, b$ are non-negative integers and $0 \le \sum_{k=0}^{8} k_i \le 6$. Since $1876896 \le |G| = 2^a.3^b.7^2 \le 1876896 + 6.592704$ It follows that the number of Sylow 2-subgroups of G is 1, 3, 7, 9, 15, 21, 27, 49, 63, 81, 147, 189, 243, 441, 189, 243, 441, 567, 729, 1323, 1701, 3969, 5103, 11907, 35721 and so the number of elements of order 2 is 1, 3, 7, 9, 15, 21, 27, 49, 63, 81, 147, 189, 243, 441, 189, 243, 441, 567, 729, 1323, 1701, 3969, 5103, 11907, 35721, but none of which belongs to nse (G).

Let exp $(P_7) = 7^3$ Then $|P_7| \mid 7^3$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^b.49$ where $k_1, k_2, ..., k_8, a, b$ are non-negative integers and $0 \le \sum_{k=0}^{8} k_i \le 8$. We can rule out this case as "$\exp(P_7) = 7^2$".

Let $\exp(P_7) = 7^4$ Then $|P_7| \mid 7^4$ and $1876896 + 52136k_1 + 117306k_2 + 117648k_3 + 134064k_4 + 156408k_5 + 234612k_6 + 469224k_7 + 592704k_8 = 2^a.3^b.7^4$ where $k_1, k_2, ..., k_8, a, b$ are non-negative integers and $0 \le \sum_{k=0}^{8} k_i \le 9$. We can rule out this case as "$\exp(P_7) = 7^2$".

Case c. $\pi(G) = \{2, 3, 7, 19\}$.

In the following, we first show that $|G| = 2^5.3^2.7^3.19, or |G| = 2^6.3^2.7^3.19$, secondly prove that $G \cong L_3(7)$.

Step 1. $|G| = 2^5.3^2.7^3.19$, $or|G| = 2^6.3^2.7^3.19$·

From the above arguments, we have that $|P_{19}| = 19$.

We know $7.19 \notin \omega(G)$. It follows that the Sylow 7-subgroup of G acts fixed freely on the set of elements of order 19 and so $|P_7| \mid s_{19}$. Therefore $|P_7| \mid 7^3$. Similarly $2.19 \notin \omega(G)$ and $|P_2| \mid 2^6$;$3.7 \notin \omega(G)$ and $|P_3| \mid 3^2$.

Therefore we have $|G| = 2^m.3^n.7^p.19$. But $1876896 = 2^5.3^2.7^3.19 \leq 2^m.3^n.7^p.19$. Therefore $|G| = 2^5.3^2.7^3.19$, $|G| = 2^5.3^3.7^3.19$, $|G| = 2^6.3^2.7^3.19$, or $|G| = 2^6.3^3.7^3.19$.

Step 2. $G \cong L_3(7)$

First show that there is no group such that $|G| = 2^6.3^2.7^3.19$, and nse(G)= $nse(L_3(7))$. Then get the result by (Shao *et al.,* 2008).

Let $|G| = 2^6.3^2.7^3.19$ and nse (G)= $nse(L_3(7))$.

Let G be soluble. Since $s_{19} = 592704$, then $n_{19} = s_{19}/\varphi(19) = 2^5.3.7^3$. By Lemma 4, $3 \equiv 1$ (mod 19), a contradiction. So G is insoluble.

Therefore we can suppose that G has a normal series $1 \leq K \leq L \leq G$ such that L/K is isomorphic to a simple K_i-group with i= 3, 4 as $19^2 \nmid |G|$.

If L/K is isomorphic to a simple K_3 -group, then from (Herzog, 1968), $L/K \cong L_2(7), L_2(8)$.

From (Conway *et al.,* 1985), $n_7(L/K) = n_7(L_2(7)) = 8$, and so $n_7(G) = 8t$ for some integer t and $7 \times t$. Hence the number of elements of order 7 in $G^{is:} s_7 = 8t.6 = 48t$. Since $s_7 \in$ nse(G), then $s_7 = 117648$ and so $t = 2451$. Therefore $3.19.43 \mid |K| \mid 2^2.3.7^2.19$, which is a contradiction. For $L_2(8)$, similarly we can rule out.

If L/K is isomorphic to a K_4-group, then by Lemma 7, $L/K \cong L_3(7)$.

Let $\bar{G} = G/K$ and $\bar{L} = L/K$. Then

$L_3(7) \leq \bar{L} \cong \bar{L}C_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq \bar{G}/C_{\bar{G}}(\bar{L}) = N_{\bar{G}}(\bar{L})/C_{\bar{G}}(\bar{L}) \leq Aut(\bar{L})$

Set $M = \{xK \mid xK \in C_{\bar{G}}(\bar{L})\}$, then $G/M \cong \bar{G}/C_{\bar{G}}(\bar{L})$ and so $L_3(7) \leq G/M \leq$Aut $L_3(7)$

Therefore $G/M \cong L_3(7), G/M \cong 2.L_3(7), G/M \cong 3.L_3(7)$. or $G/M \cong S_3.L_3(7)$.

If $G/M \cong L_3(7)$, then order consideration $|M| = 2$. It follows that M is a normal subgroup generated by a 2-central element of G. So there exists an element of order 2.19, which is a contradiction.

If $G/M \cong 3.L_3(7)$, or $G/M \cong S_3.L_3(7)$, order consideration rules out these cases.

If $G/M \cong 2.L_3(7)$, then $M = 1$. But nse $(2.L_3(7)) \neq$nse (G).

Therefore $|G| = 2^5.3^2.7^3.19 = |L_3(7)|$and by assumption, nse $(G) = nse(L_3(7))$, then by (Shao *et al.,* 2008), $G \cong L_3(7)$.

This completes the proof of the theorem.

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REFERENCES

Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. & Wilson. R. A. 1985. Atlas of finite groups: Maximal subgroups and ordinary characters for simple groups. Clarendon Press, London

Frobenius, G. 1895. Verallgemeinerung des Sylowschen Satze. Berliner Sitz 981-993.

Hall, M. 1959. The theory of groups. AMS, Macmilan.

Herzog, M. 1968. Finite simple groups divisible by only three primes. Journal of Algebra 10: 383-388.

Asboei, A. K., Amiri, S. S. S., Iranmanesh, A. & Tehranian, A. 2013. A characterization of sporadic simple groups by NSE and order, Journal of Algebra and its Applications **12**(2): 1250158 (3 pages).

Khatami, M., Khosravi, B. & Akhlaghi, Z. 2011. A new characterization for some linear groups. Monatshefte fur Mathematik **163**: 39-50.

Liu, S. & Zhang, R. 2012. A new characterization of A_{12}.Mathematical Sciences **6**: 30.

Liu, S. 2013. A characterization of $L_3(4)$. ScienceAsia. **39**: 436-439.

Liu, S. A. Characterization of projective special group$L_3(5)$. Italian Journal of Pure and Applied Mathematics. In press.

Miller, G. 1904. Addition to a theorem due to Frobenius. Bulletin of the Americal Mathematical Society **11**(1): 6-7.

Shao, C. G., Shi, W. J. & Jiang, Q. H. 2009. A new characterization of simple K_3-groups. Advances in Mathematics **38**: 327-330(in chinese).

Shao, C. G., Shi, W. J. & Jiang, Q. H. 2008. A new characterization of simple K_4 -groups. Frontiers Mathematics in China. **3** (3): 355-370.

Shao, C. & Jiang, Q. 2010. A new charcterization of Mathieu groups. Archivum Mathematicum (BRNO) **46**: 13-23.

Shao, C. & Jiang, Q. 2014. A new characterization of some linear groups by NSE. Journal of Algebra and its Applications **13**(2): 1350094.

Shen, R., Shao, C. G. Jiang, Q. H., Shi, W. J. & Mazurov, V. 2010. A new characterization of A_5. Monatshefte fur Mathematik **160**: 337-341.

Shi, W. J. 1989. A new characterization of the sporadic simple groups. Proceeding of the 1987 Singapore Conference on Group Theory. Walter de Gruyter, pp. 531-540.

Shi, W. J. 1991. On simple K_4-group. Chinese Science Bulletin **36**(17): 1281-1283(in chinese).

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خلاصة

لتكن G زمرة وتكون W, G مجموعة عناصر مرتبات G نثبت في هذا البحث بأنه إذا كان $nse(G) = nse(L_3(7))$ فإن G تكافئ $(L_3(7))$.