

Minimal locating-paired-dominating sets in triangular and king grids

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Abstract

Let $G = (V, E)$ be a finite or infinite graph. A set $S \subseteq V$ is paired-dominating if S induces a matching in G and dominates all vertices of G . A set $S \subseteq V$ is locating if for any two distinct vertices $u, v \in V \setminus S, N(u) \cap S \neq N(v) \cap S$, where $N(u)$ indicates the set of neighbor vertices from u . We find the minimal density of locating-paired-dominating sets in the infinite triangular grid, which is equal to $4/15$. We also present bounds for the minimal density D of locating-paired-dominating sets in the infinite king grid, which is $3/14 \leq D \leq 2/9$.

Keywords: King grid; locating-dominating set; triangular grid.

1. Introduction

The problem of placing monitoring devices in a system when every monitor is paired with a backup monitor serves as a motivation for the concept of locating paired-dominating sets (LPDS). The concept of locating paired-dominating sets was introduced in (McCoy & Henning, 2009) as an extension of paired-dominating sets described in (Haynes & Slater, 1995) (Haynes, Hedetniemi, & Slater, 1998).

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of undirected edges. A complete graph is a graph such that all the vertices are connected to each other. $H(V', E')$ is a subgraph of a graph G if $V' \subseteq V$ and $E' \subseteq E$. A set S of vertices, where $S \subseteq V$, is called a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S . The open neighborhood of a vertex u is defined as $N(u) = \{v \in V \mid (u, v) \in E\}$, the closed neighborhood of a vertex u is defined as $N[u] = N(u) \cup \{u\}$. The closed neighborhood of a set of vertices T is defined as $\bigcup_{u \in T} N[u]$. A dominating set is called locating-dominating if for any pair of distinct vertices $u, v \in V \setminus S$, it holds that $N(u) \cap S \neq N(v) \cap S$.

A subset of edges in a graph G is independent if no two edges in the subset share a vertex of G . A matching in a graph G is a set of independent edges in G . A perfect matching M in G is a matching in G such that every vertex of G is incident to an edge of M . A paired-dominating set, abbreviated PDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to some vertex in S , and the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). A set that is PDS and for any distinct vertices $u, v \in V \setminus S$, it holds that $N(u) \cap S \neq N(v) \cap S$ is called a locating-paired-dominating set (LPDS). The size of the smallest locating-paired-dominating set in a graph G is denoted as $\gamma_{pr}^l(G)$.

Consider a complete graph on 4 vertices. Possible perfect matchings are illustrated in Figure 1.

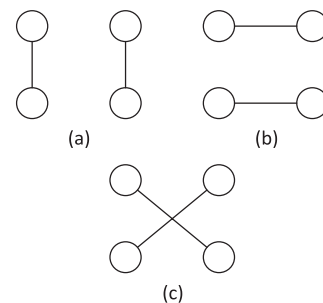


Fig. 1. Possible perfect matching.

We illustrate the notion of PDS and LPDS in Figure 2, where (a) is a graph with 8 vertices, and the PDS vertices are highlighted in (b). In Figure 2b is an example of PDS, which is also LPDS of the graph from Figure 2a. The set is PDS since the set $S = \{a, c, f, h\}$ induces a perfect matching. In addition, it is LPDS since $N(b) \cap S \neq N(d) \cap S \neq N(e) \cap S \neq N(g) \cap S$.

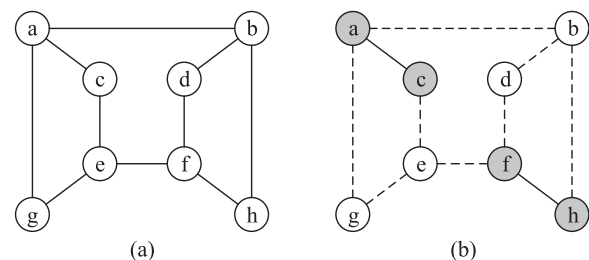


Fig. 2. (a) Graph, (b) PDS and LPDS of (a)

The triangular grid \mathbb{T} is an infinite 6-regular graph, where the vertex set is $V = \{v(i, j) = i(1, 0) + j(1/2, \sqrt{3}/2) \mid i, j \in \mathbb{Z}\}$. Two vertices in \mathbb{T} are adjacent if their Euclidean distance is equal to 1. Let T_n be the set of vertices $v(i, j)$ with $|i| \leq n$.

and $|j| \leq n$. Clearly $|T_n| = (2n + 1)^2$. The king grid \mathbb{K} is an infinite 8-regular graph where $V = \mathbb{Z}^2$ and two vertices are adjacent if their Euclidean distance is equal to 1 or $\sqrt{2}$. Let K_n be the set of vertices $(i, j) \in \mathbb{Z}^2$ with $|i| \leq n$ and $|j| \leq n$. Clearly $|K_n| = (2n + 1)^2$.

For notation and graph terminology, we in general follow (Haynes, Hedetniemi, & Slater, 1998). The problem of finding an optimal locating-dominating set in an arbitrary graph is known to be NP-complete. This problem has been studied for infinite grids in (Honkala, 2006) (Honkala & Laihonen, 2006) (Slater, 2002). The basic properties of locating paired-dominating sets in graphs and bounds of their size can be found in (McCoy & Henning, 2009). Characterization of optimal locating-paired-dominating sets in infinite square grids is discussed in (Niepel, 2015).

The rest of the paper is organized as follows. Section 2 discusses the share of a vertex and the density of a set of vertices. Finding the locating-paired-dominating set in triangular grid is given in Section 3. Finally, in Section 4, we illustrate how to find the LPDS in the king grid.

2. The Share

We consider the density of locating-paired-dominating sets in the triangular grid \mathbb{T} and king grid \mathbb{K} . The density of the dominating set S in \mathbb{T} is defined in (Honkala & Laihonen, 2006) as follows:

$$D(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap T_n|}{|T_n|} \quad (1)$$

The density of the dominating set $S \in \mathbb{K}$ is defined in (Honkala & Laihonen, 2006) as:

$$D(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap K_n|}{|K_n|}$$

If the locating-paired-dominating set in an infinite graph G has the smallest possible density, then it is called an *optimal* LPDS. Following Slater (2002), the *share* of the vertex in a dominating set is defined as follows:

$$sh(S) = \sum_{u \in N[v]} \frac{1}{|S \cap N[u]|}$$

We define the average share of vertices in triangular grid \mathbb{T} as:

$$Avg(S, T) = \limsup_{n \rightarrow \infty} \frac{\sum_{u \in S \cap T_n} sh(u)}{|S \cap T_n|}$$

Observation 1. *There is a relationship between the density of dominating set S in triangular grid \mathbb{T} and the average share of all vertices in the corresponding paired-dominating set.*

$$D(S) = \frac{1}{Avg(S, T)}$$

Proof. Let S be a dominating set in T . Each vertex in the set T_{n-1} is dominated by a vertex from $S \cap T_n$. Counting

the contribution of vertices in T_{n-1} for shares of vertices in $S \cap T_n$ we have

$$|T_{n-1}| \leq \sum_{u \in S \cap T_n} sh(u)$$

All vertices that are dominated by vertices from $S \cap T_n$ form a subset of T_{n+1} , so we have

$$\sum_{u \in S \cap T_n} sh(u) \leq |T_{n+1}|$$

As

$$\limsup_{n \rightarrow \infty} \frac{|T_{n+1}| - |T_{n-1}|}{|T_n|} = 0$$

Thus,

$$\limsup_{n \rightarrow \infty} \frac{|T_n| - \sum_{u \in S \cap T_n} sh(u)}{|T_n|} = 0$$

So, in expression (1) we can replace $|T_n|$ with $\sum_{u \in S \cap T_n} sh(u)$ and we have

$$D(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap T_n|}{\sum_{u \in S \cap T_n} sh(u)}$$

A similar result can be proved analogously for the density of a dominating set in the king grid \mathbb{K} . If we need to minimize the density of an LPDS in a grid, then we should maximize the average share of its vertices.

Let S be an LPDS in graph G , and C_i be a connected component of $G[S]$. We define a *pattern* P_i as a weighted graph that is created by vertices of neighborhood $N[C_i]$. The weight of each vertex v in P_i is $a(v) = |S \cap N[v]|$. The share of the pattern we define as the average share of dominating vertices in the pattern.

In this paper, we find the minimal density of locating-paired-dominating sets in the triangular grid \mathbb{T} and give bounds for the minimal density in the king grid \mathbb{K} .

3. LPDS in a Triangular Grid

Let S be an LPDS in the triangular grid \mathbb{T} and M a matching with locating dominating property. Each edge of matching M is included in a pattern, so we can consider the LPDS as a (partly overlapping) tiling of the triangulated plane with patterns. In Figure 3, we illustrate two different tilings of \mathbb{T} , where the black vertices are in LPDS. Figure 4 shows that in both cases, the plane is covered by patterns of the same types A and B, respectively, where the shaded vertices are the weighted black vertices. The shares of both patterns are equal to $15/4 = 3.75$. Using densities of these LPDS, we bound the minimal density of an LPDS in the infinite triangular grid as $\min D(S) \leq 4/15$.

Now we show that we can restrict ourselves to patterns created by single edges of the perfect matching.

Lemma 1. *Let P be a pattern created by a component with at least three vertices. Then the share of P is less than $15/4 = 3.75$.*

Proof. Suppose that C contains exactly three vertices. We have the following constraints for the vertices in the patterns. Each vertex of C is adjacent to at most one vertex of weight 1. If two vertices of a pattern are adjacent to the

same pair of vertices in C then at least one of these vertices should have weight at least 3. These constraints follow from the locating property of dominating set S . All possible patterns consisting of three vertices with maximal possible shares 3.5, 3.666, and 3.444 are illustrated in Figure 5. Adding more points to component C leads in decreasing the share of the corresponding pattern.

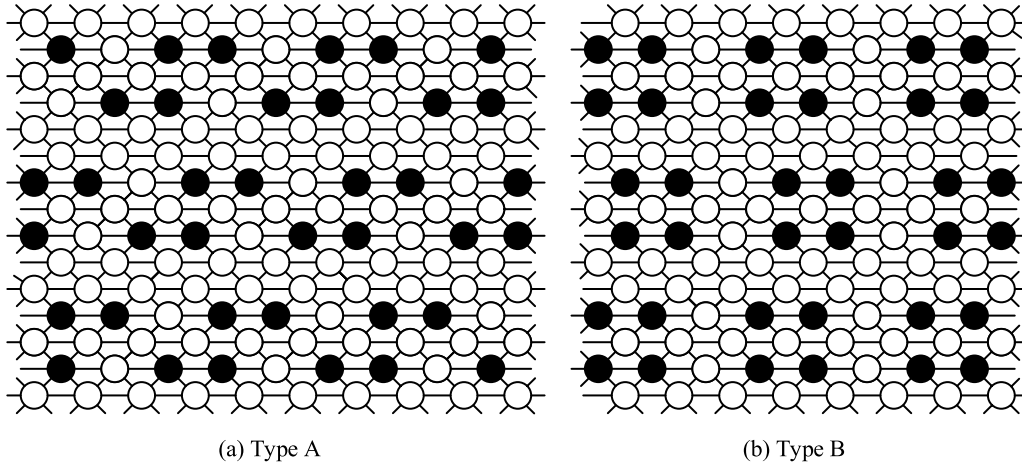


Fig. 3. Tiling with patterns of type A and B in \mathbb{T} .

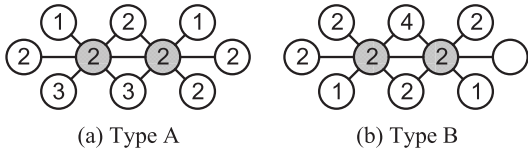


Fig. 4. Types of patterns in \mathbb{T} .

From the above lemma it follows that all patterns in an optimal LPDS are created by edges of matching which is induced by dominating set S . In the rest of the paper, we shall suppose that matching of LPDS is induced.

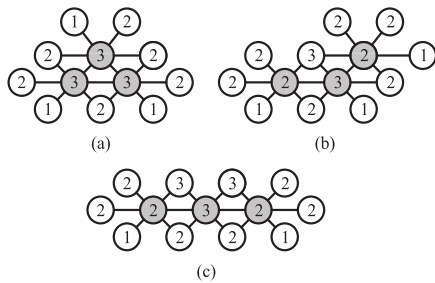


Fig. 5. Different possible patterns.

Observation 2. *Let S be an LPDS in \mathbb{T} and the corresponding matching. Each pattern created by an edge has at most two vertices with weight 1.*

Proof. Suppose that a pattern contains more than two vertices with assigned weight 1. Then, there exist two vertices with weight 1 adjacent to the same vertex in set S which violates the locating property.

Let P be a pattern corresponding edge (u, v) of the matching corresponding to a paired-dominating set and (a_1, a_2, \dots, a_k) be the weights of vertices in $N[u] - N[v]$, (b_1, b_2, \dots, b_l) be the weights of vertices in $N[v] - N[u]$, and (c_1, c_2, \dots, c_m) be the weights of vertices in $N[u] \cap N[v]$. The tuple $(a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_l; c_1, c_2, \dots, c_m)$ is what we call a *signature* of edge (u, v) .

From the definition of the share, it follows:

$$sh(P) = \frac{1}{2} \left(\sum_{i=0}^k \frac{1}{a_i} + \sum_{i=0}^l \frac{1}{b_i} + 2 \sum_{i=0}^m \frac{1}{c_i} \right)$$

The vertices in $N(u) - N(v)$ and $N(v) - N(u)$ are called *pendant* vertices of the pattern, and the vertices in $N(u) \cap N(v)$ are called *internal* vertices of the pattern.

Observation 3. *Each pattern has at least one weight greater or equal to 3 assigned to at least one of its internal vertices.*

Proof. Let (u, v) be the edge inducing a pattern and w, t are internal vertices adjacent to both u and v . Then, $\{u, v\}$ is a subset of $N[w]$ and $N[t]$. As $N[w] \cap S \neq N[t] \cap S$, there exists at least one vertex in S different from u, v and adjacent to w or t , so at least one of the weights $a(w)$ or $a(t)$ is at least 3.

Observation 4. *Let S be an LPDS in \mathbb{T} . The largest possible share of a pattern created by S is $23/6$, the second largest possible share is $15/4$, and the third largest possible share is $11/3$.*

Proof. This follows from Observation 2 and 3. The possible pattern with signature $(1, 2, 2; 1, 2, 2; 2, 2, 2, 3)$ has the maximum share equal to $23/6$. The pattern with signature $(1, 2, 2; 1, 2, 3; 2, 2, 2, 3)$ or $(1, 2, 2; 1, 2, 2; 2, 2, 2, 4)$ has the second maximum share equal to $15/4$. The pattern with signature $(1, 2, 3; 1, 2, 3; 2, 2, 2, 3)$, $(1, 3, 3; 1, 2, 2; 2, 2, 2, 3)$, $(1, 2, 3; 1, 2, 2; 2, 2, 2, 4)$, or $(1, 2, 2; 1, 2, 2; 2, 2, 3, 3)$ has the third maximum share equal to $11/3$. That is, the share is maximized if the weight values are the smallest possible. Furthermore, any pattern created by edge (u, v) , where $u, v \in V$, in a triangular grid \mathbb{T} cannot contain an internal vertex with weight 5 or pendant vertex with weight 4. Otherwise, one of the vertices u or v is adjacent to another vertex in S , and the corresponding matching is not induced.

These values of share are used in the following lemma.

Lemma 2. Let S be an LPDS of \mathbb{T} , then the density of the set S is at least $4/15$.

Proof. Let \mathbb{T} be the tiling of a grid corresponding to set S . If all patterns in \mathbb{T} have shares at most $15/4$, then the density of S is at least $4/15$.

Suppose that \mathbb{T} contains patterns with signature $(1, 2, 2; 1, 2, 2; 2, 2, 2, 3)$ and share $23/6$. To show that the average share of all patterns in \mathbb{T} is at most $15/4$ we construct an auxiliary digraph D . The nodes of this digraph correspond to patterns of \mathbb{T} . Let p and r be nodes in D corresponding to patterns P and R of tiling \mathbb{T} , respectively. The pair $[p, r]$ is an arc in D if and only if the pattern P contains exactly one internal vertex with weight 3, which is a pendant vertex of pattern R . The second internal vertex of P has a weight equal to 2.

Let p_0 be a node corresponding to a pattern P_0 with signature $(1, 2, 2; 1, 2, 2; 2, 2, 2, 3)$. From the above, it follows that the node p_0 has one outgoing arc and no incoming arcs. Thus, p_0 has $indeg(p_0) = 0$ and $outdeg(p_0) = 1$, where $indeg$ means the number of incoming arcs and $outdeg$ means the number of outgoing arcs. In what follows, we shall estimate the average value of shares of patterns that correspond to nodes of a connected component containing node p_0 . We define a contribution of a node u as the difference of the share of the pattern corresponding to u and the target value $15/4$.

The contribution of a node u corresponding to a pattern with signature $(1, 2, 2; 1, 2, 2; 2, 2, 2, 3)$ is $1/12$ and $outdeg(u) - indeg(u) = 1$. The contribution of a node u corresponding to a pattern with signature $(1, 2, 3; 1, 2, 2; 2, 2, 2, 3)$ is 0 and $outdeg(u) - indeg(u) = 0$. The contribution of nodes corresponding to patterns with signatures $(1, 2, 3; 1, 2, 3; 2, 2, 2, 3)$ is 0 and $outdeg(u) - indeg(u) = 0$.

The contribution of nodes corresponding to patterns with signatures $(1, 3, 3; 1, 2, 2; 2, 2, 2, 3)$, $(1, 2, 3; 1, 2, 2; 2, 2, 2, 4)$, or $(1, 2, 3; 1, 2, 2; 2, 2, 3, 3)$ is less or equal to $-1/12$ and their $outdeg(u) - indeg(u) = -1$. We can observe that the input degree of a node u corresponding to a pattern with k pendant vertices with weight 3 is equal to k , and the contribution of a node u ($contrib(u)$) is less or equal $(outdeg(u) - indeg(u)) \times 1/12$. It is known that for each finite directed graph C holds the followings.

$$\sum_{u \in C} outdeg(u) = \sum_{u \in C} indeg(u)$$

That is,

$$\sum_{u \in C} outdeg(u) - \sum_{u \in C} indeg(u) = 0$$

We get inequality,

$$\begin{aligned} & \sum_{u \in C} contrib(u) \\ & \leq \sum_{u \in C} \left(\frac{1}{12} \times (outdeg(u) - indeg(u)) \right) \\ & \leq \frac{1}{12} \times \left(\sum_{u \in C} outdeg(u) - \sum_{u \in C} indeg(u) \right) = 0 \end{aligned}$$

From the above, it follows that the summation of contributions of all vertices in the connected component containing node p_0 is less than or equal to 0. So, the average value of shares of all corresponding patterns is not larger than $15/4$.

We can conclude that the average share of patterns created by S is asymptotically not greater than $15/4$. The previous examples and lemma will conclude to the following theorem.

Theorem 1. The minimal possible density of an LPDS in the triangular grid is equal to $4/15$.

4. LPDS in King Grid

Similar to the previous section, we try to find the minimal possible density for an LPDS in a king grid. To estimate the minimal density of an LPDS in the king grid, \mathbb{K} , we can use the tiling as shown in Figure 6. As all patterns in the tiling have share with signature $(1, 2, 2, 2, 3; 1, 2, 2, 2, 3; 2, 2, 2, 3, 3)$ the average share is $9/2$. Thus, it means that the density of the LPDS is equal to $2/9$. The pattern of type B in Figure 7 shows the corresponding pattern associated with this tiling.

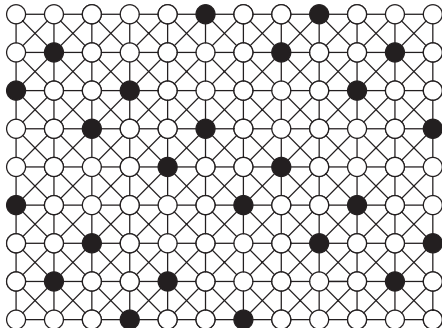


Fig. 6. Valid tiling in \mathbb{K} .

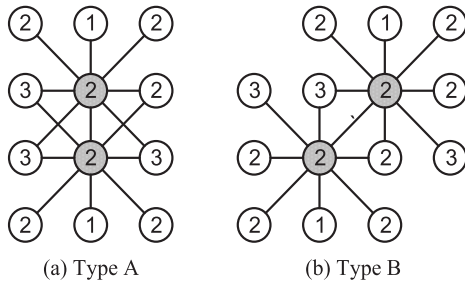


Fig. 7. Patterns types in \mathbb{K} .

In the king grid with an LPDS, the following two types of patterns are possible.

- Patterns created by horizontal or vertical edges with 6 pendant vertices, and four internal vertices. We shall classify such patterns as type A as illustrated in Figure 7.
- Patterns created by diagonal edges with 10 pendant vertices, and two internal vertices. We shall classify such patterns as type B as illustrated in Figure 7.

Now, we shall discuss maximum possible shares of patterns in the king grid \mathbb{K} .

Observation 5. *Let S be an LPDS in \mathbb{K} and M be the corresponding matching. Then each pattern created by an edge in M has at most two pendant vertices with weight 1.*

Proof. We use the same argument as in Observation 1.

Lemma 3. *Let S be an LPDS and M be the corresponding matching, each pattern of type A created by S has share ≤ 4.5 .*

Proof. Let P be a pattern of type A, then P contains 4 internal vertices. From the locating property it follows that at most, one of the internal vertices has weight 2, and the remaining internal vertices have weights at least 3. The maximum possible share of a pattern of type A corresponds to signature (1, 2, 2; 1, 2, 2; 2, 2, 2, 3, 3, 3) and its share is 4.5.

From Lemma 3 it follows that using patterns of type A in the tiling cannot improve the estimation of the density of an LPDS that is obtained by tiling in Figure 6. In the following lemma, we estimate the maximum possible share of a pattern of type B that corresponds to an LPDS in \mathbb{K} .

Lemma 4. *Let be an LPDS in \mathbb{K} and M be the corresponding matching. If a matching edge creates a pattern of type B, then it has at least two pendant vertices with weight 3.*

Proof. Let us assume that the matching edge (u, v) in M creates a pattern of type B, as shown in Figure 8. It follows from Observation 5 that at most, one pendant vertex adjacent to a dominating vertex u can have weight 1.

In Figure 8, we label pendant vertices adjacent to u as a, b, c, d , and e , and possible positions of dominating vertices as 1, 2, 3, 4, 5, 6, and 7.

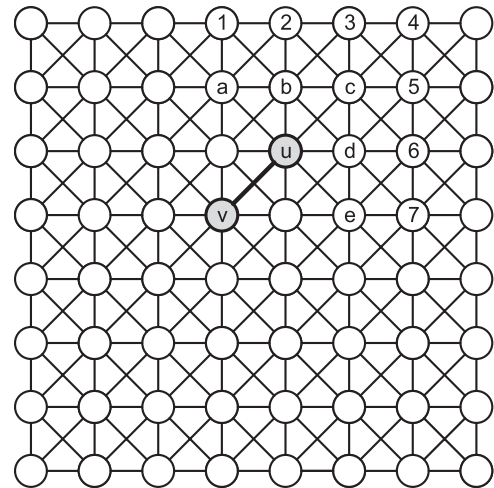


Fig. 8. Part of the proof of Lemma 4.

We can assume that the weight of vertex b is not equal to 1 (otherwise, apply the following argument to vertices c, d , and e). We show that at least one of the vertices a, b, c should have weight at least 3. Suppose that all vertices a, b, c have weight less or equal to 2. We can consider the following three cases:

- If vertex 1 is a dominating vertex, then both a and b are dominated with the same set $S = \{u, 1\}$, which is a contradiction with the locating property.
- If vertex 2 is a dominating vertex, then vertices a, b, c have the same set $S = \{u, 2\}$ of dominating vertices which violates the locating property.
- If vertex 3 is a dominating vertex, then vertices b and c have the same dominating vertices which violates the locating property.

From the above, it follows that if vertex b does not have weight 1, then at least one of the vertices a, b, c has weight at least 3. We can apply the same argument for pendant vertices in the neighborhood of vertex v to finish the proof. From Lemma 4 and the locating property of a dominating set, it follows that a pattern of type B has at most one internal vertex with weight 2, at most 2 pendant vertices have weight 1, and at most 6 pendant vertices have weight 2. So, the

largest possible share value is attained for signature (1, 2, 2, 2, 3; 1, 2, 2, 2, 3; 2, 2, 2, 3) and is equal to $14/3$. Thus, we have the following corollary:

Corollary 1. *Let \mathcal{P} be a pattern created by an LPDS in the king grid. The share of the pattern is not greater than $14/3$.*

The tiling of type B from Figure 7 creates an LPDS with density $2/9$. From Corollary 1 it follows that the density of any LPDS in the king grid is at least $3/14$. So, we can conclude the following:

Theorem 2. *Let D be the density of an optimal LPDS in the king grid. Then, the density is $3/14 \leq D \leq 2/9$.*

We finish our discussion about the king grid with the following:

Conjecture 1. *The density of an optimal LPDS in the king grid is equal to $2/9$.*

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الحد الأدنى من مجموعات تحديد موقع سيادية النقاط في شبكات king والشبكات المثلثية

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الملخص

في علم الحاسوب، يمكن التعبير عن الشبكات من خلال الرسم البياني والذي يدرس في حقل نظرية الرسم البياني. وهناك العديد من الأبحاث التي تدرس سيادية النقاط في شبكات الحاسوب. ومن هذا المنطلق، فقد قمنا بدراسة النقاط ذو سيادية مقترنة في بعض الشبكات المثلثية وشبكات ال King الغير محدودة معتمدة على بناء الأشكال المبلطة. وفي هذا البحث، نبرهن أن كثافتها هي الأقرب نظرياً لأقل حد أدنى تم العثور عليه.