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Variants of *R*-weakly commuting mappings and common fixed point theorems in intuitionistic fuzzy metric spaces

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ABSTRACT

In this paper, we utilize the notion of common limit range property with variants of *R*-weakly commuting mappings and obtain some common fixed point theorems in intuitionistic fuzzy metric spaces. We give some illustrative examples which support our results. Consequently a host of common fixed theorems are generalized and improved.

Keywords: Intuitionistic fuzzy metric space; variants of *R*-weakly commuting mapping; common limit range property; property (E.A); non-compatible mappings.

INTRODUCTION

The concept of fuzzy sets was initially investigated by Zadeh (1965) as a new way to represent vagueness in everyday life. As a generalization of fuzzy sets, Atanassov (1986) introduced the idea of intuitionistic fuzzy set. Further, Çoker & Demirsi (1996) and Çoker (1997) defined the topology of intuitionistic fuzzy sets. Samanta & Mondal (1997, 2002) introduced the definition of the intuitionistic gradation of openness. Park (2004) introduced and discussed a notion of intuitionistic fuzzy metric spaces (briefly, IFM-spaces), which is based both on the idea of intuitionistic fuzzy sets and the concept of a fuzzy metric space given by George & Veeramani (1994).

Motivated by the idea of intuitionistic fuzzy sets, Alaca *et al.* (2006) defined the notion of IFM-spaces as Park (2004) with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to

Kramosil & Michalek (1975). They (Alaca *et al.,* 2006) proved Intuitionistic fuzzy Banach and Intuitionistic fuzzy Edelstein contraction theorems with the different definition of Cauchy sequences and completeness than the ones given in Park (2004). Turkoglu *et al.* (2006) extended the definition of compatible mappings and its variants namely compatible mappings of types (α) and (β) to IFM-spaces which is equivalent to compatible mappings under some conditions. By using the same, they (Turkoglu *et al.,* 2006) and proved common fixed point theorems in IFM-spaces. Alaca *et al.* (2008) defined the concept of compatible mappings type (*I*) and (*II*) and proved some fixed point results for four mappings in IFM-spaces. Alaca (2006) weakened the notion of compatibility by introducing weakly compatible mappings in framework of IFM-spaces and showed that every pair of compatible mappings is weakly compatible but the converse is not true. Many mathematicians obtained several fixed point theorems in IFM-spaces employing different contractive conditions (Turkoglu *et al.,* 2006; Alaca *et al.,* 2008; Ćirić *et al.,* 2008; Kutukcu *et al.,* 2007; Huang *et al.,* 2010; Pant *et al.,* 2010; Sharma & Deshpande, 2009; Chauhan *et al.,* 2013). Recently, Kumar & Kutukcu (2010) proved some common fixed point theorems for expansion mappings in IFM-spaces.

Kumar (2009) utilized the notion of property (E.A) due to Aamri & Moutawakil (2002) in IFM-spaces and improved the results of Alaca *et al.* (2006) and others. Since the property (E.A) always requires the completeness (or closedness) of the underlying subspaces for the existence of coincidence point, hence Sintunavarat & Kumam (2011) coined the idea of 'common limit range property' which relaxes the requirement of completeness (or closedness) of the underlying subspace. Most recently, Chauhan *et al.* (2013) extended the notion of common limit range property in IFM-spaces and proved some fixed point theorems.

The object of this paper is to obtain a common fixed point theorem in IFM-space by using common limit range property with variants of *R*-weakly commuting mappings. We furnish some examples to show the validity of the main results. Further, we highlight the importance of our main result over several recent literature.

PRELIMINARIES

Definition 2.1. (Schweizer & Sklar, 1983) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0, 1]$;

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norm are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2 [Schweizer & Sklar, 1983] A binary operation $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-conorm if $\diamond$ is satisfying the following conditions:

(1) $\diamond$ is commutative and associative;

(2) $\diamond$ is continuous;

(3) $a \diamond 0 = a$ for all $a \in [0, 1]$;

(4) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Examples of t-conorm are $a \diamond b = \max\{a, b\}$ and $a \diamond b = \min\{1, a + b\}$.

Remark 2.1. The concepts of t-norm and t-conorm are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger (1942) in his study of statistical metric spaces.

Atanassov (1986); Kramosil & Michalek (1975) and Alaca *et al.* (2006) gave the following definition.

Ddefinition 2. 3. A 5-tuple $(X, M, N, *, \diamond)$ is said to be an IFM-space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$

(1) $M(x, y, t) + N(x, y, t) \leq 1$;

(2) $M(x, y, 0) = 0$;

(3) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(4) $M(x, y, t) = M(y, x, t)$;

(5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X, s, t > 0$;

(6) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is left continuous;

(7) $\lim_{t \to \infty} M(x, y, t) = 1$;

(8) $N(x, y, 0) = 1$;

(9) $N(x, y, t) = 0$ for all $t > 0$ iff $x = y$;

(10) $N(x, y, t) = N(y, x, t)$;

(11) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;

(12) $N(x, y, \cdot) : (0, \infty) \to [0, 1]$ is right continuous;

(13) $\lim_{t \to \infty} N(x, y, t) = 0$.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2. 2. Every fuzzy metric space $(X, M, *)$ is an IFM-space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e., $x \diamond y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in X$.

Example 1 (Park, 2004) (Induced intuitionistic fuzzy metric) Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy set on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in N$. Then $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

Remark 2.3 (Park, 2004). Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \diamond b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norms and continuous t-conorms. In the above example by taking $h = k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then $(X, M, N, *, \diamond)$ is an IFM-space induced by the metric d. It is obvious that $N(x, y, t) = 1 - M(x, y, t)$.

Remark 2. 4. In an IFM-space $(X, M, N, *, \diamond)$, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.

Let f and g be two mappings from an IFM-space $(X, M, N, *, \diamond)$ into itself. The mappings f and g are said to be

(1) commuting, if $gfx = fgx$, for all $x \in X$.

(2) weakly commuting, if

$$M(fgx, gfx, t) \geq M(fx, gx, t), N(fgx, gfx, t) \leq N(fx, gx, t),$$

for all $x \in X$ and $t > 0$.

(3) R-weakly commuting on X, if there exists a positive real number R such that

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), N(fgx, gfx, t) \leq N\left(fx, gx, \frac{t}{R}\right),$$

for all $x \in X$ and $t > 0$.

(4) pointwise R-weakly commuting on X if given x in X and $t > 0$, there exists $R > 0$ such that

$$M(fgx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), N(fgx, gfx, t) \leq N\left(fx, gx, \frac{t}{R}\right).$$

(5) R-weakly commuting mappings of type (A_g), if there exists some $R > 0$ such that

$$M(ffx, gfx, t) \geq M\left(fx, gx, \frac{t}{R}\right), N(ffx, gfx, t) \leq N\left(fx, gx, \frac{t}{R}\right),$$

for all $x \in X$ and $t > 0$.

(6) R-weakly commuting mappings of type (A_f), if there exists some $R > 0$ such that

$$M(fgx, ggx, t) \geq M\left(fx, gx, \frac{t}{R}\right), N(fgx, ggx, t) \leq N\left(fx, gx, \frac{t}{R}\right),$$

for all $x \in X$ and $t > 0$.

(7) R-weakly commuting mappings of type (P), if there exists some $R > 0$ such that

$$M(ffx, ggx, t) \geq M\left(fx, gx, \frac{t}{R}\right), N(ffx, ggx, t) \leq N\left(fx, gx, \frac{t}{R}\right),$$

for all $x \in X$ and $t > 0$.

Notice that every pair of weakly commuting mappings is R-weakly commuting with $R = 1$. Moreover, at coincidence points, all the notions of R-weak commutativity and R-weak commutativity of types (A_g), (A_f) and (P) coincide. Our next examples show that all the variants of R-weak commutativity are distinct.

Example 2. Let $(X, M, N, *, \diamond)$ be an IFM-space, where $X = [1, \infty)$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$, $N(x, y, t) = \dfrac{|x - y|}{t + |x - y|}$ for all $t > 0$ and $x, y \in X$. Define the self mappings f and g by $f(x) = 2x - 1$ and $g(x) = x^2$ for all $x \in X$. Then

$$M(fgx, gfx, t) = M\left(\frac{t}{t + 2 \mid x - 1 \mid^2}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{2 \mid x - 1 \mid^2}{t + 2 \mid x - 1 \mid^2}\right),$$

$$M(ffx, gfx, t) = M\left(\frac{t}{t + 4 \mid x - 1 \mid^2}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{4 \mid x - 1 \mid^2}{t + 4 \mid x - 1 \mid^2}\right),$$

$$M\left(fx, gx, \frac{t}{R}\right) = M\left(\frac{\frac{t}{R}}{\frac{t}{R} + \mid x - 1 \mid^2}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{\mid x - 1 \mid^2}{tR + \mid x - 1 \mid^2}\right).$$

Therefore, we conclude that

(1) for $R = 2$, the mappings f and g are R-weakly commuting, but neither weakly commuting mappings nor R-weakly commuting mappings of type (A_g).

(2) for $R = 4$, the mappings f and g are R-weakly commuting mappings of type (A_g), but not weakly commuting mappings.

Example 3. Let $(X, M, N, *, \diamond)$ be an IFM-space, where $X = [0, 1]$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$, $N(x, y, t) = \dfrac{|x - y|}{t + |x - y|}$ for all $t > 0$ and $x, y \in X$. Define the self mappings f and g by $f(x) = x$ and $g(x) = x^2$ for all $x \in X$. Then

$$M(ffx, gfx, t) = M\left(\frac{t}{t + \mid x(x - 1) \mid}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{\mid x(x - 1) \mid}{t + \mid x(x - 1) \mid}\right),$$

$$M(fgx, ggx, t) = M\left(\frac{t}{t + \mid x^2(x - 1)(x + 1) \mid}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{\mid x^2(x - 1)(x + 1) \mid}{t + \mid x^2(x - 1)(x + 1) \mid}\right),$$

$$M(ffx, ggx, t) = M\left(\frac{t}{t+ \mid x(x-1)(x^2+x+1) \mid}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{\mid x(x-1)(x^2+x+1) \mid}{t+ \mid x(x-1)(x^2+x+1) \mid}\right),$$

$$M\left(fx, gx, \frac{t}{R}\right) = M\left(\frac{\frac{t}{R}}{\frac{t}{R}+ \mid x(x-1) \mid}\right),$$

$$N(fgx, gfx, t) = N\left(\frac{\mid x(x-1) \mid}{\frac{t}{R}+ \mid x(x-1) \mid}\right).$$

We conclude the following:

(1) the pair (f, g) is R-weakly commuting for all positive real values of R,

(2) for $R = 3$, the pair (f, g) is R-weakly commuting of type (A_g), R-weakly commuting of type (A_f) and R-weakly commuting of the type (P),

(3) for $R = 2$, the pair (f, g) is R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f), but not R-weakly commuting of type (P) (for this take $x = \frac{3}{4}$).

Definition 2.5. (Kumar, 2009). A pair (f, g) of self mappings of an IFM-space $(X, M, N, *, \diamond)$ is said to satisfy the property (E.A), if there exits a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$.

Definition 2.6. (Chauhan *et al.*, 2013). A pair (f, g) of self mappings of an IFM-space $(X, M, N, *, \diamond)$ is said to satisfy the (CLRg) property, if there exits a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$.

Example 4. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, where $X = [0, 2]$ and $M(x, y, t) = \frac{t}{t + |x - y|}$, $N(x, y, t) = \frac{|x - y|}{t + |x - y|}$ for all $x, y \in X$ and $t > 0$. Define the self mappings f and g by

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ \frac{1}{2}, & \text{if } 1 < x \leq 2. \end{cases} \qquad g(x) = \begin{cases} 2 - x, & \text{if } 0 \leq x < 1; \\ \frac{1}{4}, & \text{if } 1 \leq x < 2. \end{cases}$$

Consider a sequence $\{x_n\} = \{1 - 1n\}_{n \in N}$ in X, one can verify that the pair (f, g) enjoys the property (E.A) as

$$\lim_{n\to\infty} f(1 - 1n) = 1 (= z) = \lim_{n\to\infty} g(1 - 1n).$$

However, there does not exist any point u in X for which $z = gu$.

Example 5. In the setting of Example (4), replace the mapping g (besides retaining the rest):

$$g(x) = \begin{cases} 2 - x, & \text{if } 0 \leq x \leq 1; \\ \frac{1}{4}, & \text{if } 1 < x < 2. \end{cases}$$

If we consider the sequence as in Example (4), then one can verify that the pair (f, g) satisfy the (CLRg) property as

$$\lim_{n\to\infty} f\left(1 - \frac{1}{n}\right) = g(1) = \lim_{n\to\infty} g\left(1 - \frac{1}{n}\right).$$

Notice that the (CLRg) property implies the property (E.A), but the converse implication is not true in general.

Lemma 2. 1. (Alaca *et al.,* 2008). Let $(X, M, N, *, \diamond)$ be an IFM-space. There exists a constant $k \in (0, 1)$ such that

$$M(x, y, kt) \geq M(x, y, t) \quad and \quad N(x, y, kt) \leq N(x, y, t),$$

for some $x, y \in X$ and $t > 0$, then $x = y$.

RESULTS

Theorem 3. 1. Let f and g be two self mappings of an IFM-space $(X, M, N, *, \diamond)$ with $t * t \geq t$ and $(1 - t) \diamond (1 - t) \leq 1 - t$ for all $t \in [0, 1]$. Suppose that

(1) the pair (f, g) enjoys the (CLRg) property,

(2) there exists a constant $k \in (0, 1)$ such that

$$(1) \qquad M(fx, fy, kt) \geq \left\{ \begin{array}{c} M(gx, gy, t) * M(fx, gx, t) * M(fy, gy, t) \\ * M(fx, gy, t) * M(fy, gx, t) \end{array} \right\}$$

and

$$(2) \qquad N(fx, fy, kt) \leq \left\{ \begin{array}{c} N(gx, gy, t) \diamond N(fx, gx, t) \diamond N(fy, gy, t) \\ \diamond N(fx, gy, t) \diamond N(fy, gx, t) \end{array} \right\},$$

for all $x, y \in X$ and $t > 0$.

Then the pair (f, g) has a point of coincidence. If the mappings f and g are either R-weakly commuting or R-weakly commuting mappings of type (A_g) or R-weakly commuting mappings of type (A_f) or R-weakly commuting mappings of type (P), then f and g have a unique common fixed point in X.

proof. If the pair (f, g) satisfies the (CLRg) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$. We assert that $fu = gu$. On using inequalities (1)-(2) with $x = u$, $y = x_n$, we get

$$M(fu, fx_n, kt) \geq \left\{ \begin{array}{c} M(gu, gx_n, t) * M(fu, gu, t) * M(fx_n, gx_n, t) \\ * M(fu, gx_n, t) * M(fx_n, gu, t) \end{array} \right\}$$

and

$$N(fu, fx_n, kt) \leq \left\{ \begin{array}{c} N(gu, gx_n, t) \diamond N(fu, gu, t) \diamond N(fx_n, gx_n, t) \\ \diamond N(fu, gx_n, t) \diamond N(fx_n, gu, t) \end{array} \right\},$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} M(fu, gu, kt) &\geq \left\{ \begin{array}{c} M(gu, gu, t) * M(fu, gu, t) * M(gu, gu, t) \\ * M(fu, gu, t) * M(gu, gu, t) \end{array} \right\} \\ &= \{1 * M(fu, gu, t) * 1 * M(fu, gu, t) * 1\} \\ &\geq M(fu, gu, t) \end{aligned}$$

and

$$N(fu, gu, kt) \leq \left\{ \begin{array}{c} N(gu, gu, t) \diamond N(fu, gu, t) \diamond N(gu, gu, t) \\ \diamond N(fu, gu, t) \diamond N(gu, gu, t) \end{array} \right\}$$
$$= \{0 \diamond N(fu, gu, t) \diamond 0 \diamond N(fu, gu, t) \diamond 0\}$$
$$\leq N(fu, gu, t).$$

Owing to Lemma 2.1, we have $fu = gu$.

Suppose that the mappings f and g are R-weakly commuting, there exists some $R > 0$ such that

$$M(fgu, gfu, t) \geq M\left(fu, gu, \frac{t}{R}\right), N(fgu, gfu, t) \leq N\left(fu, gu, \frac{t}{R}\right),$$

for $t > 0$. It implies $fgu = gfu$. Therefore, $ffu = fgu = gfu = ggu$. Next we assert that $ffu = fu$. On using inequalities (1)-(2) with $x = u$, $y = fu$, we have

$$M(fu, ffu, kt) \geq \left\{ \begin{array}{c} M(gu, gfu, t) * M(fu, gu, t) * M(ffu, gfu, t) \\ * M(fu, gfu, t) * M(ffu, gu, t) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} M(fu, ffu, t) * M(fu, fu, t) * M(ffu, ffu, t) \\ * M(fu, ffu, t) * M(ffu, fu, t) \end{array} \right\}$$
$$= \{M(fu, ffu, t) * 1 * 1 * M(fu, ffu, t) * M(ffu, fu, t)\}$$
$$\geq M(fu, ffu, t)$$

$$N(fu, fx_n, kt) \leq \left\{ \begin{array}{c} N(gu, gfu, t) \diamond N(fu, gu, t) \diamond N(ffu, gfu, t) \\ \diamond N(fu, gfu, t) \diamond N(ffu, gu, t) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} N(fu, ffu, t) \diamond N(fu, fu, t) \diamond N(ffu, ffu, t) \\ \diamond N(fu, ffu, t) \diamond N(ffu, fu, t) \end{array} \right\}$$
$$= \{N(fu, ffu, t) \diamond 0 \diamond 0 \diamond N(fu, ffu, t) \diamond N(ffu, fu, t)\}$$
$$\leq N(fu, ffu, t).$$

On employing Lemma 2.1, we have $fu = ffu = gfu$. Hence fu is a common fixed point of f and g.

Next suppose that f and g are R-weakly commuting of type (A_g), we have

$$M(ffu, gfu, t) \geq M\left(fu, gu, \frac{t}{R}\right), N(ffu, gfu, t) \leq N\left(fu, gu, \frac{t}{R}\right),$$

for $t > 0$, i.e., $ffu = fgu = ggu = gfu$. Putting $x = u$ and $y = fu$ in inequalities (1)-(2), we can get $fu = ffu = gfu$. Therefore, fu is a common fixed point of the f and g.

Next assume that f and g are R-weakly commuting of type (A_f), we have

$$M(fgu, ggu, t) \geq M\left(fu, gu, \frac{t}{R}\right), N(fgu, ggu, t) \leq N\left(fu, gu, \frac{t}{R}\right),$$

for $t > 0$, i.e., $fgu = ffu = gfu = ggu$. Similarly using inequalities (1)-(2) with $x = u$, $y = fu$, we can obtain $fu = ffu = gfu$ which shows that fu is a common fixed point of the f and g.

Finally, suppose that f and g are R-weakly commuting of type (P), we have

$$M(ffu, ggu, t) \geq M\left(fu, gu, \frac{t}{R}\right), N(ffu, ggu, t) \leq N\left(fu, gu, \frac{t}{R}\right),$$

for $t > 0$, i.e., $ffu = fgu = gfu = ggu$. Again using inequalities (1)-(2) with $x = u$, $y = fu$, similarly we can get $fu = ffu = gfu$. Hence fu is a common fixed point of the f and g.

Uniqueness of the common fixed point is an easy consequence of inequalities (1)-(2).

Now, we give an example which illustrates Theorem 3.1.

Example 6. Let $X = [2, 20)$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if t>0;} \\ 0, & \text{if t=0.} \end{cases} \quad N(x, y, t) = \begin{cases} \dfrac{|x - y|}{t + |x - y|}, & \text{if t>0;} \\ 1, & \text{if t=0.} \end{cases}$$

for all $x, y \in X$. Clearly $(X, M, N, *, \diamond)$ be an IFM-space, where $*$ and $\diamond$ are continuous t-norm ($a * b = \min\{a, b\}$) and continuous t-conorm ($a \diamond b = \max\{a, b\}$) respectively. Define the self mappings f and g by

$$f(x) = \begin{cases} 2, & \text{if x=2 or x>5;} \\ 8, & \text{if 2<x≤5.} \end{cases} \qquad g(x) = \begin{cases} 2, & \text{if x=2;} \\ 6, & \text{if 2<x≤5;} \\ \dfrac{x+1}{3}, & \text{if x>5.} \end{cases}$$

Consider the sequence $\{x_n\} = \left\{5 + \frac{1}{n}\right\}$ (or $\{x_n\} = 2$), the the pair (f, g) satisfies the property (E.A)

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g(2).$$

Thus $f(X) = \{2, 8\} \nsubseteq [2, 7) = g(X)$. By a routine calculation, one can easily verify inequalities (1)-(2). Then f and g satisfy all the conditions of Theorem 3.1 and have a unique common fixed point at $x = 2$. It can also be verified that f and g are R-weakly commuting or R-weakly commuting mappings of type (A_g) or R-weakly commuting mappings of type (A_f) or R-weakly commuting mappings of type (P).

Theorem 3.2. Let f and g be two self mappings of an IFM-space $(X, M, N, *, \diamond)$ with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq 1-t$ for all $t \in [0, 1]$ satisfying inequalities (1)-(2). Suppose that the pair (f, g) satisfies the property (E.A) and $g(X)$ is a closed subset of X. Then the pair (f, g) has a point of coincidence. If the mappings f and g are either R-weakly commuting or R-weakly commuting mappings of type (A_g) or R-weakly commuting mappings of type (A_f) or R-weakly commuting mappings of type (P), then f and g have a unique common fixed point in X.

Proof. Since the pair (f, g) satisfies the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$. It is assumed that $g(X)$ is a closed subset of X, therefore $\lim_{n\to\infty} gx_n = z \in g(X)$. Then there exists a point $u \in X$ such that $z = gu$ and then f and g satisfy the (CLRg) property. By Theorem 3.1, we can prove that f and g have a unique common fixed point.

Example 7. In the setting of Example 6, replace the mapping g by the following, besides retaining the rest:

$$g(x) = \begin{cases} 2, & \text{if x=2;} \\ 7, & \text{if 2x≤5;} \\ x+13, & \text{if x5.} \end{cases}$$

Also $g(X) = [2, 7]$ which is closed subset of X. Thus all the conditions of Theorem 3.2 are satisfied and 2 is a unique common fixed point of f and g. Here, it may be pointed out that Theorem 3.1 is not applicable to this example as $g(X)$ is a closed subset of X. Also, notice that all the mappings in this example are even discontinuous at their unique common fixed point 2.

Corollary 3.1. Let f and g be two self mappings of an IFM-space $(X, M, N, *, \diamond)$ with $t * t \geq t$ and $(1-t) \diamond (1-t) \leq 1-t$ for all $t \in [0, 1]$ satisfying inequalities (1)-(2). Suppose that the mappings f and g are non-compatible and $g(X)$ is a closed subset of X. Then the pair (f, g) has a point of coincidence. If the mappings f and g are either R-weakly commuting or R-weakly commuting mappings of type (A_g) or R-weakly commuting mappings of type (A_f) or R-weakly commuting mappings of type (P) then f and g have a unique common fixed point in X.

CONCLUSION

Theorem 3.1 improved the result of Abu-Donia & Nase (2008); Huang *et al.* (2010) and Kumar (2009) without any requirements of completeness of the whole space (or underlying subspace), continuity of the mapping and containment of ranges amongst involved mappings. Theorem 3.2 is obtained for a pair of mappings under property (E.A) which also generalized and extended a host of previously known results.

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تباينات التطبيقات ضعيفة الابدال و مبرهنات النقطة الثابتة المشتركة في الفضاءات المقاسية الحدسية المشوشة

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خلاصة

نقوم في هذا البحث باستخدام خاصية نهاية المدى المشترك للتطبيقات ضعيفة الابدال وذلك للحصول على مبرهنات النقطة الثابتة المشتركة في الفضاءات المقاسية الحدسية المشوشة. كما نقوم برد عدد من الأمثلة التي تشرح نتائجنا وتدعم صحتها. بناء على ذلك، نقوم بتعميم وتحسين عدد من مبرهنات النقطة الثابتة المشتركة.