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Strongly almost summable sequences of order α

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ABSTRACT

In this paper, we introduce the concept $\hat{S}_{\lambda}^{\alpha}-$ statistical convergence of order α. Also some relations between $\hat{S}_{\lambda}^{\alpha}$ -statistical convergence of order α and strong $\hat{w}_{p}^{\beta}(\mu)$ -summability of order β are given. Furthermore some relations between the spaces $\hat{w}_{(p)}^{\alpha}[\lambda,f]$ and $\hat{S}_{\lambda}^{\alpha}$ are examined.

Keywords: Statistical convergence; almost convergence; Cesàro summability.

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INTRODUCTION

The idea of statistical convergence was given by Zygmund (1979) in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951) and later reintroduced by Schoenberg (1959) independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor (1988); Dutta (2009, 2010); Dutta & Bilgin (2011); Dutta *et al.* (2010); Et & Nuray (2001); Güngör *et al.* (2004); Fridy (1985); Kolk (1991); Miller & Orhan (2001); Mursaleen (2000); Rath & Tripathy (1994); Šalát (1980); Savaş (2000); Tripathy & Dutta (2012) and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Cech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

Let w denote the set of all real sequences $x = (x_n)$. By ℓ_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_n)$ normed by

$\|x\| = \sup_n |x_n|$, respectively. A linear functional L on ℓ_∞ is said to be a Banach limit if it has the properties

i) $L(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),

ii) $L(e) = 1$, where $e = (1, 1, \ldots)$,

iii) $L(Dx) = L(x)$,

where D is the shift operator defined by $(Dx_n) = (x_{n+1})$ Banach (1955).

Let B be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $L(x) = L$ for all $L \in B$. Lorentz (1948) has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \ldots + x_{m+k}}{k+1} \to L \quad as\ k \to \infty, \text{ uniformly in } m.$$

Subsequently Banach limits and almost convergence have been studied by Çolak & Çakar (1989).

Let f denote the set of all almost convergent sequences. We write $f - \lim x = L$ if x is almost convergent to L. Maddox (1978) and (independently) Freedman *et al.* (1978) has defined x to be strongly almost convergent to a number L if

$$\frac{1}{k+1} \sum_{i=0}^{k} |x_{i+m} - L| \to 0 \quad as \quad k \to \infty, \text{ uniformly in } m.$$

Let $[f]$ denote the set of all strongly almost convergent sequences. If x is strongly almost convergent to L, we write $[f] - \lim x = L$. It is easy to see that $[f] \subset f \subset \ell_\infty$. Das & Sahoo (1992) defined the sequence space

$$[\hat{w}(p)] = \left\{ x \in w : \frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x) - L|^{p_k} \to 0 \ as\ n \to \infty, \text{uniformly in } m \right\}$$

and investigated some of its properties, where $p = (p_k)$ is a sequence of positive real numbers.

The order of statistical convergence of a sequence of numbers was given by Gadjiev & Orhan (2002) and after then statistical convergence of order α and strong $p-$Cesàro summability of order α studied by Çolak (2010); Çolak (2011) and generalized by Çolak & Bektas (2011).

The statistical convergence of order α is defined as follows. Let $0 < \alpha \leq 1$ be given. The sequence (x_k) is said to be statistically convergent of order α if there is a real number L such that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0,$$

for every $\varepsilon > 0$, in which case we say that x is statistically convergent of order α, to L. In this case we write $S^{\alpha}-\lim x_k = L$. The set of all statistically convergent sequences of order α will be denoted by S^{α}.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1$. The set of all such sequences will be denoted by Λ. The generalized de la Vallée-Poussin mean is defined by $t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k$, where $I_n = [n-\lambda_n+1, n]$, Leindler (1965).

Throughout the paper, unless stated otherwise, by "for all $n \in N_{n_o}$" we mean "for all $n \in N$ except finite numbers of positive integers" where $N_{n_o} = \{n_o, n_o+1, n_o+2, ...\}$ for some $n_o \in N = \{1, 2, 3, ...\}$.

MAIN RESULTS

In this section we give the main results of this paper. In Theorem 2.3 we give the inclusion relations between the set of $\hat{S}^{\alpha}_{\lambda}-$statistically convergent sequences of order α and the set of $\hat{S}^{\beta}_{\mu}-$statistically convergent sequences of order β. In Theorem 2.6 we give the relationship between the strong $\hat{w}^{\alpha}_{p}(\lambda)-$summability of order α and the strong $\hat{w}^{\beta}_{p}(\mu)-$summability of order β. In Theorem 2.9 we give the relationship between the strong $\hat{w}^{\beta}_{p}(\mu)-$summability of order β and the $\hat{S}^{\alpha}_{\lambda}-$statistical convergence of order α.

Definition 2.1 Let $\lambda = (\lambda_n) \in \Lambda$ and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k) \in w$ is said to be $\hat{S}^{\alpha}_{\lambda}-$statistically convergent of order α if there is a real number L such that

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}|\{k\in I_n:|t_{km}(x)-L|\geq\varepsilon\}|=0, \text{uniformly in } m,$$

where $I_n = [n-\lambda_n+1, n]$ and λ_n^{α} denote the α^{th} power $(\lambda_n)^{\alpha}$ of λ_n, that is $\lambda^{\alpha} = (\lambda_n^{\alpha}) = (\lambda_1^{\alpha}, \lambda_2^{\alpha}, ..., \lambda_n^{\alpha}, ...)$. In this case we write $\hat{S}^{\alpha}_{\lambda}-\lim x_k = L$. The set of all $\hat{S}^{\alpha}_{\lambda}-$statistically convergent sequences of order α will be denoted by $\hat{S}^{\alpha}_{\lambda}$. For $\lambda_n = n$ for all $n \in N$, we shall write $\hat{S}^{\alpha}$ instead of $\hat{S}^{\alpha}_{\lambda}$ and in the special case $\alpha = 1$ we shall write $\hat{S}_{\lambda}$ instead of $\hat{S}^{\alpha}_{\lambda}$ and also in the special case $\alpha = 1$ and $\lambda_n = n$ for all $n \in N$ we shall write $\hat{S}$ instead of $\hat{S}^{\alpha}_{\lambda}$.

The $\hat{S}_{\lambda}^{\alpha}-$statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$ in general. For this let $\lambda_n = n$ for all $n \in N$ and $x = (x_k)$ be defined as follows:

$$x_k = \begin{cases} 1, & if\ k\ is\ \ even \\ 0, & if\ k\ is\ odd \end{cases}$$

then both

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |t_{km}(x) - 1| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{[\lambda_n] + 1}{2\lambda_n^{\alpha}} = 0\ ,\ \text{uniformly in}\, m$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |t_{km}(x) - 0| \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{[\lambda_n] + 1}{2\lambda_n^{\alpha}} = 0,\ \text{uniformly in}\, m$$

for $\alpha > 1$ and so $x = (x_k)$, $\hat{S}_{\lambda}^{\alpha}-$statistically converges of order α, both to 1 and 0, i.e $\hat{S}_{\lambda}^{\alpha} - \lim x_k = 1$ and $\hat{S}_{\lambda}^{\alpha} - \lim x_k = 0$, where $t_{0m}(x) = x_m$. But this is impossible.

Definition 2.2 Let $\lambda = (\lambda_n) \in \Lambda$ be given, $\alpha \in (0, 1]$ be any real number and let p be a positive real number. A sequence x is said to be strongly $\hat{w}_p^{\alpha}(\lambda)-$summable of order α, if there is a real number L such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \sum_{k\in I_n} |t_{km}(x) - L|^p = 0\ , \quad \text{uniformly in}\, m,$$

where $I_n = [n - \lambda_n + 1, n]$. In this case we write $\hat{w}_p^{\alpha}(\lambda) - \lim x_k = L$. The strong $\hat{w}_p^{\alpha}(\lambda)-$summability of order α reduces to the strong $\hat{w}_p(\lambda)-$summability for $\alpha = 1$. The set of all strongly $\hat{w}_p^{\alpha}(\lambda)-$summable sequences of order α will be denoted by $\hat{w}_p^{\alpha}(\lambda)$.

Theorem 2.3 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$ and let α and β be such that $0 < \alpha \leq \beta \leq 1$.

(i) If

$$\liminf_{n\to\infty} \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} > 0 \tag{1}$$

then $\hat{S}_{\mu}^{\beta} \subseteq \hat{S}_{\lambda}^{\alpha}$,

(ii) If

$$\lim_{n\to\infty}\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}=1 \qquad and \qquad \lim_{n\to\infty}\frac{\mu_n}{\mu_n^{\beta}}=1 \tag{2}$$

then $\hat{S}_{\lambda}^{\alpha}=\hat{S}_{\mu}^{\beta}$.

Proof (i) Suppose that $\lambda_n\le\mu_n$ for all $n\in N_{n_o}$ and let (1) be satisfied. Since $I_n\subset J_n$, for given $\varepsilon>0$ we have

$$\{k\in J_n:|t_{km}(x)-L|\ge\varepsilon\}\supset\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}$$

and so

$$\frac{1}{\mu_n^{\beta}}|\{k\in J_n:|t_{km}(x)-L|\ge\varepsilon\}|\ge\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\alpha}}|\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}|$$

for all $n\in N_{n_o}$, where $J_n=[n-\mu_n+1,n]$. Now taking the limit as $n\to\infty$ in the last inequality and using (1) we get $\hat{S}_{\mu}^{\beta}\subseteq\hat{S}_{\lambda}^{\alpha}$.

(ii) Let $(x_k)\in\hat{S}_{\lambda}^{\alpha}$ and (2) be satisfied. Since $I_n\subset J_n$, for $\varepsilon>0$ we may write

$$\begin{aligned}
\frac{1}{\mu_n^{\beta}}|\{k\in J_n:|t_{km}(x)-L|\ge\varepsilon\}| &= \frac{1}{\mu_n^{\beta}}|\{n-\mu_n+1\le k\le n-\lambda_n:|t_{km}(x)-L|\ge\varepsilon\}|\\
&+\frac{1}{\mu_n^{\beta}}|\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}|\\
&\le\left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)+\frac{1}{\mu_n^{\beta}}|\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}|\\
&\le\left(\frac{\mu_n-\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)+\frac{1}{\mu_n^{\beta}}|\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}|\\
&\le\left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)+\frac{1}{\lambda_n^{\alpha}}|\{k\in I_n:|t_{km}(x)-L|\ge\varepsilon\}|
\end{aligned}$$

for all $n\in N_{n_o}$. Since $\lim_n\frac{\mu_n}{\mu_n^{\beta}}=1$ and $\lim_n\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}=1$ by (2) the first term and since $x=(x_k)\in\hat{S}_{\lambda}^{\alpha}$ the second term of right hand side of above inequality tend to 0 as $n\to\infty$ (Note that $\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\ge0$ for all $n\in N_{n_o}$). This implies that $\hat{S}_{\lambda}^{\alpha}\subseteq\hat{S}_{\mu}^{\beta}$. Since (2) implies (1) we have the equality $\hat{S}_{\lambda}^{\alpha}=\hat{S}_{\mu}^{\beta}$.

From Theorem 2.3 we have following results.

Corollary 2.4 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$.

i) If (1) holds with $\beta = \alpha$, then $\hat{S}_\mu^\alpha \subseteq \hat{S}_\lambda^\alpha$ for each $\alpha \in (0, 1]$,

ii) If (1) holds with $\beta = 1$, then $\hat{S}_\mu \subseteq \hat{S}_\lambda^\alpha$ for each $\alpha \in (0, 1]$.

Corollary 2.5 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$.

i) If (2) holds with $\beta = \alpha$, then $\hat{S}_\lambda^\alpha \subseteq \hat{S}_\mu^\alpha$ for each $\alpha \in (0, 1]$,

ii) If (2) holds with $\beta = 1$, then $\hat{S}_\lambda^\alpha \subseteq \hat{S}_\mu$ for each $\alpha \in (0, 1]$.

Theorem 2.6 Given for $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ suppose that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$ and let $0 < \alpha \leq \beta \leq 1$. Then

(*i*) If (1) holds then $\hat{w}_p^\beta(\mu) \subset \hat{w}_p^\alpha(\lambda)$,

(*ii*) If (2) holds then $\ell_\infty \cap \hat{w}_p^\alpha(\lambda) \subseteq \hat{w}_p^\beta(\mu)$.

Proof (*i*) Omitted.

(*ii*) Let $x = (x_k) \in \ell_\infty \cap \hat{w}_p^\alpha(\lambda)$ and suppose that (2) holds. Since $x = (x_k) \in l_\infty$ then there exists some $M > 0$ such that $|t_{km}(x) - L| \leq M$ for all k. Now, since $\lambda_n \leq \mu_n$ and so that $\frac{1}{\mu_n^\beta} \leq \frac{1}{\lambda_n^\alpha}$, and $I_n \subset J_n$ for all $n \in N_{n_o}$, we may write

$$\begin{aligned}
\frac{1}{\mu_n^\beta} \sum_{k \in J_n} |t_{km}(x) - L|^p &= \frac{1}{\mu_n^\beta} \sum_{k \in J_n - I_n} |t_{km}(x) - L|^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |t_{km}(x) - L|^p \\
&\leq \left(\frac{\mu_n - \lambda_n}{\mu_n^\beta} \right) M^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |t_{km}(x) - L|^p \\
&\leq \left(\frac{\mu_n - \lambda_n^\alpha}{\mu_n^\beta} \right) M^p + \frac{1}{\mu_n^\beta} \sum_{k \in I_n} |t_{km}(x) - L|^p \\
&\leq \left(\frac{\mu_n}{\mu_n^\beta} - \frac{\lambda_n^\alpha}{\mu_n^\beta} \right) M^p + \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} |t_{km}(x) - L|^p
\end{aligned}$$

for every $n \in N_{n_o}$. Therefore $\ell_\infty \cap \hat{w}_p^\alpha(\lambda) \subseteq \hat{w}_p^\beta(\mu)$.

Corollary 2.7 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$.

i) If (1) holds with $\beta = \alpha$,then $\hat{w}_p^\alpha(\mu) \subset \hat{w}_p^\alpha(\lambda)$ for each $\alpha \in (0, 1]$, *ii*) If (1) holds with $\beta = 1$,then $\hat{w}_p(\mu) \subset \hat{w}_p^{\alpha}(\lambda)$ for each $\alpha \in (0, 1]$.

Corollary 2.8 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$.

i) If (2) holds with $\beta = \alpha$, then $\ell_\infty \cap \hat{w}_p^{\alpha}(\lambda) \subseteq \hat{w}_p^{\alpha}(\mu)$ for each $\alpha \in (0, 1]$,

ii) If (2) holds with $\beta = 1$, then $\ell_\infty \cap \hat{w}_p^{\alpha}(\lambda) \subseteq \hat{w}_p(\mu)$ for each $\alpha \in (0, 1]$.

Theorem 2.9 Let α,$\beta \in (0, 1]$ be fixed real numbers such that $\alpha \leq \beta$, $0 < p < \infty$ and $\lambda = (\lambda_n)$, $\mu = (\mu_n) \in \Lambda$ such that $\lambda_n \leq \mu_n$ for all $n \in N_{n_o}$.

(*i*) Let (1) holds, if a sequence is strongly $\hat{w}_p^{\beta}(\mu)-$summable of order β, to L, then it is $\hat{S}_\lambda^{\alpha}-$statistically convergent of order α, to L,

(*ii*) Let (2) holds, then if a sequence is bounded and $\hat{S}_\lambda^{\alpha}-$statistically convergent of order α, to L then it is strongly $\hat{w}_p^{\beta}(\mu)-$summable of order β, to L.

Proof. (*i*) For any sequence $x = (x_k)$ and $\varepsilon > 0$, we have

$$\begin{aligned}
\sum_{k \in J_n} |t_{km}(x) - L|^p &= \sum_{\substack{k \in J_n \\ |t_{km}(x)-L| \geq \varepsilon}} |t_{km}(x) - L|^p + \sum_{\substack{k \in J_n \\ |t_{km}(x)-L| < \varepsilon}} |t_{km}(x) - L|^p \\
&\geq \sum_{\substack{k \in I_n \\ |t_{km}(x)-L| \geq \varepsilon}} |t_{km}(x) - L|^p + \sum_{\substack{k \in I_n \\ |t_{km}(x)-L| < \varepsilon}} |t_{km}(x) - L|^p \\
&\geq \sum_{\substack{k \in I_n \\ |t_{km}(x)-L| \geq \varepsilon}} |t_{km}(x) - L|^p \\
&\geq |\{k \in I_n : |t_{km}(x) - L| \geq \varepsilon\}| \,.\, \varepsilon^p
\end{aligned}$$

and so that

$$\frac{1}{\mu_n^{\beta}} \sum_{k \in J_n} |t_{km}(x) - L|^p \geq \frac{1}{\mu_n^{\beta}} |\{k \in I_n : |t_{km}(x) - L| \geq \varepsilon\}| \ \varepsilon^p$$

$$\geq \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : |t_{km}(x) - L| \geq \varepsilon\}| \ \varepsilon^p.$$

Since (1) holds it follows that if $x = (x_k)$ is strongly $\hat{w}_p^{\beta}(\mu)-$summable of order β, to L, then it is $\hat{S}_\lambda^{\alpha}-$statistically convergent of order α, to L.

(*ii*) Suppose that $\hat{S}_\lambda^{\alpha} - \lim x_k = L$ and $x = (x_k) \in \ell_\infty$. Then there exists some $M > 0$ such that $|t_{km}(x) - L| \leq M$ for all k, then for every $\varepsilon > 0$ we may write

$$
\begin{aligned}
\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}|t_{km}(x)-L|^p &= \frac{1}{\mu_n^{\beta}}\sum_{k\in J_n-I_n}|t_{km}(x)-L|^p+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}|t_{km}(x)-L|^p\\
&\leq \left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)M^p+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}|t_{km}(x)-L|^p\\
&\leq \left(\frac{\mu_n-\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M^p+\frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}|t_{km}(x)-L|^p\\
&= \left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M^p+\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|\geq\varepsilon\}}}|t_{km}(x)-L|^p\\
&\quad+\frac{1}{\mu_n^{\beta}}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|<\varepsilon}}|t_{km}(x)-L|^p\\
&\leq \left(\frac{\mu_n}{\mu_n^{\beta}}-\frac{\lambda_n^{\alpha}}{\mu_n^{\beta}}\right)M^p+\frac{M^p}{\lambda_n^{\alpha}}|\{k\in I_n:|t_{km}(x)-L|\geq\varepsilon\}|+\frac{\lambda_n}{\mu_n^{\beta}}\,\varepsilon^p
\end{aligned}
$$

for all $n\in N_{n_o}$. Using (2) we obtain that $\hat{w}_p^{\beta}(\lambda)-\lim x_k=L$, whenever $\hat{S}_{\lambda}^{\alpha}-\lim x_k=L$.

Corollary 2.10 Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ for all $n\in N_{n_o}$.

i) If (1) holds with $\beta=\alpha$, then $\hat{w}_p^{\alpha}(\mu)\subset\hat{S}_{\lambda}^{\alpha}$ for each $\alpha\in(0,1]$,

ii) If (1) holds with $\beta=1$, then $\hat{w}_p(\mu)\subset\hat{S}_{\lambda}^{\alpha}$ for each $\alpha\in(0,1]$.

Corollary 2.11 Let $\lambda=(\lambda_n)$ and $\mu=(\mu_n)$ be two sequences in Λ such that $\lambda_n\leq\mu_n$ for all $n\in N_{n_o}$.

i) If (2) holds with $\beta=\alpha$, then $\ell_\infty\cap\hat{S}_{\lambda}^{\alpha}\subset\hat{w}_p^{\alpha}(\mu)$ for each $\alpha\in(0,1]$,

ii) If (2) holds with $\beta=1$, then $\ell_\infty\cap\hat{S}_{\lambda}^{\alpha}\subset\hat{w}_p(\mu)$ for each $\alpha\in(0,1]$.

RESULTS RELATED TO MODULUS FUNCTION

In this section we give the inclusion relations between the sets of $\hat{S}_{\lambda}^{\alpha}-$statistically convergent sequences of order α and strongly $\hat{w}_{(p)}^{\alpha}[\lambda,f]-$summable sequences of order α with respect to the modulus function f.

The notion of a modulus was introduced by Nakano (1951). We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. Maddox (1986) and Ruckle (1973) used a modulus function to construct some sequence spaces. Later on using a modulus function different sequence spaces have been studied by Altin (2009); Bhardwaj & Bala (2008); Çolak (2003); Et (2003, 2006); Gaur & Mursaleen (1998); Nuray & Savas (1993) and many others.

Definition 3.1 Let f be a modulus function, $p = (p_k)$ be a sequence of strictly positive real numbers and let $\alpha \in (0,1]$ be any real number. Now we define

$$\hat{w}^{\alpha}_{(p)}[\lambda,f] = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{\lambda^{\alpha}_n} \sum_{k\in I_n} [f(|t_{km}(x) - L|)]^{p_k} = 0 \,\text{uniformly in m, for some}\, L \right\}.$$

In the following theorems we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$.

Theorem 3.2 Let $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \le \beta$, f be a modulus function and $\lambda = (\lambda_n)$ be a sequence in Λ. Then $\hat{w}^{\alpha}_{(p)}[\lambda,f] \subset \hat{S}^{\beta}_{\lambda}$.

Proof. Let $x \in \hat{w}^{\alpha}_{(p)}[\lambda,f]$ and $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_n$, $|t_{km}(x) - L| \ge \varepsilon$ and $k \in I_n$, $|t_{km}(x) - L| < \varepsilon$, respectively. Since $\lambda^{\alpha}_n \le \lambda^{\beta}_n$ for each n we may write

$$\begin{aligned}
\frac{1}{\lambda^{\alpha}_n} \sum_{k\in I_n} [f(|t_{km}(x) - L|)]^{p_k} &= \frac{1}{\lambda^{\alpha}_n} \left[\sum_1 [f(|t_{km}(x) - L|)]^{p_k} + \sum_2 [f(|t_{km}(x) - L|)]^{p_k} \right] \\
&\ge \frac{1}{\lambda^{\beta}_n} \left[\sum_1 [f(|t_{km}(x) - L|)]^{p_k} + \sum_2 [f(|t_{km}(x) - L|)]^{p_k} \right] \\
&\ge \frac{1}{\lambda^{\beta}_n} \sum\nolimits_1 [f(\varepsilon)]^{p_k} \\
&\ge \frac{1}{\lambda^{\beta}_n} \sum\nolimits_1 \min([f(\varepsilon)]^h, [f(\varepsilon)]^H) \\
&\ge \frac{1}{\lambda^{\beta}_n} |\{k \in I_n : |t_{km}(x) - L| \ge \varepsilon\}| \min([f(\varepsilon)]^h, [f(\varepsilon)]^H).
\end{aligned}$$

Since $x \in \hat{w}^{\alpha}_{(p)}[\lambda,f]$, the left hand side of the above inequality tends to zero as $n \to \infty$ uniformly in m. Therefore the right hand side tends to zero as $n \to \infty$ uniformly in m and hence $x \in \hat{S}^{\beta}_{\lambda}$.

Theorem 3.3 Let $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \leq \beta$ and $\lambda = (\lambda_n)$ be a sequence in Λ. If the modulus f is bounded and $\lim \frac{\lambda_n}{\lambda_n^\alpha} = 1$,then $\hat{S}_\lambda^\alpha \subset \hat{w}_{(p)}^\beta[\lambda, f]$.

Proof. Let $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \leq \beta$, $\lambda = (\lambda_n)$ be a sequence in Λ, $x \in \hat{S}_\lambda^\alpha$ and suppose that f is bounded and $\varepsilon > 0$ be given. Since f is bounded there exists an integer K such that $f(x) \leq K$, for all $x \geq 0$. Then since $\lambda_n^\alpha \leq \lambda_n^\beta$ for each n we may write

$$\frac{1}{\lambda_n^\beta}\sum_{k\in I_n}[f(|t_{km}(x)-L|)]^{p_k} \leq \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}[f(|t_{km}(x)-L|)]^{p_k}$$

$$=\frac{1}{\lambda_n^\alpha}\left(\sum_1 [f(|t_{km}(x)-L|)]^{p_k} + \sum_2 [f(|t_{km}(x)-L|)]^{p_k}\right)$$

$$\leq \frac{1}{\lambda_n^\alpha}\sum_1 \max\left(K^h, K^H\right) + \frac{1}{\lambda_n^\alpha}\sum_2 [f(\varepsilon)]^{p_k}$$

$$\leq \max\left(K^h, K^H\right)\frac{1}{\lambda_n^\alpha}\left|\{k \in I_n : f(|t_{km}(x)-L|) \geq \varepsilon\}\right|$$

$$+\frac{\lambda_n}{\lambda_n^\alpha}\max\left(f(\varepsilon)^h, f(\varepsilon)^H\right)$$

Since $x \in \hat{S}_\lambda^\alpha$ the first term of the right hand side of above inequality tends to zero as $n \to \infty$ uniformly in m, and the second term can be made as small as desired. Therefore the left hand side of above inequality tends to zero as $n \to \infty$ uniformly in m. Hence $x \in \hat{w}_{(p)}^\beta[\lambda, f]$.

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متتاليات قوية قرب - الجمعية من المرتبة α

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خلاصة

ندخل في هذا البحث مفهوم التقارب الاحصائي من المرتبة α ثم نقوم بإيجاد بعض العلاقات بين هذا التقارب من جهة وخاصية الجمع القوية من المرتبة β من جهة أخرى.بالاضافة إلى ذلك. نبحث بعض العلاقات بين فضاءات $\hat{w}_{(p)}^{\alpha}[\lambda,f]$ وفضاءات $\hat{s}_{\lambda}^{\alpha}$.

حوليات الآداب والعلوم الاجتماعية

ANNALS OF THE ARTS AND SOCIAL SCIENCES

- مجلة فصلية محكمة.
- تصدر عن مجلس النشر العلمي بجامعة الكويت.
- صدر العدد الأول سنة ١٩٨٠م.
- تنشر الموضوعات التي تدخل في مجالات اهتمام الأقسام العلمية لكليتي الآداب والعلوم الاجتماعية.
- تنشر الابحاث والدراسات باللغتين العربية والإنجليزية شريطة أن لا يقل حجم البحث عن ٥٠ صفحة وأن لا يزيد عن ٢٠٠ صفحة مطبوعة من ثلاث نسخ.
- لا يقتصر النشر في الحوليات على أعضاء هيئة التدريس لكليتي الآداب والعلوم الاجتماعية فحسب ، بل يشمل ما يعادل هذه التخصصات في الجامعات والمعاهد الاخرى داخل الكويت وخارجها.
- تمنح المجلة الباحث خمسين نسخة من بحثه المنشور كإهداء.

رئيس هيئة التحرير

د. فاطمة إبراهيم آل خليفة

ثمن الرسالة للأفراد

(٥٠٠ فلس)

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