Kuwait J. Sci. **41** (2) pp. 15-33, 2014

Fuzzy soft Γ-hemirings

XUELING MA AND JIANMING ZHAN

Department of Mathematics, Hubei University for Nationalities, Enshi, Hubei province 445000, China
E-mail: zjmhbmy@126.com, zhanjianming@hotmail.com

ABSTRACT

Maji *et al.* (2001) introduced the concept of a fuzzy soft set, which is an extension to the concept of a soft set. The concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideals (right h-ideals, h-interior-ideals) in Γ-hemirings are introduced. Some new characterization theorems of these kinds of fuzzy soft h-ideals of a Γ-hemiring are also given. Finally, we show that h-hemiregular and h-semisimple Γ-hemirings can be described by $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideals.

Keywords: Γ-hemirings, Fuzzy soft set; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft (left, right) h-ideal; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideal; (h-hemiregular, h-semisimple) Γ-hemiring.

INTRODUCTION

Uncertainties, which could be caused by information incompleteness, data randomness limitations of measuring instruments, etc., are pervasive in many complicated problems in biology, engineering, economics, environment, medical science and social science. Alternatively, mathematical theories, such as probability theory (Zadeh, 1965), fuzzy set theory, vague set theory, rough set theory and interval mathematics, have been proven to be useful mathematical tools for dealing with uncertainties. However all these theories have their inherent difficulties, as pointed out by Molodtsov (1999). Nowadays, works on the soft set theory are progressing rapidly. Maji *et al.* (2003) discussed soft set theory. Ali *et al.* (2009) proposed some new operations on soft sets. Qin & Hong (2010) investigated soft equality. In particular, fuzzy soft set theory has been investigated by some researchers, for examples, (Maji *et al.,* 2001; Majumdar & Samanta, 2010). In the same time, this theory has proven useful in many different fields such as decision making (Cagman & Enginoglu, 2010; Feng *et al.,* 2010; Maji *et al.,* 2002), data analysis, forecasting and so on. Recently, the algebraic structures of soft sets have been studied increasingly. Aktas & Cagman (2007) defined the notion of soft groups and derived some related properties.

Feng *et al.* (2008) investigated soft semirings by using the soft set theory. Jun (2008) introduced and investigated the notion of soft BCK/BCI-algebras. Jun (2008) discussed the applications of soft sets in ideal theory of BCK/BCI-algebras. Zhan & Jun (2010) characterized the (implicative, positive implicative and fantastic) filteristic soft *BL*-algebras based on $\in$-soft sets and q-soft sets. Yin & Zhan (2012) introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h -ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-quasi-ideals and investigated their fundamental properties and mutual relationships. They also investigated the characterization of (left) h-hemiregular and (left) duo hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals. The other important results can be found in (Inan & Ozturk, 2012; Sezgin & Atagun, 2011; Yang, 2011; Yin *et al.,* 2011).

We note that, the ideals of semirings play a crucial role in the structure theory and ideals in semirings do not in general coincide with the ideals of a ring. For this reason, the usage of ideals in semirings is somewhat limited. Jun *et al.* (2004) considered the fuzzy h-ideals of hemirings. The h-hemiregular hemirings were described by Zhan & Dudek (2007) by using the fuzzy h-ideals. Furthermore, Yin & Li (2008) introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings. As a continuation of these investigations, Ma & Zhan (2009) introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. Recently, Ma *et al.* (2012) introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-) ideals of hemirings. In particular, some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings were investigated by these kinds of fuzzy h-ideals. The general properties of fuzzy h-ideals have been considered by Dudek, Dutta, Jun, Ma, Zhan, Yin and others. The reader is refereed to (Dudek *et al.,* 2009; Dudek *et al.,* 2010; Jun, 1995; Ma & Zhan, 2007; Ma *et al.,* 2011; Yin *et al.,* 2009; Zhan & Davvaz, 2007).

The concept of Γ-rings was first introduced in 1964 by Barnes (1996). Further, Kyuno *et al.* (1987) discussed regular Γ-rings. Also, the notion of fuzzy ideals in a Γ-ring was introduced by Jun (1995) and Jun & Lee (1992). They gave some properties of fuzzy ideals of Γ-rings. Furthermore, Ozturk *et al.* (2003); Ozturk *et al.* (2002) characterized the Artinian and Noetherian Γ-rings. In particular, Dutta & Chanda (2005) studied the fuzzy ideals of a Γ-ring and characterized the Γ-fields and Noetherian Γ-rings by considering the fuzzy ideals via operator rings of Γ-rings. The concept of Γ-semirings was then introduced by Rao (1995) and some properties of such Γ-semirings have been studied, for example, Dutta & Sardar (2002) and Sardar & Mandal (2009). Recently, Ma & Zhan (2010) and Zhan & Shum (2011) described the characterizations of Γ-hemirings.

The present paper is organized as follows. In section 2, we recall some concepts and properties of Γ-hemirings, fuzzy sets and fuzzy soft sets. In section 3, we introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft *h*-ideals (*h*-interior-ideals) of Γ-hemirings and investigate some related properties. In section 4, we describe the characterizations of *h*-hemiregular Γ-hemirings and *h*-semisimple Γ-hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft *h*-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft *h*-interior-ideals.

PRELIMINARIES

In this section, we recall some basic notions and results about Γ-hemirings, fuzzy sets and fuzzy soft sets (Jun *et al.,* 2004; Ma & Zhan, 2010; Maji *et al.,* 2003; Yin *et al.,* 2009; Yin & Zhan, 2012; Zhan & Dudek, 2007; Zhan & Shum, 2011).

2.1 Γ-hemirings

Let S and Γ be two commutative semigroups. Then S is said to be a Γ-semiring if there exists a mapping $S\times\Gamma\times S\rightarrow S$ (images to be denoted by $a\alpha b$ for $ab \in S$ and$\alpha \in \Gamma$) satisfies the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$;

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a zero of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with a zero is said to be a Γ-hemiring.

Throughout this paper, S is a Γ-hemiring and we use 0_S to denote the zero element of S.

A left (resp., right) ideal of a Γ-hemiring S is a subset A of S which is closed under addition such that $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$), where $S\Gamma A = \{x\alpha y | x, y \in S, \alpha \in \Gamma\}$. Naturally, a subset A of S is called an ideal of S, if it is both a left ideal and a right ideal of S.

A subset I of S is called an interior ideal of S if I is closed under addition such that $I\Gamma I \subseteq I$ and $S\Gamma I\Gamma S \subseteq I$.

A left ideal (right ideal, ideal) A of S, is called a left *h*-ideal (right *h*-ideal, *h*-ideal) of S, respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z \rightarrow x \in A$.

The *h*-closure $\overline{A}$ of A in S is defined by

$$\overline{A} = \{x \in S | x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A, z \in S\}.$$

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

An interior ideal I of S is called an h-interior ideal of S if I is closed under addition such that $\overline{I\Gamma I} \subseteq I$, $\overline{S\Gamma I\Gamma S} \subseteq I$ and for any $x, z \in S$ and $a, b \in I$, then from $x + a + z = b + z$ it follows that $x \in I$.

Definition 2.1 (Ma & Zhan, 2010; Zhan & Shum, 2011) (i) Let μ and ν be fuzzy sets of S. Then the *h-sum* of μ and ν is defined by

$$(\mu +_h \nu)(x) = \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}.$$

(ii) Let μ and ν be fuzzy sets of S. Then the *h-product* of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigcup_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + a_1\gamma_1 b_1 + z = a_2\gamma_2 b_2 + z$.

A fuzzy subset μ of X of the from

$$\mu(y) = \{r(\neq 0) \quad \text{if y = x}, 0 \qquad \text{otherwise}$$

is said to be a fuzzy point with support x and value r and is denoted by x_r, where $r \in (0, 1]$.

In what follows, let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $Y \subseteq X$, define $\chi_{\gamma Y}^{\delta}$ be the fuzzy subset of X by $\chi_{\gamma Y}^{\delta}(x) \geq \delta$ for all $x \in Y$ and $\chi_{\gamma Y}^{\delta}(x) \leq \gamma$ otherwise. Clearly, $\chi_{\gamma Y}^{\delta}$ is the characteristic function of Y if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and fuzzy subset μ of X, we say that

(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.

(2) $x_r q_\delta \mu$ if $\mu(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta$ if $x_r \in_\gamma \mu$ or $x_r q_\delta \mu$.

(4) $x_r \overline{\in_\gamma \vee q_\delta} \mu$ if $x_r \in_\gamma \vee q_\delta \mu$ does not hold.

Let us now introduce an ordered relation on $F(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $\mu, \nu \in F(x)$, by $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta \nu$

for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, μ and ν are said to be (γ, δ)-equal, denoted by $\mu =_{(\gamma,\delta)} \nu$, if $\mu \subseteq \vee q_{(\gamma,\delta)}\nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)}\mu$.

The following is similar to Proposition (3.5) in Zhan & Shum (2011).

Lemma 2.2 *Let S be a* Γ*-hemiring and* $X, Y \subseteq S$. *Then we have*

(1) $X \subseteq Y$ *if and only if* $\chi_{\gamma}^{\delta} \subseteq \vee q_{(\gamma,\delta)}\chi_{\gamma Y}^{\delta}$.

(2) $\chi_{\gamma X}^{\delta} \cap \chi_{\gamma Y}^{\delta} =_{(\gamma,\delta)} \chi_{\gamma(X\cap Y)}^{\delta}$.

(3) $\chi_{\gamma X}^{\delta} +_h \chi_{\gamma Y}^{\delta} =_{(\gamma,\delta)} \chi_{\gamma\overline{X+Y}}^{\delta}$.

(4) $\chi_{\gamma X}^{\delta}\Gamma_h\chi_{\gamma Y}^{\delta} =_{(\gamma,\delta)} \chi_{\gamma\overline{X\Gamma Y}}^{\delta}$.

2.2 Fuzzy soft sets

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. As a generalization of soft set introduced, Molodtsov (1999) and Maji *et al.* (2001) defined fuzzy soft set in the following way.

Definition 2.3 (Maji *et al.,* 2001) A pair $\langle F, A\rangle$ is called a fuzzy soft set over $U\}$, *where* $A \subseteq E$ *and* F is a mapping given by $F : A \to F(U)$.

In general, for every$\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called fuzzy value set of parameter ε. The set of all fuzzy soft sets over U with parameters from E is called a fuzzy soft classes, and it is denote by $FJ(U, E)$.

Definition 2.4 (Ali *et al.,* 2009; Maji *et al.,* 2001; Yin & Zhan, 2012) (1) Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that $\langle F, A\rangle$ is an fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle\langle G, B\rangle$ if

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$. $\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be fuzzy soft equal and write $\langle F, A\rangle = \langle G, B\rangle$ if $\langle F, A\rangle\langle G, B\rangle$ and $\langle G, B\rangle\langle F, A\rangle$.

The extended intersection of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is a fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \widetilde{\cap} \langle G, B\rangle$.

(3) Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U such that $A \cap B \neq \emptyset$.

The restricted intersection of $\langle F, A\rangle$ and $\langle G, B\rangle$ is defined to be the fuzzy soft set $\langle H, C\rangle$, where $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle\langle G, B\rangle$.

(4) Let $V \subseteq U$. A fuzzy soft set $\langle F, A\rangle$ over V is said to be a relative whole (γ, δ) -fuzzy soft set (with respect to universe set V and parameter set A), denoted by $\sum(V, A)$, $F(\varepsilon) = \chi_{\gamma V}^{\delta}$ for all $\varepsilon \in A$.

Definition 2.5 (Yin & Zhan, 2012) Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that is an (γ, δ)-fuzzy soft subset of $\langle G, B\rangle$ and write $\langle F, A\rangle_{(\gamma,\delta)}\langle G, B\rangle$ if

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}G(\varepsilon)$. $\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be (γ, δ)-fuzzy soft equal and write $\langle F, A\rangle \asymp_{(\gamma,\delta)} \langle G, B\rangle$ if $\langle F, A\rangle_{(\gamma,\delta)}\langle G, B\rangle$ and $\langle G, B\rangle_{(\gamma,\delta)}\langle F, A\rangle$.

We point out that $_{\cancel{(\gamma,\delta)}}$ means $_{(\gamma,\delta)}$ does not hold.

Definition 2.6 (i) The h-sum of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over S is a fuzzy soft set over S, denoted by $\langle F +_h G, C\rangle$, where $C = A \cup B$ and

$$(F +_h G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) +_h G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F +_h G, C\rangle = \langle F, A\rangle +_h \langle G, B\rangle$.

(ii) The h-product of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over S is a fuzzy soft set over S, denoted by $\langle F\Gamma_h G, C\rangle$, where $C = A \cup B$ and

$$(F\Gamma_h G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon)\Gamma_h G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F\Gamma_h G, C\rangle = \langle F, A\rangle\Gamma_h\langle G, B\rangle$.

$(\in_\gamma, \in_\gamma \vee \boldsymbol{q}_\delta)$-FUZZY SOFT *h*-IDEALS(*h*-INTERIOR-IDEALS)

In this section, we will introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals (h -interior-ideals) over a Γ-hemiring and investigate their fundamental properties.

Definition (3.1)*A fuzzy soft set* $\langle \boldsymbol{F}, \boldsymbol{A}\rangle$ *over* $\boldsymbol{S}$ *is called an* $(\in_\gamma, \in_\gamma \vee \boldsymbol{q}_\delta)$*-fuzzy soft* *left (resp., right)* ***h*****-*ideal over*** ***S*** *if it satisfies:*

(F1a) $\langle F, A\rangle +_h \langle F, A\rangle_{(\gamma,\delta)}\langle F, A\rangle$,

(F1b) $\sum(S, A)\Gamma_h\langle F, A\rangle_{(\gamma,\delta)}\langle F, A\rangle$(resp., $\langle F, A\rangle\Gamma_h \sum(S, A)\langle F, A\rangle$),

(F1c) $x + a + z = b + z, a_r, b_s \in_\gamma F(\varepsilon) \rightarrow x_{\min\{r,s\}} \in_\gamma \vee q_\delta F(\varepsilon)$ for all $a, b, x, z \in S, \varepsilon \in A$ and $r, s \in (\gamma, 1]$.

A fuzzy soft set over S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S.

It is worth attending that for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h-ideal $\langle F, A\rangle$ over S, we obtain $\max\{F(\varepsilon)(0), \gamma\} \geq \min\{F(\varepsilon)(x), \delta\}$ for all $\varepsilon \in A$ and $x \in S$.

Example 3.2 Let $(S, +)$ and $(\Gamma, +)$ be two semigroups, where S and Γ are the sets of all non-negative integers and the operations are the usual additive operations. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\gamma b = a \cdot \gamma \cdot b$ for all $a, b \in S$ and $\gamma \in \Gamma$, where "·" is the usual multiplication. Then it can be easily verified that S, under the above multiplication and the structure Γ-mapping, is a Γ-hemiring. Define a fuzzy soft set $\langle F, A\rangle$ over S, where $A = S$, by

$$F(\varepsilon)(x) = \begin{cases} 0.6 & \text{if } x \in (4), \\ 0.8 & \text{if } x \in (2) - (4), \\ 0.2 & \text{otherwise}, \end{cases}$$

for all $\varepsilon, x \in S$, where $(2) = \{2n|n \in N\}$, $(4) = \{4n|n \in N\}$, $(2) - (4) = \{4n - 2|n \in N^+\}$. Here, N and N^+ are an set of all non-negative integer numbers and an set of all positive integer numbers, respectively. Then one can easily check that $\langle F, A\rangle$ is an $(\in_{0.6}, \in_{0.6} \vee q_{0.8})$-fuzzy soft h-interior-ideal over S.

Theorem (3.3) *A fuzzy soft set* $\langle \boldsymbol{F}, \boldsymbol{A}\rangle$ *over* ***S*** *is an* $(\in_\gamma, \in_\gamma \vee \boldsymbol{q}_\delta)$*-fuzzy soft left (resp., right)* ***h*****-*ideal over*** ***S*** *if and only if it satisfies:*

(F2a) $\max\{F(\varepsilon)(x + y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ *for all* $\}x, y \in S$ *and* $\varepsilon \in A$.

(F2b) $\max\{F(\varepsilon)(x\alpha y), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$(*resp.*, $\max\{F(\varepsilon)(x\alpha y), \gamma\} \geq \min\{F(\varepsilon)(x), \delta\}$) *for all* $x, y \in S$,$\alpha \in \Gamma$ *and* $\varepsilon \in A$.

(F2c) $x + a + z = b + z \rightarrow \max\{F(\varepsilon)(x), \gamma\} \geq \min\{F(\varepsilon)(b), \delta\}$ *for all* $a, b, x, z \in S$ *and* $\varepsilon \in A$.

Proof. Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S. For any $x, y \in S$, $\varepsilon \in A$. If $\max\{F(x)(x + y), \gamma\} < t < \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$, then $F(\varepsilon)(x) > t$, $F(\varepsilon)(y) > t$ and $F(\varepsilon)(x + y) < t < \delta$, that is, $(x + y)_t\overline{\in_\gamma \vee q_\delta}F(\varepsilon)$. On the other hand, we have $\max\{F(\varepsilon)(0),\gamma\} \geq \min\{F(\varepsilon)(x),\delta\}$ and so $F(\varepsilon)(0) \geq \min\{F(\varepsilon)(x),\delta\}$ since $\gamma < \min\{F(\varepsilon)(x),\delta\}$ Thus

$$\begin{aligned}
&(F(\varepsilon) +_h F(\varepsilon))(x+y)\\
&= \min\Big\{\bigcup\nolimits_{x+y+a_1+b_1+z=a_2+b_2+z} \min\{F(\varepsilon)(a_1), F(\varepsilon)(a_2), F(\varepsilon)(b_1), F(\varepsilon)(b_2)\}\Big\}\\
&\geq \min\{F(\varepsilon)(0), F(\varepsilon)(x), F(\varepsilon)(y)\}\\
&\geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}\\
&> \min\{t,\delta\} = t,
\end{aligned}$$

Which implies, $(x+y)_t \in_\gamma F(\varepsilon) +_h F(\varepsilon)$, and so $(x+y)_t \in_\gamma \vee q_\delta F(\varepsilon)$. This is a contradiction. This implies that (F2a) holds.

Now, if there exist $x, y \in S$, $\alpha \in \Gamma$, $\varepsilon \in A$ such that $\max\{F(\varepsilon)(x\alpha y),\gamma\} < t < \min\{F(\varepsilon)(y),\delta\}$, then $F(\varepsilon)(y) > t$, and $F(\varepsilon)(x\alpha y) < t < \delta$, that is , $(x\alpha y)_t \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$. On the other hand,

$$\begin{aligned}
&(F(\varepsilon)\Gamma_h F(\varepsilon))(x\alpha y)\\
&= \min\Big\{\bigcup\nolimits_{x\alpha y+a_1+b_1+z=a_2+b_2+z} \min\Big\{\chi_\gamma^\delta(\varepsilon)(a_1), \chi_\gamma^\delta(\varepsilon)(a_2), F(\varepsilon)(b_1), F(\varepsilon)(b_2)\Big\}\Big\}\\
&\geq \min\{F(\varepsilon)(0), F(\varepsilon)(y)\}\\
&\geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}\\
&> \min\{t,\delta\} = t,
\end{aligned}$$

which implies, $(x\alpha y)_t \in_\gamma F(\varepsilon) +_h F(\varepsilon)$, and so $(x\alpha y)_t \in_\gamma q_\delta F(\varepsilon)$. This is a contradiction. This implies that (F2b) holds.

Finally, if there exist $a, b, x, z \in S$ with $x + a + z = b + z$, $\varepsilon \in A$ and $r \in (\gamma, 1]$ shch that $\max\{F(\varepsilon)(x),\gamma\} < r < \min\{F(\varepsilon)(a), F(\varepsilon)(b),\delta\}$, then $F(\varepsilon)(b) > r$ and $F(\varepsilon)(x) < r < \delta$, this implies $a_r, b_r \in_\gamma F(\varepsilon)$ but $x_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, which contradicts with the condition F(1c). Hence the condition (F2c) is valid.

Conversely, assume that the given conditions (F2a), (F2b) and (F2c) hold. If $\langle F, A\rangle +_h \langle F, A\rangle \overline{(\gamma,\delta)} \langle F, A\rangle$, then there exist $\varepsilon \in A$ and $x_r \in_\gamma F(\varepsilon) +_h F(\varepsilon)$ such that $x_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$. Hence $F(\varepsilon)(x) < r$ and $F(\varepsilon)(x) + r \leq 2\delta$, which gives $F(\varepsilon)(x) < \delta$. If there exist $a_1, a_2, b_1, b_2 x, z \in S$ with $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by the conditions F(2a) and F(2c), we have

$$
\begin{aligned}
\delta > \max\{F(\varepsilon)(x),\gamma\} &\geq \max\{\min\{F(\varepsilon)(a_1+b_1),F(\varepsilon)(a_2+b_2),\delta\},\gamma\}\\
&= \min\{\max\{F(\varepsilon)(a_1+b_1),\gamma\}\max\{F(\varepsilon)(a_2+b_2),\gamma\},\delta\}\\
&\geq \min\{\min\{F(\varepsilon)(a_1),F(\varepsilon)(b_1),\delta\},\min\{F(\varepsilon)(a_2),F(\varepsilon)(b_2),\delta\},\delta\}\\
&= \min\{F(\varepsilon)(a_1),F(\varepsilon)(a_2),F(\varepsilon)(b_1),F(\varepsilon)(b_2),\delta\}
\end{aligned}
$$

which implies $\max\{F(\varepsilon)(x),\gamma\} \geq \min\{F(\varepsilon)(a_1),F(\varepsilon)(a_2),F(\varepsilon)(b_1),F(\varepsilon)(b_2)\}$. Hence we have

$$
\begin{aligned}
&r \leq (F(\varepsilon)+_h F(\varepsilon))(x) = \textstyle\bigcup_{x+a_1+b_1+z=a_2+b_2+z} \min\{F(\varepsilon)(a_1),F(\varepsilon)(a_2),F(\varepsilon)(b_1),F(\varepsilon)(b_2)\}\\
&\leq \textstyle\bigcup_{x+a_1+b_1+z=a_2+b_2+z} \max\{F(\varepsilon)(x),\gamma\} = \max\{F(\varepsilon)(x),\gamma\},
\end{aligned}
$$

a contradiction. Therefore, (F1a) holds.

Similarly, we can prove (F1b) holds.

Finally, if there exist $a,b,x,z \in S$, $\varepsilon \in A$ and $r,s \in (\gamma,1]$ with $x+a+z=b+z$ and $a_r,b_s \in F(\varepsilon)$ such that $x_{\min\{r,s\}}\overline{\in_\gamma \vee q_\delta}F(\varepsilon)$, then $F(\varepsilon)(a) \geq r > \gamma$, $F(\varepsilon)(b) \geq s > \gamma$ but $F(\varepsilon)(x) < \min\{r,s\}$ and $F(\varepsilon)(x)+\min\{r,s\} < 2\delta$, it follows that $F(\varepsilon)(x) < \delta$. Hence $\max\{F(\varepsilon)(x),\gamma\} < \min\{F(\varepsilon)(a),F(\varepsilon)(b),\delta\}$, a contradiction. Thus the condition (F1c) is valid.

For any fuzzy soft set $\langle F,A\rangle$ over a Γ-hemiring S, $\varepsilon \in A$ and $r \in (\gamma,1]$. Denote $F(\varepsilon)_r = \{x \in S | x_r \in_\gamma F(\varepsilon)\}$, $\langle F(\varepsilon)\rangle_r = \{x \in S | x_r q_\delta F(\varepsilon)\}$ and $[F(\varepsilon)]_r = \{x \in S | x_r \in_\gamma \vee q_\delta F(\varepsilon)\}$. The next theorem presents the relationships between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (resp., right) h-ideals and crisp left (resp., right) h-ideals of a Γ-hemiring S.

Theorem 3.4 *Let* S *be a* Γ*-hemiring and* $\langle F,A\rangle$ *a fuzzy soft set over* S*. Then*

(1) $\langle F,A\rangle$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left (resp., right) h-ideal over* S *if and only if non-empty subset* $F(\varepsilon)_r$ *is a left (resp., right) h-ideal of* S *for all* $\varepsilon \in A$ *and* $r \in (\gamma,\delta]$.

(2) *If* $2\delta = 1+\gamma$*, then* $\langle F,A\rangle$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left (resp., right) h-ideal over* S *if and only if non-empty subset* $\langle F(\varepsilon)\rangle_r$ *is a left (resp., right) h-ideal of* S *for all* $\varepsilon \in A$ *and* $r \in (\gamma,\delta]$.

(3) $\langle F,A\rangle$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left (resp., right) h-ideal over* S *if and only if non-empty subset* $\langle F(\varepsilon)\rangle_r$ *is a left (resp., right) h-ideal of* S *for all* $\varepsilon \in A$ *and* $r \in (\gamma, \min\{2\delta-\gamma,1\}]$.

Proof. We only prove (2) and (3). The conclusion (1) can be easily proved.

(2) Assume that $2\delta = 1+\gamma$. Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S and assume that $\langle F(\varepsilon)\rangle_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in (\delta, 1]$. Let $x, y \in \langle F(\varepsilon)\rangle_r$. Then $x_r q_\delta F(\varepsilon)$ and $y_r q_\delta F(\varepsilon)$, that is, $F(\varepsilon)(x) + r > 2\delta$ and $F(\varepsilon)(y) + r > 2\delta$. Since $\langle F, A\rangle$ is an (γ, δ)-fuzzy soft h-left ideal over S, we have $\max\{F(\varepsilon)(x+y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$. Hence, by $r > \delta$,

$$\begin{aligned}\max\{F(\varepsilon)(x+y)+r, \gamma + r\} &= \max\{F(\varepsilon)(x+y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} + r\\ &= \min\{F(\varepsilon)(x)+r, F(\varepsilon)(y)+r, \delta+\gamma\} > 2\delta\end{aligned}$$

From $r \leq 1 = 2\delta - \gamma$, that is $r + \gamma \leq 2\delta$, we have $F(\varepsilon)(x+y) + r > 2\delta$ and so $x + y \in \langle F(\varepsilon)\rangle_r$. Similarly, we can show that $x\alpha y, y\alpha x \in \langle F(\varepsilon)\rangle_r$ for all $x \in \langle F(\varepsilon)\rangle_r$, $\alpha \in \Gamma$ and $y \in S$ and that $x + a + z = b + z$ for $x, z \in S$ and $a, b \in \langle F(\varepsilon)\rangle_r$ implies $x \in \langle F(\varepsilon)\rangle_r$. Therefore, $\langle F(\varepsilon)\rangle_r$ is a left h-ideal of S.

Conversely, assume that the given conditions hold. If here exist $\varepsilon \in A$ and $x, y \in S$ such that $\max\{F(\varepsilon)(x+y), \gamma\} < \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$. Take $r = 2\delta - \max\{F(\varepsilon)(x+y), \gamma\}$. Then $r \in (\delta, 1]$, $F(\varepsilon)(x+y) \leq 2\delta - r$, $F(\varepsilon)(x) > \max\{G(\varepsilon)(x+y), \gamma\} = 2\delta - r$ and $F(\varepsilon)(y) > \max\{G(\varepsilon)(x+y), \gamma\} = 2\delta - r$, that is $x, y \in \langle F(\varepsilon)\rangle_r$ but $x + y \notin \langle F(\varepsilon)\rangle_r$, a contradiction. Hence $\langle F, A\rangle$ satisfies condition (F2a). Similarly we may show that $\langle F, A\rangle$ satisfies conditions (F2b) and (F2c). Therefore, $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S.

(3) Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S and assume that $[F(\varepsilon)]_r \neq \emptyset$ for some $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta - \gamma, 1\}]$. Let $x, y \in [F(\varepsilon)]_r$. Then $x_r \in_\gamma \vee q_\delta F(\varepsilon)$ and $y_r \in_\gamma \vee q_\delta F(\varepsilon)$ that is $F(\varepsilon)(x) \geq r > \gamma$ or $F(\varepsilon)(x) > 2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$, and $F(\varepsilon)(y) \geq r > \gamma$ or $F(\varepsilon)(y) > 2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$. Since $\langle F, A\rangle$ is an $(\in_r, \in_r \vee q_\delta)$-fuzzy soft left h-ideal over S, we have $2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$. Since $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S, we have $\max\{F(\varepsilon)(x+y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ and so $F(\varepsilon)(x+y) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ since $\gamma < \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ in any case. Now we consider the following cases.

Case 1: $r \in (\gamma, \delta]$ Then $2\delta - r \geq \delta \geq r$.

(1) If $F(\varepsilon)(x) \geq r$ or $F(\varepsilon)(y) \geq r$, then

$F(\varepsilon)(x+y) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} \geq r$. Hence $(x+y)_r \in_\gamma F(\varepsilon)$.

(2) If $F(\varepsilon)(x) + r > 2\delta$ and $F(\varepsilon)(y) + r > 2\delta$, then

$F(\varepsilon)(x+y) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} = \delta \geq r$. Hence $(x+y)_r \in_\gamma F(\varepsilon)$.

Case 2: $r \in (\delta, \min\{2\delta - \gamma, 1\}]$. Then $r > \delta > 2\delta - r$.

(1) If $F(\varepsilon)(x) \geq r$ and $F(\varepsilon)(y) \geq r$, then

$F(\varepsilon)(x+y) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} = \delta > 2\delta - r$. Hence $(x+y)_r q_\delta F(\varepsilon)$.

(2) If $F(\varepsilon)(x) + r > 2\delta$ or $F(\varepsilon)(y) + r > 2\delta$, then

$F(\varepsilon)(x+y) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} > 2\delta - r$. Hence $(x+y)_r q_\delta F(\varepsilon)$.

Thus, in any case, $(x+y)_r \in_\gamma \vee q_\delta F(\varepsilon)$, that is, $x + y \in [F(\varepsilon)]_r$. Similarly, we can show that $x\alpha y, y\alpha x \in [F(\varepsilon)]_r$ for all $x \in [F(\varepsilon)]_r$, $\alpha \in \Gamma$ and $y \in S$ and that $x + a + z = b + z$ for $x, z \in S$ and $a, b \in [F(\varepsilon)]_r$ implies $x \in [F(\varepsilon)]_r$. Therefore, $[F(\varepsilon)]_r$ is a left h-ideal of S.

Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in S$ such that $\max\{F(\varepsilon)(x+y), \gamma\} < r = \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$. Then $x_r, y_r \in_\gamma F(\varepsilon)$ but $(x+y)_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, that is, $x, y \in [F(\varepsilon)]_r$ but $x + y \overline{\in} [F(\varepsilon)]_r$, a contradiction. Hence $\langle F, A\rangle$ satisfies the condition (F2a). Similarly we may show that $\langle F, A\rangle$ satisfies the conditions (F2b) and (F2c). Therefore, $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S.

The case for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideals over S can be similarly proved.

Ddfinition 3.5 *A fuzzy soft set* $\langle \boldsymbol{F}, \boldsymbol{A}\rangle$ *over* $\boldsymbol{S}$ *is called an* $(\in_\gamma, \in_\gamma \vee \boldsymbol{q}_\delta)$*-fuzzy soft left (resp., right)* ***h****-ideal over* $\boldsymbol{S}$ *if it satisfies (F1a), (F1c) and*

(F3a) $\langle F, A\rangle \Gamma_h \langle F, A\rangle_{(\gamma,\delta)} \langle F, A\rangle$,

(F3b) $\sum(S, A)\Gamma_h \langle F, A\rangle \Gamma_h \sum(S, A)_{(\gamma,\delta)} \langle F, A\rangle$.

Example 3.6 Let S be a hemiring with the multiplicative identity 1. Then S is a Γ-hemiring, where $\Gamma = S$ and $a\alpha b$ denotes the product of elements a, α, b in S. Now any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideal of the hemiring S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior-ideal of the Γ-hemiring S.

From Definitions 3.1 and 3.5, we can obtain:

Theorem 3.7 *Every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft h-ideal over S is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft h-interior-ideal over S.*

Theorem 3.8 *A fuzzy soft set* $\langle F, A\rangle$ *over S is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft h-interior-ideal over S if and only if it satisfies (F2a), (F2c) and*

(F4a) $\max\{F(\varepsilon)(x\alpha y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ *for all* $x, y \in S$, $\alpha \in \Gamma$ *and* $\varepsilon \in A$.

(F4b) $\max\{F(\varepsilon)(x\alpha y\beta z), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$ *for all* $x, y, z \in S$, $\alpha, \beta \in \Gamma$ *and* $\varepsilon \in A$.

Proof. The proof is similar to the proof of Theorem 3.3.

Theorem 3.9 *Let S be a Γ-hemiring and $\langle F, A\rangle$ a fuzzy soft set over S. Then*

(1) $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideal over S if and only if *non*-empty subset $F(\varepsilon)_r$ is an h-interior-ideal of S for all $\varepsilon \in A$ and $r \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideal over S if and only if *non*-empty subset $\langle F(\varepsilon)\rangle_r$ is an h-interior-ideal of S for all $\varepsilon \in A$ and $r \in (\delta, 1]$.

(3) $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideal over S if and only if *non*-empty subset $[F(\varepsilon)]_r$ ia an h-interior-ideal of S for all $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta - \gamma, 1\}]$.

Proof. The proof is similar to the proof of Theorem 3.4.

Characterizations of Γ-hemirings

In this section, we describe the characterizations of h-hemiregular (h-semisimple) Γ-hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideals.

Definition 4.1 (Ma & Zhan, 2010; Zhan & Shum, 2011) *A Γ-hemiring S is said to be h-hemiregular if for each $x \in S$, there exist $a, a', z \in S$ and $\alpha, \alpha', \beta, \beta' \in \Gamma$ such that*

$$x + x\alpha a\beta x + z = x\alpha' a'\beta' x + z.$$

Example 4.2 (Ma & Zhan, 2010) *Let S and Γ be two sets of all non-negative integers N_0 with an element $\infty \geq x$ for all $x \in N_0$. Define the operation "+" by*

$$a + b = a \vee b \text{ for all } a, b \in S, \Gamma.$$

Then S and Γ are additive semigroups.

Now, define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = a \wedge b$ for all $a, b \in S$ and $\gamma \in \Gamma$.

Then, by routine verification, we can easily see that S is an h-hemiregular Γ-hemiring.

Lemma 4.3 (Ma & Zhan, 2010) *A Γ-hemiring S is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B, we have $\overline{A\Gamma B} = A \cap B$.*

Theorem 4.4 *A Γ-hemiring S is h-hemiregular if and only if $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \underset{\cap}{\cup} (\gamma, \delta)\langle F, A\rangle \Gamma_h \langle G, B\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideal $\langle F, A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal $\langle G, B\rangle$ over S.*

Proof. Let S be an h-hemiregular Γ-hemiring, $\langle F, A\rangle$ any $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy soft right h-ideal and $\langle G, B\rangle$ any $\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy soft left h-ideal over S, respectively.

Thus, we have

$\langle F, A\rangle\Gamma_h\langle G, B\rangle_{(\gamma,\delta)}\langle F, A\rangle\Gamma_h \sum(S, A)_{(\gamma,\delta)}\langle F, A\rangle$

and $\langle F, A\rangle\Gamma_h\langle G, B\rangle_{(\gamma,\delta)} \sum(S, A)\Gamma_h\langle G, B\rangle_{(\gamma,\delta)}\langle G, B\rangle$.

This proves that $\langle F, A\rangle\Gamma_h\langle G, B\rangle_{(\gamma,\delta)}\langle F, A\rangle\widetilde{\cap}\langle G, B\rangle$. (*)

Now let x be any element of $S, \varepsilon \in A \cup B$ and $\langle F, A\rangle\widetilde{\cap}\langle G, B\rangle = \langle H, A \cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A - B$. Then $H(\varepsilon) = F(\varepsilon) = (F\Gamma_h G)(\varepsilon)$.

Case 2: $\varepsilon \in B - A$. Then $H(\varepsilon) = G(\varepsilon) = (F\Gamma_h G)(\varepsilon)$.

Case 3: $\varepsilon \in A \cap B$. Then $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $(F\Gamma_h G)(\varepsilon) = F(\varepsilon)\Gamma_h G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)\Gamma_h G(\varepsilon)$.

Since S is h-hemiregular, there exist $\alpha, \alpha', \beta, \beta' \in \Gamma$ and $a, a', z \in S$ such that $x + x\alpha a\beta x + z = x\alpha' a'\beta' x + z$.

Then we have

$$\begin{aligned}
&\max\{(F(\varepsilon)\Gamma_h G(\varepsilon))(x), \gamma\}\\
&= \max\left\{\bigcup\nolimits_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{F(\varepsilon)(a_1), F(\varepsilon)(a_2), G(\varepsilon)(b_1), G(\varepsilon)(b_2)\}, \gamma\right\}\\
&\geq \max\{\min\{F(\varepsilon)(x\alpha a), F(\varepsilon)(x\alpha' a'), G(\varepsilon)(x)\}, \gamma\}\\
&= \min\{\max\{F(\varepsilon)(x\alpha a), \gamma\}, \max\{F(\varepsilon)(x\alpha' a'), \gamma\}, \max\{G(\varepsilon)(x), \gamma\}\}\\
&\geq \min\{\min\{F(\varepsilon)(x), \delta\}, \min\{G(\varepsilon)(x), \delta\}\} = \min\{(F(\varepsilon) \cap G(\varepsilon))(x), \delta\}.
\end{aligned}$$

It follows that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)\Gamma_h G(\varepsilon)$.

Thus, in any case, we have

$$\langle F, A\rangle\widetilde{\cap}\langle G, B\rangle_{(\gamma,\delta)}\langle F, A\rangle\Gamma_h\langle G, B\rangle. \qquad (**)$$

Combling (*) and (**), we have $\langle F, A\rangle\widetilde{\cap}\langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle\Gamma_h\langle G, B\rangle$.

Conversely, let R and L be any right h-ideal and any left h-ideal of S, respectively. Then $\sum(L, A)$ and $\sum(R, A)$ are an $(\in_r, \in_r \vee q_\delta)$-fuzzy soft right h-ideal and an $(\in_r, \in_r \vee q_\delta)$-fuzzy soft left h-ideal over S, respectively. Hence by Lemma 2.2, we have

$$\chi^{\delta}_{\gamma(R\cap L)} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma R} \cap \chi^{\delta}_{\gamma L} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma R}\Gamma_h \chi^{\delta}_{\gamma L} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma\overline{R\Gamma L}}.$$

It follows from Lemma 2.2 that $R \cap L = \overline{R\Gamma L}$. Therefore S is h-hemiregular by Lemma 4.3.

Definition 4.5 (Ma & Zhan, 2010) *Let S be a Γ-hemiring. Then the following statements are equivalent:*

(1) S is h-semisimple;

(2) $a \in \overline{S\Gamma a\Gamma S\Gamma a\Gamma S}$, for all $a \in S$;

(3) $A \subseteq \overline{S\Gamma A\Gamma S\Gamma A\Gamma S}$, for all $A \subseteq S$.

Theorem 4.7 *Let S be an h-semisimple Γ-hemiring, $\langle F, A\rangle$ a fuzzy soft set over S. Then $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S if and only if it is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S.*

Proof. If $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S, then it follows from Theorem 3.7 that it is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior-ideal.

Conversely, Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior-ideal over S. For any $x, y \in S$ and $\varepsilon \in A$. Since S is h-semisimple, by Lemma 4.6, there exist $a_i, a_i', z \in S(i = 1, 2, 3, 4)$ and $\beta_i, \beta_i' \in \Gamma(i = 1, 2, 3, 4, 5)$ such that

$$x + a_1\beta_1 x\beta_2 a_2\beta_3 a_3\beta_4 x\beta_5 a_4 + z = a_1'\beta_1' x\beta_2' a_2'\beta_3' a_3'\beta_4' x\beta_5' a_4' + z,$$

and so

$$x\alpha y + a_1\beta_1 x\beta_2 a_2\beta_3 a_3\beta_4 x\beta_5 a_4\alpha y + z\alpha y = a_1'\beta_1' x\beta_2' a_2'\beta_3' a_3'\beta_4' x\beta_5' a_4'\alpha y + z\alpha y.$$

Thus, we have

$$\begin{aligned}
&\max\{F(\varepsilon)(x\alpha y), \gamma\}\\
&\geq \max\big\{\min\big\{F(\varepsilon)(a_1\beta_1 x\beta_2 a_2\beta_3 a_3\beta_4 x\beta_5 a_4\alpha y), F(\varepsilon)\big(a_1'\beta_1' x\beta_2' a_2'\beta_3' a_3'\beta_4' x\beta_5' a_4'\alpha y\big), \delta\big\}, \gamma\big\}\\
&\geq \min\big\{\max\{F(\varepsilon)(a_1\beta_1 x\beta_2 a_2\beta_3 a_3\beta_4 x\beta_5 a_4\alpha y), \gamma\}, \max\big\{F(\varepsilon)\big(a_1'\beta_1' x\beta_2' a_2'\beta_3' a_3'\beta_4' x\beta_5' a_4'\alpha y\big), \gamma\big\}, \max\{\delta, \gamma\}\big\}\\
&\geq \min\{F(\varepsilon)(x), \delta\}.
\end{aligned}$$

It follows that $\langle F, A\rangle\Gamma_h \sum(S, A)_{(\gamma,\delta)}\langle F, A\rangle$ holds. Thus, by Definition 3.1, we know that $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideal over S. Similarly, we can prove it is also an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S. Therefore, $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S.

Theorem 4.8 *A Γ-hemiring S is h-semisimple if and only if for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior-ideals $\langle F, A\rangle$ and $\langle G, B\rangle$ over S, we have $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle\Gamma_h\langle G, B\rangle$.*

Proof. Let S be an h-semisimple Γ-hemiring, $\langle F, A\rangle$ and $\langle G, B\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals over S. Thus, we have

$$\langle F, A\rangle \Gamma_h \langle G, B\rangle_{(\gamma,\delta)} \langle F, A\rangle \Gamma_h \sum(S, A)_{(\gamma,\delta)} \langle F, A\rangle$$

and $\langle F, A\rangle \Gamma_h \langle G, B\rangle_{(\gamma,\delta)} \sum(S, A) \Gamma_h \langle G, B\rangle_{(\gamma,\delta)} \langle G, B\rangle$.

This proves that $\langle F, A\rangle \Gamma_h \langle G, B\rangle_{(\gamma,\delta)} \langle F, A\rangle \widetilde{\cap} \langle G, B\rangle$. (*)

Now let $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle = \langle H, A \cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A - B$. Then $H(\varepsilon) = F(\varepsilon) = (F\Gamma_h G)(\varepsilon)$.

Case 2: $\varepsilon \in B - A$. Then $H(\varepsilon) = G(\varepsilon) = (F\Gamma_h G)(\varepsilon)$.

Case 1: $\varepsilon \in A \cap B$. Then $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $(F\Gamma_h G)(\varepsilon) = F(\varepsilon)\Gamma_h G(\varepsilon)$.

Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)\Gamma_h G(\varepsilon)$. For any $x, y \in S$, since S is h-semisimple, there exist $a_i, a'_i, z \in S(i = 1, 2, 3, 4)$ and $\beta_i, \beta'_i \in \Gamma(i = 1, 2, 3, 4, 5)$ such that

$$x + a_1\beta_1 x\beta_2 a_2\beta_3 a_3\beta_4 x\beta_5 a_4 + z = a'_1\beta'_1 x\beta'_2 a'_2\beta'_3 a'_3\beta'_4 x\beta'_5 a'_4 + z.$$

Then we have

$$\begin{aligned}
&\max\{(F(\varepsilon)\Gamma_h G(\varepsilon))(x), \gamma\}\\
&= \max\Big\{\bigcup_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{F(\varepsilon)(a_1), F(\varepsilon)(a_2), G(\varepsilon)(b_1), G(\varepsilon)(b_2)\}, \gamma\Big\}.\\
&\geq \max\{\min\{F(\varepsilon)(a_1\beta_1 x\beta_2 a_2), F(\varepsilon)(a'_1\beta'_1 x\beta'_2 a'_2), G(\varepsilon)(\beta_4 x\beta_5 a_4), G(\varepsilon)(\beta'_4 x\beta'_5 a'_4)\}, \gamma\}\\
&= \min\{\min\{F(\varepsilon)(x), \delta\}, \min\{G(\varepsilon)(x), \delta\}\}\\
&= \min\{(F(\varepsilon) \cap G(\varepsilon))(x), \delta\}.
\end{aligned}$$

It follows that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)\Gamma_h G(\varepsilon)$, that is, $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F\Gamma_h G)(\varepsilon)$. Thus, in any case, $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)}(F\Gamma_h G)(\varepsilon)$.

This proves that $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle_{(\gamma,\delta)} \langle F, A\rangle \Gamma_h \langle G, B\rangle$. (*)

Combling (*) and (**), we have $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \Gamma_h \langle G, B\rangle$.

Conversely, let I be any h-ideal of S, then it is h-interior-ideal. Then $\sum(I, A)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S. Now, by the assumption, we have $\sum(I, A) \widetilde{\cap} \sum(I, A) \asymp_{(\gamma,\delta)} \sum(I, A) \Gamma_h \sum(I, A)$. Hence by Lemma 2.2, we have

$$\chi^\delta_{\gamma I} =_{(\gamma,\delta)} \chi^\delta_{\gamma I} \cap \chi^\delta_{\gamma I} =_{(\gamma,\delta)} \chi^\delta_{\gamma I} \Gamma_h \chi^\delta_{\gamma I} =_{(\gamma,\delta)} \chi^\delta_{\gamma \overline{I^2}}.$$

It follows from Lemma 2.2 that $I = \overline{I^2}$. Therefore S is h-semisimple by Definition 4.5.

CONCLUSION

In this paper, our aim is to promote the research and development of fuzzy soft technology by studying fuzzy soft Γ-hemirings. The goal is to explain new methodological development in Γ-hemirings which will also be of growing importance in the future.

In the future study of fuzzy Γ-hemirings, perhaps the following topics are worth to be considered:

(1) To describe roughness soft Γ-hemirings;

(2) To establish an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy spectrum of Γ-hemirings and its applications;

(3) To discuss fuzzy rough soft Γ-hemirings and some of its applications in computer science.

ACKNOWLEDGEMENT

This research is partially supported by a grant of National Natural Science Foundation of China (61175055), Innovation Term of Higher Education of Hubei Province, China (T201109), Natural Science Foundation of Hubei Province(2012FFB01101).

REFERENCES

Aktas, H. & Cagman, N. 2007. Soft sets and soft groups. Information Sciences **177**: 2726-2735.

Ali, M. I., Feng, F., Liu, X., Min, W. K. & Shabir, M. 2009. On some new operations in soft set theory. Computers and Mathematics with Applications **57**: 1547-1553.

Barnes, W. E. 1996. On theΓ-rings of Nobusawa. Pacific Journal of Mathematics **18**: 411-422.

Cagman, N. & Enginoglu, S. 2010. Soft matrix theory and its decision making. Computers and Mathematics with Applications **59**: 3308-3314.

Dudek, W. A., Shabir, M. & Ali, M. I. 2009. (α, β)-fuzzy ideals of hemirings. Computers and Mathematics with Applications **58**: 310-321.

Dudek, W. A., Shabir, M. & Anjum, R. 2010. Characterizations of hemirings by their *h*-ideals. Computers and Mathematics with Applications **59**: 3167-3179.

Dutta, T. K. & Chanda, T. 2005. Structures of fuzzy ideals of Γ-rings. Bulletin of Malaysian Mathematical Sciences Society **28**: 9-15.

Dutta, T. K & Sardar, S. K. 2002. On the operator semirings of a Γ-semirings. Southeast Bulletin of Mathematics **26**: 203-213.

Feng, F., Jun, Y. B., Liu, X. & Li, L. 2010. An adjusta ble approach to fuzzy soft set based decision making. Journal of Computation Applied Mathematics **234**: 10-20.

Feng, F., Jun, Y. B. & Zhao, X. 2008. Soft semirings. Computers and Mathematics with Applications **56**: 2621-2628.

Inan, E. & Ozturk, M. A. 2012. Fuzzy soft rings and fuzzy soft ideals. Neural Computation and Applications **21**:s1-s8.

Jun, Y. B. 1995. On fuzzy prime ideals of Γ-rings. Soochow Journal Mathematics **21**: 41-48.

Jun, Y. B. 2008. Soft BCK/BCI-algebras. Computers and Mathematics with Applications **56**: 1408-1413.

Jun, Y. B. & Lee, C.Y. 1992. Fuzzy Γ-rings. Pusan Kyongnan Math. J. (presently, East Asian Mathematics Journal) **8**: 163-170.

Jun, Y. B., Ozturk, M. A. & Song, S. Z. 2004. On fuzzy *h*-ideals in hemirings. Information Sciences **162**: 211-226.

Kyuno, S., Nobusawa, N. & Sith, N. B. 1987. Regular gamma rings. Tsukuba Journal Mathematics **11**: 371-382.

Ma, X., Yin, Y. & Zhan, J. 2012. Characterizations of *h*-intra- and *h* -quasi-hemiregular hemirings. Computers and Mathematics with Applications **63**: 783-793.

Ma, X. & Zhan, J. 2007. On fuzzy *h*-ideals of hemirings. Journal of Systems and Sciences Complexity **20**: 470-478.

Ma, X. & Zhan, J. 2009. Generalized fuzzy *h*-bi-ideals and *h*-quasi-ideals of hemirings. Information Sciences **179**: 1249-1268.

Ma, X. & Zhan, J. 2010. Fuzzy *h*-ideals in *h*-hemiregular and *h*-semisimple Γ-hemirings. Neural Computation and Application **19**: 477-485.

Ma, X., Zhan, J. & Shum, K. P. 2011. Generalized fuzzy *h*-ideals of hemirings. Bulletin of Malaysian Mathematical Sciences Society **34**: 561-574.

Maji, P. K., Biswas, R. & Roy, A. R. 2001. Fuzzy soft sets. Journal of Fuzzy Mathematics **9**: 589-602.

Maji, P. K., Biswas, R. & Roy, A. R. 2003. Soft set theory. Computers and Mathematics with Applications **45**: 555-562.

Maji, P. K., Roy, A. R. & Biswas, R. 2002. An application of soft sets in a decision making problem. Computers and Mathematics with Applications **44**: 1077-1083.

Majumdar, P. & Samanta, S. K. 2010. Generalised fuzzy soft sets. Computers and Mathematics with Applications **59**: 1425-1432.

Molodtsov, D. 1999. Soft set theory-first results. Computers and Mathematics with Applications **37**: 19-31.

Ozturk, M. A., Uckum, M. & Jun, Y. B. 2002. Characterizations of Artinian and Noetherian gamma-rings in terms of fuzzy ideals. Turkish Journal of Mathematics **27**: 199-205.

Ozturk, M. A., Uckum, M. & Jun, Y. B. 2003. Fuzzy ideals in Gamma-rings. Turkish Journal of Mathematics **27**: 369-374.

Qin, K. Y. & Hong, Z. 2010. On soft equality. Journal of Computation Applied Mathematics **234**: 1347-1355.

Rao, M. K. 1995. Γ-semirings-1. Southeast Bulletin of Mathematics **19**: 49-54.

Sardar, S. K. & Mandal, D. 2009. Fuzzy *h*-ideal in Γ-hemiring. International Journal of Pure and Applied Mathematics **56**: 439-450.

Sezgin, A. & Atagun, A. O. 2011. Soft groups and normalistic soft groups. Computers and Mathematics with Applications **62**: 685-698.

Yang, C. F. 2011. Fuzzy soft semigroups and fuzzy soft ideals. Computers and Mathematics with Applications **61**: 255-261.

Yin, Y., Huang, X., Xu, D. & Li, H. 2009. The characterizations of *h*-semisimple hemirings. International Journal of Fuzzy Systems **11**: 116-122.

Yin, Y., Jun, Y. B. & Zhan, J. 2011. Vague soft hemirings. Computers and Mathematics with Applications **62**: 199-213.

Yin, Y. & Li, H. 2008. The characterizations of *h*-hemiregular hemirings and *h*-intra- hemiregular hemirings. Information Sciences **178**: 3451-3464.

Yin, Y. & Zhan, J. 2012. The characterizations of hemirings in terms of fuzzy soft *h*-ideals, Neural Computation and Applications **21**: S43-S57.

Zadeh, place City L. A. 1965. Fuzzy sets. Information and Control **8**: 338-353.

Zhan, J. & Davvaz, B. 2007. *L*-fuzzy *h*-ideals with operators in hemirings. Northeast Mathematics Journal **23**: 1-14, in Chinese.

Zhan, J. & Dudek, W. A. 2007. Fuzzy *h*-ideals of hemirings. Information Sciences **177**: 876-886.

Zhan, J. & Jun, Y. B. 2010. Soft BL-algebras based on fuzzy sets. Computers and Mathematics with Applications **59**: 2037-2046.

Zhan, J. & Shum, K. P. 2011. On fuzzy *h*-ideals in Γ-hemirings. Neural Computation and Applications **20**: 495-505.

Submitted : *16/10/2012*
Revised : *06/05/2013*
Accepted : *29/08/2013*

نظر الحلقات المشوشة الرخوة من النوع Γ

زولنج ما و جيانمنج زهان

قسم الرياضيات - جامعة هوبي - إنشي، مقاطعة هوبي445000 - الصين

خلاصة

قام ماجي وآخرون (2001) بإدخال مفهوم المجموعة المشوشة الرخوة الذي هو إمتداد لمفهوم المجموعة الرخوة. نقوم في هذا البحث بإدخال مفاهيم المثالية اليسرى، المثالية اليمنى، المثالية الداخية في نظر الحلقات من النوع Γ. كما نعطي مبرهنات تمييز لهذه الانواع من المثاليات. أخيراً، نثبت أن نظر الحلقات شبة البسيطة ونظر المنتظمة يمكن وصفها بواسطة مثاليات مشوشة رخوة ومثاليات داخلية مشوشة رخوة.

مجلة دراسات الخليج والجزيرة العربية

مجلة علمية فصلية محكمة تصدر عن مجلس النشر العلمي - جامعة الكويت
صدر العدد الأول منها في يناير عام 1975م

رئيس التحرير

أ. د. بـدر عمـر العمـر

ترحـب المجلة بنشر البحوث والدراسات العلمية المتعلقة بشؤون منطقة الخليج والجزيرة العربية في مختلف علوم البحث والدراسة .

ومن أبوابها

- البحوث العربية.
- البحوث الإنجليزية.
- عرض الكتب ومراجعتها .
- البيبلوجرافيا العربية.
- ملخصات الرسائل الجامعية: ماجستير - دكتوراه.
- التقارير : مؤتمرات - ندوات

الاشتراكات

ترسل قيمة الاشتراك مقدماً بشيك لأمر - جامعة الكويت
مسحوب على أحد المصارف الكويتية

داخل دولة الكويت : للأفراد : 3 دنانير - للمؤسسات : 15 ديناراً
الدول العربية : للأفراد : 4 دنانير - للمؤسسات : 15 ديناراً
الدول الغير عربية : للأفراد : 4 دنانير - للمؤسسات : 15 ديناراً

توجه جميع المراسلات باسم رئيس تحرير مجلة دراسات الخليج والجزيرة العربية

Tel.: (+965) 24833215 - 24984066 - 24984067 Fax: (+965) 24833705 P.O. Box 17073 Al-Khaldiah, 72451 Kuwait
www.pubcouncil.kuniv.edu.kw/jgaps E - mail : jgaps@ku.edu.kw
w w w . k u . e d u . k w